Lv's Distribution for Time-Frequency Analysis

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Abstract: - A frequency-chirp rate (FCR) distribution, known as Lv's distribution (LVD), is introduced in this paper. The LVD can directly represents the information of centroid frequency and chirp rate of a linear frequency modulated (LFM) signal without using any searching process. It is able to suppress interference to represent clearly the signal peak in the FCR plane. Comparing to polynomial Fourier transform (PFT) and fractional Fourier transform (FrFT), the LVD has better performance on FCR resolution and energy concentration. Based on LVD, a new method of time-frequency transform named inverse LVD (ILVD) is also introduced to obtain a time-frequency representation (TFR). Simulation results show that the ILVD has better energy distribution concentration than the Wigner-Ville distribution (WVD) and local polynomial time-frequency transform (LPTFT).

Key-Words: - Lv's distribution, inverse Lv's distribution, time-frequency transform

1 Introduction

Many signals in music, speech, geology, wireless communication and radar are non-stationary and belong to frequency modulated (FM) signals. The instantaneous frequency (IF) of the signal, defined as the derivative of the phase, usually varies with time. The Fourier transform (FT) is not suitable to process this kind of signals because it cannot tell when the frequencies occur or disappear. It is more useful to characterize the signals with methods of time-frequency analysis, such as short-time Fourier transform and Wigner-Ville distribution (WVD) [1].

Recently, a frequency-chirp rate (FCR) distribution, known as Lv's distribution (LVD), is proposed for linear FM (LFM) signals [2]. It can directly provide accurate centroid frequency-chirp rate representation on the FCR plane by using a scaling operation and a 2-dimension (2D) FT, without using any searching steps required by the polynomial Fourier transform (PFT) [3, 4, 5] and fractional FT (FrFT) [6]. The LVD is simple on computation and asymptotic linear to be applied on multi-component signal. Based on the LVD, inverse LVD (ILVD) is proposed in [2] as a time-frequency transform method to generate the time-frequency representation (TFR) of an LFM signal. It does not suffer from noise seriously, and can present a clear TFR in premise of the LVD provides a good estimation of centroid frequency-chirp rate.

This paper presents the principles of the LVD and ILVD and their FCR and TFR planes of a multi-component LFM signal. Moreover, we give some performance analysis on energy distribution concentration in terms of distribution concentration rate (DCR). Comparing to PFT and FrFT, the LVD has better performance on FCR resolution and higher value of DCR. Experiments based on Monte Carlo simulation show that the ILVD also provides higher DCR than the WVD and local polynomial time-frequency transform (LPTFT) [7, 8].

2 Review of the LVD

The LVD is a frequency-chirp rate distribution for LFM signals proposed in [2]. An LFM signal \( x(t) \) can be expressed as

\[
x(t) = \sum_{k=1}^{K} A_k \exp\left( j2\pi f_k t + j\pi\gamma_k t^2 \right).
\]

where \( K \) is the number of signal components, \( A_k \) is the constant amplitude, \( f_k \) and \( \gamma_k \) denote the centroid frequency and chirp rate, or the first derivatives of the IF, respectively. Its parametric symmetric instantaneous autocorrelation function (PSIAF) is defined as

\[
R_x^c(t,\tau) = x(t + \frac{\tau + a}{2})^* x^* \left( t - \frac{\tau + a}{2} \right)
\]

\[
= \sum_{k=1}^{K} A_k^2 \exp\left( j2\pi f_k (\tau + a) + j2\pi\gamma_k (\tau + a)t \right)
\]

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\[ + \sum_{i=1}^{K-1} \sum_{j=i+1}^{K} \left( R^C_{x_i x_j} (t, \tau) + R^C_{x_j x_i} (t + \tau) \right), \] (2)

where \( a \) denotes a constant time-delay and \( R^C_{x_i x_j} \) means the cross terms. We define a scaling operator \( \Gamma \) of a phase function \( G \) with respect to \((t, \tau)\) as

\[ \Gamma \left[ G(t, \tau) \right] \rightarrow G \left( \frac{t_a}{h(t + a), \tau} \right), \] (3)

where \( h \) is a scaling factor and \( t_a \) is called the scaled time as \( t_a = (t + a)h \). Let perform a scaling operation \( \Gamma \) on the PSIAF in (2) and obtain

\[ + \sum_{i=1}^{K-1} \sum_{j=i+1}^{K} \Gamma \left[ R^C_{x_i x_j} (t, \tau) + R^C_{x_j x_i} (t, \tau) \right]. \] (4)

Then we take the 2-D FT of the previous equation with respect to \( \tau \) and \( t_a \) as

\[ L_x (f, \gamma) = F^\{-1\} \left\{ F \Gamma \left[ R^C_{x_i x_j} (t, \tau) \right] \right\}, \]

\[ = \sum_{k=1}^{K} L_{x_k} (f, \gamma) + \sum_{i=1}^{K-1} \sum_{j=i+1}^{K} L_{x_{ij}} (f, \gamma). \] (5)

where \( F \{ \cdot \} \) means the FT and the auto-terms \( L_{x_k} \) is calculated by

\[ L_{x_k} (f, \gamma) = A_k e^{i2\pi f f_k} \delta \left( f - f_k \right) \delta \left( \gamma - \frac{\gamma_k}{h} \right). \] (6)

Equation (5) is called the LVD, and \( L_x (f, \gamma) \) is an FCR plane. According to [2], the optimal parameters are \( a = h = 1 \). For a multi-component LFM signal inputs, the LVD has a property of asymptotic linearity because the cross-terms of components are very small comparing to their auto-terms. We therefore have

\[ x(t) = \sum_{k=1}^{K} x_k (t) \rightarrow L_x (f, \gamma) \approx \sum_{k=1}^{K} L_{x_k} (f, \gamma). \] (7)

From the LVD plane, all the parameters of an LFM signal, i.e., centroid frequencies and chirp rates, are obtained by the locations of the peaks \((f_k, \gamma_k)\).

3 Review of the ILVD

The ILVD is a time-frequency transform generating signal's TFR based on LVD [2], and defined as

\[ X_x (t, f) = F^\{-1\} \left\{ F \Gamma \left[ F^\{-1\} \left\{ M \left[ L_x (f, \gamma) \right] \right\} \right] \right\}. \] (8)

where the \( F \{ \cdot \}, \Gamma \{ \cdot \} \) and \( M \{ \cdot \} \) represent the inverse FT, inverse scaling transform and masking operation, respectively. The masking operator is defined as

\[ M \left[ L_x (f, \gamma) \right] = \begin{cases} L_x (f, \gamma), & \text{for } (f, \gamma) \in X, \\ 0, & \text{for } (f, \gamma) \notin X, \end{cases} \] (9)

where \( X \) denotes the support of the auto-term on the LVD plane. A TFR plane of a three-component LFM signal by using the ILVD is shown in Fig. 4(c). It is seen that the ILVD has good time-frequency energy concentration. In particular, it does not suffer from noise because the masking operator \( M \{ \cdot \} \) in (9) only selects the signal terms for the transform. Therefore the ILVD can always generate a clear TFR in premise of the LVD provides a good estimation of \((f, \gamma)\).

4 Experimental Results

This section presents some simulation results based on the LVD and ILVD compared with some other time-frequency methods.

4.1 The LVD

An FCR plane of a three-component LFM signal obtained by using the LVD is presented in Fig. 1 in which the three peaks located at \((f_k, \gamma_k)\) are clearly seen. In this section, we will discuss some performance of the LVD, the FCR resolution and DCR, and compare with the FrFT and PFT. The PFT is defined as [3, 4]
4.1.1 FCR Resolution

According to [2], the resolutions of the centroid frequency and chirp rate achieved by the LVD are \( \Delta f = 1/T \) and \( \Delta \gamma = 2/(hT) \), where \( T \) is the time interval. The signal used in the simulation is denoted as Example 1 and given by

\[
s(t) = \sum_{k=1}^{K} A_k \exp \left( j2\pi f_k t + j\pi \gamma_k t^2 \right),
\]

where the number of LFM components \( K = 3 \), amplitudes \( A_k = 1 \) for all \( k \), centroid frequencies \( f_1 = f_2 = 51.2 \) Hz and \( f_3 = 53.2 \) Hz, the chirp rates \( \gamma_1 = \gamma_2 = 12.8 \) Hz/s and \( \gamma_3 = 14.8 \) Hz/s, the sampling frequency \( f_s = 256 \) Hz and signal length \( N = 1024 \). This signal has three LFM components with differences in centroid frequencies by 2 Hz and in chirp rates by 2 Hz/s.

Fig. 2 shows the FCRs of the signal specified in Example 1. In this figure, these LFM components are clearly represented with three peaks only by the LVD since the differences in frequencies and chirp rates are too small to be separated by the resolutions of PFT and FrFT. It is also observed in Fig. 2(a) that the interferences around the three peaks make the detection difficult. These interferences are mainly due to the finite length of the signal samples so that the large main-lobe width of the window effects may result in interferences between peak values of the signal components.

4.1.2 DCR

Next let us evaluate the energy concentration performance of the frequency-chirp rate. Energy distribution concentration is an important performa-
nce in time-frequency analysis. Similar to the concept of the distribution concentration used in [9], the DCR is defined to measure the signal-term energy concentration in the FCR domain as follows,

\[
DCR(f,\gamma) = 10 \log_{10} \left( \frac{\text{ave}\left[ FCR(f,\gamma) \right]}{\text{ave}\left[ FCR(f,\gamma)^2 \right]} \right), \quad (12)
\]

where \(\text{ave}[]\) measures the average energy in a specified region in the FCR domain, \(S\) is the region specified by the points \((f_k, \gamma_k)\) in the FCR domain. A higher DCR value indicates a better performance on energy concentration of the frequency-chirp rate. The signal, denoted as Example 2, in this simulation is expressed by the same expression as (11), with the parameters of \(K=3\), centroid frequencies \(f_1 = 29.44\) Hz, \(f_2 = 18.56\) Hz, \(f_3 = 27.52\) Hz, chirp rates \(\gamma_1 = 7.0263\) Hz/s, \(\gamma_2 = 8.6232\) Hz/s, \(\gamma_3 = -8.6232\) Hz/s, \(A_k = 1\) for all \(k\), \(f_1 = 128\) Hz and \(N = 512\).

Table 1 shows the DCR values obtained from Example 2 by using different methods. These DCR values are for signals without noise and therefore we consider them as ideal ones. It is seen that the DCRs based on the LVD are significantly larger than those from the PFT and FrFT methods. It shows that the detection performance based on the LVD should be better in low SNR environments since it has better capability of signal energy concentration than the other two methods. In the SNR range larger than 10 dB, the DCR values are approaching to those listed in Table 1.

### Table 1 DCR Comparison for Signals without Noise

<table>
<thead>
<tr>
<th>Method</th>
<th>DCR (K=1) (dB)</th>
<th>DCR (K=3) (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PFT</td>
<td>20.7298</td>
<td>17.5228</td>
</tr>
<tr>
<td>FrFT</td>
<td>26.6028</td>
<td>19.0780</td>
</tr>
<tr>
<td>LVD</td>
<td>39.8126</td>
<td>31.2373</td>
</tr>
</tbody>
</table>

Fig. 3 shows the performance of the DCR in noise environments based on the Monte Carlo simulation results for the signal specified in Example 2. In the figure, the DCRs of the LVD are larger than those from the PFT and FrFT even in negative signal-to-noise ratio (SNR) range. It means that the detection performance based on the LVD should be better in low SNR environments since it has better capability of signal energy concentration than the other two methods. In the SNR range larger than 10 dB, the DCR values are approaching to those listed in Table 1.

#### 4.2 The ILVD

In this section, the experiments based on the ILVD are implemented, and the results are discussed and compared with its counterparts, WVD [1] and LPTFT. The LPTFT [7, 8] is defined as

\[
X_L(t,\bar{f}) = \int_{-\infty}^{\infty} x(t+\tau) \eta^\ast(\tau) e^{-j\theta(\tau,\bar{f})} d\tau, \quad (13)
\]

where \(\eta(\tau)\) is a window function in \([-h,h]\), \(\theta(\tau,\bar{f}) = 2\pi(\bar{f}\tau + f_1\tau^2 / 2! + ... + f_M\tau^M / M!)\), and vector \(\bar{f} = (f_1, f_2, \ldots, f_M)\) is a set of parameters which have to be estimated for the computation defined in (13). The parameter \(f_1\) corresponds to the IF of the signal and \(f_i\), for \(i = 2,3,\ldots,M\), correspond to the \((i-1)\)th derivatives of the IF. Since the input signal is an LFM signal, we set \(M = 2\) and the \(f_2\) corresponds to the chirp rate \(\gamma\) in (1). In next simulation, we apply the real chirp rate on the LPTFT which is considered an ideal form of LPTFT, with a normalized rectangular window of length \(N/2\).

#### 4.2.1 TFR

The simulation signal here is as the same form in (11) with \(K = 3\), centroid frequencies \(f_1 = -1\) Hz, \(f_2 = 5\) Hz and \(f_3 = 0\) Hz, chirp rates \(\gamma_1 = 11.5\), \(\gamma_2 = 5\) and \(\gamma_3 = -7.5\) Hz/s, the sampling frequency...
Fig. 4 The TFRs of the multi-component LFM signal in Example 3. 

$\nu_0 = 128$ Hz and $N = 512$, denoted as Example 3. Fig. 4 shows the TFRs of the three-component LFM signal based on WVD, LPTFT and ILVD. It is clear to see that the ILVD can provide a nice representation of time-frequency, while the WVD and LPTFT are all suffer from the cross-terms more or less.

4.2.3 DCR

We also evaluate the energy concentration of the ILVD in terms of DCR. Similar to the definition in equation (12), we define the DCR in TFR plane as following,

$$DCR_{(t,f)} = 10 \log_{10} \frac{\text{ave}[TFR(t,f)]_{(t,f) \in S}^2}{\text{ave}[TFR(t,f)]_{(t,f) \notin S}^2},$$

where $\text{ave}[]^2$ means the average energy, $S$ denotes the signal domain in the TFR, i.e., the LFM signal lines here. Fig. 5 presents the DCR values of Example 3 based on WVD, LPTFT and ILVD. We can see in most SNR range the values of DCR by ILVD are larger than by other two methods. Therefore the ILVD has a better performance in the issue of energy distribution concentration.

Fig. 5 The DCRs of the multi-component LFM signal in Example 3 based on WVD, LPTFT and ILVD.

5 Conclusion

A frequency-chirp rate distribution called the LVD for LFM signals is introduced and some performance is analyzed in this paper. The LVD has better performance on FCR resolution and signal energy concentration comparing to PFT and FrFT. Then a new time-frequency transform named ILVD based on LVD is introduced. Experimental results indicate that it can generate a clearer TFR and provide high-
er value of DCR than WVD and LPTFT even in negative SNR environments.

References: