Determination of Heat Physical Characteristics by using Series along Boundary Condition Derivatives

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Abstract: - One method for calculating heat physical characteristics is discussed in this article. The method considers a model of heat conduction process when one knows the temperature at an inner point of a body. This method is based on application of temperature measurements inside a body for determination of temperature conductivity coefficient and heat conductivity coefficient. For solution of a problem, a mathematical model of heat conduction is used where temperature at an inner point is put forth into a series along boundary condition derivatives.

Key-Words: temperature conductivity coefficient, boundary conditions, inverse problem, series.

1 Introduction

There are different methods for solving inverse coefficient problems of heat conduction. Methods that use temperature measurements at regular conditions are frequently used. Conditions are considered regular if effect of initial conditions has decreased so far that it is of no consequence and the temperature field can be described by Fourier series of mathematical model first item. There are other methods, too. For example, there are methods that use temperature measurements at the beginning of the process, methods that demand modifications of the temperature on the boundary to be periodical or demand constant heat flow on the boundary. Inverse problems are solved using temperature measurements of figures of simple shape.

2 Mathematical model

One-dimensional temperature field is considered as follows:

$$\frac{\partial t}{\partial \tau} = a\left(\frac{\partial^2 t}{\partial x^2} + \frac{k-1}{x}\frac{\partial t}{\partial x}\right), \qquad (1)$$

where t is temperature, τ is time, $x \in [-b,b]$ is coordinate if $k=1$ and $x \in [0,b]$ if $k=2$ or $k=3$, a is temperature conductivity coefficient. If $k=1$, equation (1) describes heat conduction process on a plate; if $k=2$, then in a cylinder; if $k=3$, then in a sphere. Let us presume that temperature distribution is homogenous in the beginning $t(x,0) = t_0$. We shall move temperature measurements aside so that the following would be fulfilled:

$$t(x,0) = 0\,. \qquad (2)$$

It is assumed that on border $x=b$ temperature is known and temperature field is symmetric:

$$t(b,\tau) = t_1(\tau)\,, \frac{\partial t(0,\tau)}{\partial x} = 0\,. \qquad (3)$$

Passing over to non-dimensional variables

$$N = \frac{x}{b}\,,\ \ F = \frac{a\tau}{b^2} \qquad (4)$$

where N is non-dimensional coordinate and F is non-dimensional time, thus problem (1)–(3) may be written as follows:

$$\frac{\partial t}{\partial F} = \frac{\partial^2 t}{\partial N^2} + \frac{k-1}{N}\frac{\partial t}{\partial N}\,, \qquad (5)$$

$$t(1,F) = t_1(F), t(N,0) = 0\,, \frac{\partial t(0,F)}{\partial N} = 0\,. \qquad (6)$$

According to [2] solution of problem (5) – (6) is recorded in the following way:

$$t(N,F) = \sum_{n=0}^{\infty} P_n(N) t_1^{(n)}(F)\,, \qquad (7)$$

where $t_1^{(n)}(F)$ is derivative of temperature on the border. Coordinate functions $P_n(N)$ are given below.

If $k=1$, then

$P_0(N) = 1\,,$

$$P_1(N)=\frac{N^2}{2}-\frac{1}{2},$$
$$P_2(N)=\frac{N^4}{24}-\frac{N^2}{4}+\frac{5}{24},$$
$$P_3(N)=\frac{N^6}{720}-\frac{N^4}{48}+\frac{5N^2}{48}-\frac{61}{720},$$
$$P_4(N)=\frac{N^8}{40320}-\frac{N^6}{1440}+\frac{5N^4}{576}-\frac{61N^2}{1440}+\frac{277}{8064}.$$

Series (7) converges quickly as [2] ,

$0 \le |P_n(N)| \le \frac{1}{2^n}$, if $N \in [0,1]$.

If $k=2$, then

$$P_0(N)=1,$$
$$P_1(N)=\frac{N^2}{4}-\frac{1}{4},$$
$$P_2(N)=\frac{N^4}{64}-\frac{N^2}{16}+\frac{3}{64},$$
$$P_3(N)=\frac{N^6}{2304}-\frac{N^4}{256}+\frac{3N^2}{256}-\frac{19}{2304},$$
$$P_4(N)=\frac{N^8}{147456}-\frac{N^6}{9216}+\frac{3N^4}{4096}-\frac{19N^2}{9216}+\frac{211}{147456}.$$

Convergence of series (7) is characterized by inequality [2] $0 \le |P_n(N)| \le \frac{1}{4^n}$, if $N \in [0,1]$.

Passing over to real time in series (7), we obtain

$$t(N,\tau)=\sum_{n=0}^{\infty} P_n(N) t_1^{(n)}(\tau)\left(\frac{b^2}{a}\right)^n. \tag{8}$$

3 The Inverse Problem

It is assumed that temperature is known at point $N_1 \in [0,1)$. Then having taken a number of finite addends in formula (8) and denoting $\frac{b^2}{a}=y$, we obtain

$$\sum_{n=1}^{M} P_n(N_1) t_1^{(n)}(\tau) y^n + t_1(\tau) - t(N_1,\tau) = 0. \tag{9}$$

As series (8) converges quickly, one has grounds to discuss a case when temperature field may be described by the first two addends of the series ($M=1$). If $M=1$, then it follows from formula (9) [2]:

$$a=\frac{b^2 P_1(N_1) t_1'(\tau)}{t(N_1,\tau)-t_1(\tau)}. \tag{10}$$

If temperature is measured in the center, id est., $N_1=0$, then it results from (10) that the following is valid if $k=1$:

$$a=\frac{b^2 t_1'(\tau)}{2(t_1(\tau)-t(0,\tau))}. \tag{11}$$

If $k=2$, then [1], [2]

$$a=\frac{b^2 t_1'(\tau)}{4(t_1(\tau)-t(0,\tau))}. \tag{12}$$

In article [1] formula (12) is obtained by means of different approach. It is anticipated that results would be of higher quality if more than two addends are taken in sum (9). If three addends are taken in sum ($M=2$), then (9) results in:

$$\begin{aligned}&P_2(N_1) t_1''(\tau) y^2 + P_1(N_1) t_1'(\tau) y + \\ &+ t_1(\tau) - t(N_1,\tau) = 0\end{aligned}. \tag{13}$$

Equation (13) is a quadratic equation as regards y, thus it has two roots, but temperature conduction coefficient is one. One can prove that if $t_1'(\tau)>0$ and $t_1''(\tau)<0$, which can be easily provided experimentally, then root signs are different. Therefore a question, which root is valid, does not arise. If temperature measurements are made in the center of plate ($k=1$), then equation (13) is the following:

$$\begin{aligned}&\frac{5}{24} t_1''(\tau) y^2 - \frac{1}{2} t_1'(\tau) y + t_1(\tau) - \\ &- t(N_1,\tau) = 0\end{aligned}. \tag{14}$$

If four addends are summed up and temperature is measured in the center of plate ($k=1$, $M=3$), we obtain the following from equation (9)

$$\begin{aligned}&-\frac{61}{720} t_1^{(3)}(\tau) y^3 + \frac{5}{24} t_1''(\tau) y^2 - \\ &-\frac{1}{2} t_1'(\tau) y + t_1(\tau) - t(N_1,\tau) = 0\end{aligned}. \tag{15}$$

Equation (15) has three roots. We have not proved it, but calculations made at various boundary conditions show that two of those are complex roots and one is real. Consequently any problem does not appear.

If heat flow on the border $q(\tau)$ is known, then heat conductivity coefficient can be determined from coherence $q(\tau)=-\lambda\frac{\partial t(b,\tau)}{\partial x}$, where $\frac{\partial t(b,\tau)}{\partial x}$ is obtainable from formula (8) as follows:

$$\frac{\partial t(b,\tau)}{\partial x}=\frac{1}{b}\sum_{n=0}^{\infty} P_n'(1) t_1^{(n)}(\tau)\left(\frac{b^2}{a}\right)^n. \tag{16}$$

4 Numerical Example

Let us discuss an example when $a = 10^{-5} \frac{m^2}{s}$, *b=0.01m*, $f_1(\tau) = 20 \cdot \ln(1+\tau)$, $\tau \in [0,500]$ *k=1*. Solution of problem (1) – (3) under such conditions was obtained by means of software MATHEMATICA. Change of temperature in the center was used as input information for inverse problem $t(0,\tau)$. Following these data temperature conductivity coefficient was calculated, by using formula (11), (14) and (15). Summary of results is given in table 1. The table provides relative error in percents of temperature conductivity coefficient if formulas (11), (14) and (15) are used for determination of the latter.

TABLE 1
EVALUATION
OF RELATIVE ERROR IN PERCENTS
OF FORMULA (11), (14), (15)

τ	(11)	(14)	(15)
100	4.80061	0.423142	0.0495278
200	2.22947	0.0945881	0.00565302
300	1.45214	0.0405149	0.00162426
400	1.07678	0.0223744	0.000674739
500	0.85569	0.0141629	0.000342275

5 Conclusion

Many calculations under different boundary conditions were made. In all the cases it was stated that when applying formula (14) and (15), error of temperature conductivity coefficient determination is significantly less than when applying formula (11). One concluded that calculations made as per measurements in the beginning of heat conduction process are less precise than those made in the end of the process where changes of temperature are slower. It is attributable to the fact that a mathematical model with small number of addends in sum does not describe initial stage of heat conduction process precisely.

References:

[1] Shaskov A., Volohov G., Abramenko T., Kozlov V. *Methods of determination of heat and temperature transfer.* Moscow, Energia, 1973, (in Russian).

[2] Temkin *A. Inverse methods of thermal conductivity.* Moscow, Energia, 1973 (in Russian).