Conic Transformational High Dimensional Model Representation
In Comparison With Hermite-Padé Approximants

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Abstract: The basic philosophy behind the Transformational High Dimensional Model Representation (THDMR) is to transform a multivariate function to another multivariate function such that the High Dimensional Model Representation (HDMR) truncations of the new function is much more efficient. A previous work using affine transformation forms milestone to this work where a conic (affine plus quadratic) transformation is considered with an attempt to optimize its coefficients. Fundamental formulation has been completely based on that work although the derived formulae and the corresponding approximants in literature (Hermite-Padé) are completely different because of the branch points. Thus what we try to develop here is competitor not for Padé but Hermite-Padé approximants with capabilities superior to the ones in both of them.

Key–Words: High Dimensional Model Representation, Transformational High Dimensional Model Representation, Optimization, Multivariate Functions, Approximation, Hermite-Padé Approximants, Weight Functions, Weight Optimization, Constancy Maximization

1 Introduction

In an attempt to generalize Logarithmic HDMR [1, 2] M. Demiralp introduced the more general idea of Transformational HDMR [3]. The fundamental concept behind Logarithmic HDMR is to transform the multiplicativity in a multivariate function to additivity in another multivariate function. This would enable us to use plain (or additive) HDMR [4] with confidence on the resulting additive natured function. However it is not true that a function is of either additive or multiplicative or a mixture of both in nature. Trigonometric, inverse trigonometric, hyperbolic and inverse hyperbolic functions are only few such examples. In applying HDMR to such cases one can in fact think of one-to-one transformations which will serve the purpose much better than taking logarithms. A previous dissertation [5, 6] from our group was focused on the HDMR constancy optimisation under an affine transformation. With the same philosophy we focused [7] on the HDMR constancy optimisation under a conic transformation. The algebraic manipulations and certain conceptual items of the abovementioned dissertation have also been used in the previous paper because of the polynomial structure based analogies between affine and conic transformations. We observe that the approximant suggested here is rather of similar nature to the Hermite Padé approximants [8]. Examples are given in the Implementations section where comparisons of both these methods are made.

2 Transformational High Dimensional Model Representation

Consider a multivariate function \( f(x_1, x_2, ..., x_N) \). If the function is not of additive nature then the constancy, univariance, and even, the bivariance measurers will not be sufficiently close to one. The idea of Transformational HDMR is to choose an appropriate transformation \( T \) yielding a new multivariate function \( \varphi(x_1, x_2, ..., x_N) \)

\[
Tf(x) \equiv \varphi(x)
\]

\[
x = \{x_1, x_2, ..., x_N\}
\]

and then apply plain HDMR expansion to \( \varphi \)

\[
\varphi(x) = \varphi_0 + \sum_{i=1}^{N} \varphi_1(x_i) + \cdots + \varphi_{12...N}(x)
\]

With the basic philosophy in HDMR \( \varphi_0 \) can be determined as follows

\[
\varphi_0(x) = \int_{V} dW(x) \varphi_0(x)
\]
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Here, $V = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_N, b_N]$ represents the hyperprism which is the HDMR’s construction domain and $W(x)$ stands for the HDMR’s product type weight function whose factors are univariate functions each of which depends on a different independent variable. Each weight function factor will be taken to be a constant normalized over the corresponding interval. $dV$ is the product of the individual differentials, $dx_1 dx_2 ... dx_N$.

$\varphi_0(x_i)$’s can be determined as

$$\varphi_i(x_i) = \int_V dV W(x) \varphi(x)$$

Here, $V = [a_1, b_1] \times ... \times [a_{i-1}, b_{i-1}] \times [a_{i+1}, b_{i+1}] \times ... \times [a_N, b_N]$ and $dV$ is the product of the individual differentials, $dx_1 ... dx_{i-1} dx_{i+1} ... dx_N$.

Now, the additivity measurers can be defined in the usual manner however this time using the newly defined $\varphi$’s. This will give

$$\sigma_0(\varphi) = \frac{||\varphi_0||^2}{||\varphi||^2},$$

$$\sigma_1(\varphi) = \sigma_0(\varphi) + \sum_{i=1}^{N} \frac{||\varphi_i||^2}{||\varphi||^2},$$

$$\sigma_2(\varphi) = \sigma_1(\varphi) + \sum_{i_1 < i_2}^{N} \frac{||\varphi_{i_1 i_2}||^2}{||\varphi||^2},$$

... (6)

Clearly these $\sigma_i(\varphi)$’s will be quite different from those that would have been obtained from applying HDMR expansion directly to the original function $f(x_1, x_2, ..., x_N)$. The difference will depend not only on the original function but also on the specific choice of the transformation $T$. Since the essential philosophy behind HDMR is to be able to express the original function with terms in a number as small as possible by keeping the multivariate at a level as low as possible, it will be preferable to get $\sigma_0$ and $\sigma_1$ values as close to 1 as possible. Since its first proposal, various transformations were analyzed for THDMR applications. Perhaps the logarithmic and trigonometric transformations were amongst the most interesting ones. However the utilization of polynomials as THDMR’s operators seemed to be effective at least as much as their challengers are.

## 3 Conic Transformational High Dimensional Model Representation

A rational approximation based on optimization of linear transformation operators have been proven to be seriously compatible with its Padé approximant counterpart. This may be considered as expected, perhaps only since, Padé approximants are based on Taylor expansions which give all the information about the relevant function at a single point namely $x = 0$. On the other hand the approximant constructed by using THDMR uses the information about the original function throughout the entire relevant interval. That is, it is global in contrast to the pointwise locality of Padé approximants.

In the present work, the degree of the polynomial to be used is two. Linear combination coefficients of the conic transformation will be taken to vary with independent variables. In other words, the combination coefficients will be regarded as operators dependent on the algebraic operators each of which multiplies its operand with a different independent variable. Such a choice gives additional flexibility to the relevant transformation. Hence the coefficients can be selected so as to approximate the THDMR expansion in an optimal manner.

$$T f(x) \equiv \varphi(x)$$

$$\equiv a_0(x) + a_1(x) f(x) + a_2(x) f(x)^2$$

Since only $\sigma_0(\varphi)$ will be under consideration, $\varphi$ will be approximated by the constant term $\varphi_0$

$$\varphi = a_0 + a_1 f + a_2 f^2 \approx \varphi_0$$

This will give the following approximate equality

$$f \approx - \frac{a_1}{2a_2} \pm \sqrt{\frac{4\varphi_0 a_2 - 4a_2 a_0}{4a_2^2}}$$

There are two possible choices in approximating $f$, since the conic transformation is not a one-to-one transformation. The choice will be for the one which gives the better results. This will be discussed in due course. The purpose here is to find convenient forms for the coefficients $a_0$, $a_1$ and $a_2$ which maximize the constancy measurer $\sigma_0$ in (6). These three constants will be taken to be $N$-variate square integrable functions. This will lead us to choose an orthonormal basis set in the Hilbert space $H^{(N)}$. 

$$U = \{ u_i(x_1, x_2, ..., x_N) \}_{i=1}^{\infty}$$

Orthonormality is defined in terms of the inner product as

$$(u_i, u_j) = \int_V dV W(x) u_i(x) u_j(x) = \delta_{ij}, \quad 1 \leq i, j < \infty$$

Although the basis set given above has an infinite number of elements, in practical applications, a finite
The elements of the vector $a$ are

$$a_0(x) = \sum_{j=2}^{m} a_j^{(0)} u_j,$$

$$a_1(x) = \sum_{k=1}^{n} a_k^{(1)} u_k,$$

$$a_2(x) = \sum_{l=1}^{p} a_l^{(2)} u_l$$

(12)

With these expressions in hand, the constancy measure $\sigma_0(\varphi)$ will be a function of the parameters $a_j^{(0)}$, $a_k^{(1)}$ and $a_l^{(2)}$ where $2 \leq j \leq m$, $1 \leq k \leq n$, $1 \leq l \leq p$.

$$\sigma_0(\varphi) = \sigma_0(a_2^{(0)}, ..., a_m^{(0)}, a_1^{(1)}, ..., a_n^{(1)}, a_1^{(2)}, ..., a_p^{(2)})$$

(13)

Using (12) $\varphi$ can be expressed as,

$$\varphi(x) = \left( \sum_{j=2}^{m} a_j^{(0)} u_j \right) + \left( \sum_{k=1}^{n} a_k^{(1)} u_k \right) f + \left( \sum_{l=1}^{p} a_l^{(2)} u_l \right) f^2$$

(14)

Integrating both sides of the above equation with respect to all independent variables over $V$ under the product type weight function of the HDMR, the constant HDMR term $\varphi_0$ can be obtained as

$$\varphi_0 = \sum_{j=2}^{m} a_j^{(0)} \int_V dV \left( \prod_{i=1}^{N} W_i(x_i) \right) u_j + \sum_{k=1}^{n} a_k^{(1)} \int_V dV \left( \prod_{i=1}^{N} W_i(x_i) \right) u_k f + \sum_{l=1}^{p} a_l^{(2)} \int_V dV \left( \prod_{i=1}^{N} W_i(x_i) \right) u_l f^2$$

(15)

To proceed, define $a$ and $\alpha$ vectors as

$$a = \left( a_2^{(0)}, ..., a_m^{(0)}, a_1^{(1)}, ..., a_n^{(1)}, a_1^{(2)}, ..., a_p^{(2)} \right)^T,$$

$$\alpha = \left( \alpha_2^{(0)}, ..., \alpha_m^{(0)}, \alpha_1^{(1)}, ..., \alpha_n^{(1)}, \alpha_1^{(2)}, ..., \alpha_p^{(2)} \right)^T$$

(16)

The elements of the vector $\alpha$ are

$$\alpha_j^{(0)} = \int_V dV \left( \prod_{i=1}^{N} W_i(x_i) \right) u_j(x)$$

$$a_k^{(1)} = \int_V dV \left( \prod_{i=1}^{N} W_i(x_i) \right) u_k(x) f$$

$$a_l^{(2)} = \int_V dV \left( \prod_{i=1}^{N} W_i(x_i) \right) u_l(x) f^2$$

(17)

Where, $h(x)$ appearing in the first inner product in (15) is a function which has the constant value 1 for all $x_i$s in the hyperprismatic domain $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N]$. $\varphi_0$ can now be written as an inner product

$$\varphi_0 = a^T \alpha$$

(18)

The norm square of $\varphi_0$ can be expressed as

$$\|\varphi_0\|^2 = (a^T \alpha)(a^T \alpha)^T = a^T \alpha \alpha^T a$$

(19)

On the other hand the square of the norm of $\|\varphi\|^2$ can be expressed in terms of the vector $a$ and a square matrix $A$ which we will express in terms of its blocks as,

$$A = \begin{bmatrix} I & L & N \\ L^T & M & P \\ N^T & P^T & R \end{bmatrix}$$

(20)

Here

$$I_{jk} = (u_j, u_k), \quad 2 \leq j, k \leq m$$

$$L_{jk} = (u_j, f u_k), \quad 2 \leq j \leq m, 1 \leq k \leq n$$

$$M_{jk} = (u_j, f^2 u_k), \quad 1 \leq j, k \leq n$$

$$N_{jk} = (u_j, f^2 u_k), \quad 2 \leq j \leq m, 1 \leq k \leq p$$

$$R_{jk} = (u_j, f^4 u_k), \quad 1 \leq j, k \leq p$$

(21)

Here $A$ is a symmetric, positive definite matrix. The norm square of $\varphi$ can be expressed in terms of $A$ and $a$ as

$$\|\varphi\|^2 = a^T A a$$

(22)

Hence the constancy measure $\sigma_0$ can be expressed as

$$\sigma_0 = \frac{\|\varphi_0\|^2}{\|\varphi\|^2} = \frac{a^T \alpha \alpha^T a}{a^T A a}$$

(23)

Defining

$$y = A^{\frac{1}{2}} a$$

(24)
\( \sigma_0 \) can be expressed as a Rayleigh quotient [9]
\[
\sigma_0 = \frac{y^T A^{-\frac{1}{2}} \alpha \alpha^T A^{-\frac{1}{2}} y}{y^T y} \tag{25}
\]
It is well known that a Rayleigh Quotient takes its maximum value at the maximum eigenvalue of its kernel which is in this case \( A^{-\frac{1}{2}} \alpha \alpha^T A^{-\frac{1}{2}} \). Similarly \( y \) is the eigenvector which corresponds to the maximum eigenvalue. An analysis of the kernel will give the maximum eigenvalue and the corresponding eigenvector; they are respectively,
\[
\sigma_0 = \alpha^T A^{-1} \alpha, \quad y = A^{-\frac{1}{2}} \alpha \tag{26}
\]
This equation for \( y \) gives us the vector \( a \) in (23) that maximizes \( \sigma_0 \)
\[
a = A^{-\frac{1}{2}} y = A^{-1} \alpha \tag{27}
\]
Utilizing these equalities we can construct a function \( \sigma \). To proceed we define
\[
\varphi_0 = a^T \alpha = \alpha^T A^{-1} \alpha \tag{28}
\]
We define a vector \( u \) with \((m+n+p-1)\) elements
\[
u = [u_2, \ldots, u_m, u_1, \ldots, u_1, \ldots, u_p]^T \tag{29}
\]
and express \( a_0, a_1 \) and \( a_2 \) more compactly as
\[
a_0 = a^{(0)T} u^{(0)} \quad a_1 = a^{(1)T} u^{(1)} \quad a_2 = a^{(2)T} u^{(2)} \tag{30}
\]
Here the vectors \( a^{(0)}, a^{(1)}, a^{(2)} \) and \( u^{(0)}, u^{(1)}, u^{(2)} \) are explicitly defined as
\[
a^{(0)} = [a_2^{(0)}, \ldots, a_m^{(0)}]^T \quad a^{(1)} = [a_1^{(1)}, \ldots, a_n^{(1)}]^T \quad a^{(2)} = [a_1^{(2)}, \ldots, a_p^{(2)}]^T
\]
\[
u^{(0)} = [u_2, \ldots, u_m]^T \quad 
u^{(1)} = [u_1, \ldots, u_n]^T \quad 
u^{(2)} = [u_1, \ldots, u_p]^T \tag{31}
\]
To proceed we define \((m+n+p-1) \times (m+n+p-1)\) projection matrices \( P_1 \) and \( P_2 \) as
\[
P_1 = \sum_{i=1}^{m-1} e_i e_i^T \quad P_2 = \sum_{i=m}^{m+n-1} e_i e_i^T \quad P_3 = \sum_{i=m+n}^{m+n+p-1} e_i e_i^T \quad \tag{32}
\]
where \( e_i \) stands for the \( i \)th standard unit cartesian vector whose only nonzero element which has the value of 1 resides at the \( i \)th position in the vector. Utilizing these projection operators \( a_0, a_1 \) and \( a_2 \) can be approximated as
\[
a_0 \approx a^T P_1 u = \alpha^T A^{-1} P_1 u \quad a_1 \approx a^T P_2 u = \alpha^T A^{-1} P_2 u \quad a_2 \approx a^T P_3 u = \alpha^T A^{-1} P_3 u \tag{33}
\]
If now these entities are substituted into (9) we obtain
\[
f \approx -\frac{\alpha^T A^{-1} P_2 u}{2\alpha^T A^{-1} P_3 u} + \sqrt{\frac{N}{D}} \tag{34}
\]
where,
\[
N = 4\alpha^T A^{-1} \alpha \alpha^T A^{-1} P_3 u - 4\alpha^T A^{-1} P_3 u \alpha^T A^{-1} P_1 u + \left( \alpha^T A^{-1} P_2 u \right)^2 \tag{35}
\]
and,
\[
D = 4 \left( \alpha^T A^{-1} P_3 u \right)^2 \tag{36}
\]
Even though the formulation developed here is for the general case of functions of \( N \) variables the applications given here will be for the univariate case. The results will be compared with those of the Hermite-Padé approach.

### 4 Hermite-Padé Approximation

In this section, we shall give a brief account of the Hermite-Padé approximation, since we shall compare our results with those obtained from Hermite-Padé approximation in the next section.

Consider a function \( f(x) \) whose Taylor expansion around \( x = 0 \) can be expressed as
\[
f(x) = \sum_{i=0}^{\infty} f_i x^i \tag{37}
\]
where \( f_i \)'s are the expansion coefficients known to be the value of the \( i \)th derivative of \( f(x) \) at \( x = 0 \). The Hermite-Padé approximation for \( f(x) \) can be obtained by setting the conic form below equal to zero,
\[
P_m(x) + Q_n(x) f(x) + R_p(x) f(x)^2 = 0 \tag{38}
\]
where $P_m$, $Q_n$ and $R_p$ are the polynomials with degrees $m$, $n$ and $p$ respectively. Hermite-Padé approximation of a function is given by

$$f(x) \approx H_{[m,n,p]}(x) \quad (39)$$

When we insert $H_{[m,n,p]}(x)$ into (38) for $f$ we obtain the following equation

$$P_m + Q_n H_{[m,n,p]} + R_p H^2_{[m,n,p]} \approx 0 \quad (40)$$

The solution of this approximate equality gives

$$H_{[m,n,p]} \approx -\frac{Q_n(x)}{2R_p(x)} \pm \sqrt{\frac{Q_n(x)^2}{4R_p(x)^2} - \frac{P_m(x)}{R_p(x)}} \quad (41)$$

$P$, $Q$ and $R$ can also be rewritten as infinite series by simply setting their coefficients equal to zero for indices greater than the appropriate highest degree for each polynomial. This is common practice when Cauchy Products are to be applied. In other words by setting $a_j \equiv 0$ for $j$ values greater than $m$, $b_k \equiv 0$ for $k$ values greater than $n$ and $c_\ell \equiv 0$ for $\ell$ values greater than $p$ the following expressions can be written

$$P_m(x) = \sum_{j=0}^{m} a_j x^j = \sum_{j=0}^{\infty} a_j x^j;$$

$$Q_n(x) = \sum_{k=0}^{n} b_k x^k = \sum_{k=0}^{\infty} b_k x^k;$$

$$R_p(x) = \sum_{\ell=0}^{p} c_\ell x^\ell = \sum_{\ell=0}^{\infty} c_\ell x^\ell;$$

$$a_j \equiv 0, j > m$$

$$b_k \equiv 0, k > n$$

$$c_\ell \equiv 0, \ell > p \quad (42)$$

With these definitions Hermite-Padé approximation becomes,

$$\sum_{j=0}^{\infty} a_j x^j + \left( \sum_{k=0}^{\infty} b_k x^k \right) \left( \sum_{i=0}^{\infty} f_i x^i \right) + \left( \sum_{\ell=0}^{\infty} c_\ell x^\ell \right)^2 = 0 \quad (43)$$

Reorganising this approximation formula by using Cauchy Products,

$$\left( \sum_{k=0}^{\infty} b_k x^k \right) \left( \sum_{i=0}^{\infty} f_i x^i \right) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} b_k f_i x^{i+k} \quad (44)$$

and by writing $i$ instead of $i-k$

$$\sum_{k=0}^{\infty} \sum_{i=k}^{\infty} b_k f_{i-k} x^i = \sum_{i=0}^{\infty} \left( \sum_{k=0}^{i} b_k f_{i-k} \right) x^i \quad (45)$$

is obtained. In the same manner, the equality

$$\left( \sum_{\ell=0}^{\infty} c_\ell x^\ell \right)^2 = \sum_{i=0}^{\infty} \sum_{s=0}^{i} d_i f_{s-i} x^s \quad (46)$$

can be written. For practical purposes defining

$$d_i = \sum_{\ell=0}^{i} c_\ell f_{i-\ell} \quad (47)$$

then

$$\sum_{i=0}^{s} d_i f_{s-i} = \sum_{i=0}^{s} \sum_{\ell=0}^{i} c_\ell f_{i-\ell} f_{s-i} \quad (48)$$

Using these formulae the approximation can be rewritten as

$$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} \left( \sum_{k=0}^{i} b_k f_{i-k} \right) x^i + \sum_{i=0}^{\infty} \left( \sum_{\ell=0}^{i} c_\ell f_{i-\ell} f_{s-i} \right) x^i = 0 \quad (49)$$

and this approximation formula is written under the same index to obtain

$$\sum_{i=0}^{\infty} \left[ a_i + \sum_{k=0}^{i} b_k f_{i-k} + \sum_{\ell=0}^{i} c_\ell f_{i-\ell} f_{s-i} \right] x^i = 0 \quad (50)$$

Here for $i = 1, 2, 3...$ the coefficients in the equation must be set to zero leading to

$$a_i + \sum_{k=0}^{i} b_k f_{i-k} + \sum_{\ell=0}^{i} c_\ell f_{i-\ell} f_{s-i} = 0, \quad i = 0, 1, \ldots \quad (51)$$

Here,$m + n + p + 2$ equations will be taken into consideration. When we use more than $m + n + p + 2$ equations we obtain a trivial solution.

We will give formulation for $m = n = p = 0$ as an example. When these equations are solved for various $i$ and $i = 1$ values.

For $i = 0$,

$$a_0 + b_0 f_0 + c_0 f_0 f_0 = 0 \quad (52)$$

For $i = 1$,

$$a_1 + b_0 f_1 + b_1 f_0 + c_0 f_0 f_1 + c_0 f_1 f_0 + c_1 f_0 f_1 + c_1 f_1 f_0 = 0 \quad (53)$$

Organising these equations

$$b_0 f_1 + c_0 f_0 f_1 + c_0 f_1 f_0 = 0 \quad (54)$$
\[ b_0 f_1 + 2c_0 f_0 f_1 = 0 \]  
(55)

we will get

\[ \begin{align*}
  b_0 &= -2c_0 f_0 \\
  a_0 &= c_0 f_0^2
\end{align*} \]  
(56)

with these new coefficients our polynomials are given

\[ \begin{align*}
  p_0(x) &= a_0 = c_0 f_0^2 \\
  q_0(x) &= b_0 = -2c_0 f_0 \\
  r_0(x) &= c_0
\end{align*} \]  
(57)

When we insert these polynomials into our approximation we obtain for \( m = n = p = 0 \)

\[ H_{[0,0,0]} = -\frac{2c_0 f_0}{2c_0} + \sqrt{\frac{(-2c_0 f_0)^2}{4c_0^2} - \frac{c_0 f_0^2}{c_0}} = f_0 \]  
(58)

Which is the expected solution in this case. By the same way, Hermite-Padé approximation is obtained for different values of \( m, n \) and \( p \).

## 5 IMPLEMENTATION

Hermite-Padé approximation is based on the Taylor expansion at \( x = 0 \) and is therefore expected to give better results at \( x \) values close to zero. On the other hand since THDMR is of global nature it is expected to give better results globally speaking. Need it be, this can be achieved by scaling up the domain interval of the original function. In fact, THDMR utilization has the advantage of using different weights giving different importance to different parts of the relevant interval. We give some examples here for approximations obtained from Conic Transformational HDMR and compare them with corresponding Hermite-Padé approximations. Choosing different weight functions we observe different results. We did not discuss here how weight functions will be chosen. This is an optimisation problem which will be discussed later.

Analysis of error between exact functions and the two approximations are given in Table 1 and 2 for \( \sin(x) \) and \( \log(1 + x) \) respectively. As is expected for the function \( \sqrt[3]{1 - x^2} \) Hermite-Padé and Conic THDMR values give zero error. Hence the results are not given in the table. However in the next Table 3 results for \( \sqrt[3]{1 - x^2} \) are analysed and better results are obtained by Hermite-Padé as \( x \) goes to one. The \( m, n \) and \( p \) values given in the tables are the degrees of the polynomials used in the Hermite-Padé approximation. In the corresponding Conic THDMR work on the other hand \( m+1, n \) and \( p \) values were used. The reason for this is, due to the fact that in Conic THDMR \( y_0 \) is excluded in the first polynomial yielding a matrix of one less dimension.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Hermite-Padé App.</th>
<th>Conic THDMR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.0178×10^{-10}</td>
<td>6.1453×10^{-3}</td>
</tr>
<tr>
<td>0.2</td>
<td>3.8446×10^{-8}</td>
<td>3.0556×10^{-3}</td>
</tr>
<tr>
<td>0.3</td>
<td>6.5167×10^{-7}</td>
<td>1.3114×10^{-3}</td>
</tr>
<tr>
<td>0.4</td>
<td>4.8262×10^{-6}</td>
<td>4.4939×10^{-4}</td>
</tr>
<tr>
<td>0.5</td>
<td>2.2664×10^{-5}</td>
<td>1.0235×10^{-4}</td>
</tr>
<tr>
<td>0.6</td>
<td>7.9645×10^{-5}</td>
<td>6.1690×10^{-6}</td>
</tr>
<tr>
<td>0.7</td>
<td>2.2874×10^{-4}</td>
<td>2.5090×10^{-6}</td>
</tr>
<tr>
<td>0.8</td>
<td>5.6584×10^{-4}</td>
<td>5.0019×10^{-7}</td>
</tr>
<tr>
<td>0.9</td>
<td>1.2467×10^{-3}</td>
<td>1.8132×10^{-7}</td>
</tr>
<tr>
<td>1.0</td>
<td>2.5037×10^{-3}</td>
<td>2.4903×10^{-7}</td>
</tr>
</tbody>
</table>

Table 1: Error between exact value and approximant value compared for Hermite-Padé and Conic THDMR values for \( \sin(x) \) in the interval \([0, 1]\), \( m=2, n=2 \) and \( p=1 \), weight function is \( w(x) = x^{18} \).

<table>
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<th>Conic THDMR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.8636×10^{-10}</td>
<td>3.1242×10^{-3}</td>
</tr>
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Table 2: Error between exact value and approximant value compared for Hermite-Padé and Conic THDMR values for \( \log(1 + x) \) in the interval \([0, 1]\) \( m=2, n=p=1 \), weight function is \( w(x) = (x - 0.1)^{10} \).

## 6 Conclusion

In this paper we focused on the transformational version of High Dimensional Model Representation (THDMR). We have used a conic transformation and we took into consideration the HDMR expansion of new function under transformation and we tried to find optimal parameters which provide the best approximation. We have seen that the Conic THDMR method works better than Hermite-Padé approximation which is well-known in literature. We gave some implementation results in order to compare the methods.
Table 3: Error between exact value and approximant value compared for Hermite-Padé and Conic THDMR values for \( \sqrt[3]{1-x^2} \) in the interval \([0,1]\) and \(m=3, n=p=1\), weight function is \(w(x) = x(1-x)\).

<table>
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<th>Hermite-Padé App.</th>
<th>Conic THDMR</th>
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7 Acknowledgments

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References:


