New identity of $a^n \pm b^n$ expression

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Abstract A polynomial formula depending on $a + b$ and ab arguments of $a^n \pm b^n$ expression was first put forward. The coefficients in the formula are easily determined from the table written by a special algorithm. The sums of the figures in each row of the triangles which are called Abiyev triangles make Lucas and Fibonacci sequences. New analytic formulae were found to write the table. The identities proposed in the article allow facilitating mathematical calculations in some fields of science and technology.
Keywords: Lucas**,** Fibonacci, sequences, identity, Binet formula, polynom**.**

1.INTRODUCTION

The binomial expression $a^n \pm b^n$ is one of the most used expressions in algebra. It has evidently been seen up to now that one of the powers of a and b coefficients increases and another one decreases in the identity of this expression [1]. The following formula can serve as an example.

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ... + ab^{n-2} + b^{n-1}\right) \quad (1)$$

When a and b letters are conjugate expressions, algebraic calculus become more difficult by using (1) identity. As the sum and multiplication of the conjugate expressions are real numbers, writing the right side of (1) identity as a function depending on $a+b$ and ab arguments simplifies the calculus. This function can be easily written for small values of n:

$$a^3 + b^3 = (a+b)\left[(a+b)^2 - 3ab\right]$$

$$a^3 - b^3 = (a-b)\left[(a+b)^2 - ab\right]$$

But there are no formulae for big values of n in literature. The purpose of the article is to write new identities for the binomial expression $a^n \pm b^n$ and study the fields of their application.

2. ABIYEV TRIANGLES

The identity $a^n \pm b^n = F\left[(a+b), ab\right]$ was first revealed and a special table was compiled to determine the coefficients in this identity [3]. Let's explain the algorithm first to write the table: insert columns to the right and left of 0 at the top of the table 1 and write numbers beginning from 1 in sequence in column 0. Write odd numbers to the right side and figure 1 to the left side of this column; write figure 2 before even numbers at the right and figure 0 at the left. In consequence, 2 columns formed at the right and at the left of column 0.

Afterwards, write figure 1 in the 2nd and 3rd rows of the 2nd columns; by summing the figure 1 in the 2nd row of the 2nd column with the figure 3 in the 3rd row of the 1st column write figure 4 in the 4th row of the 2nd column. So by continuing this method figures 4, 9, 16, 25 etc. squares of sequent numbers are written at the right in the even rows of the 2nd column.

Summing the figure 1 in the 3rd row of the 2nd column with the figure 4 underneath figure 5 is written in the 5th row; later summing this last figure 5 with the figure 9 underneath figure 14 is written in the 7th row. At the result, the 2nd column is formed at the right.

In order to write the 3rd column figure 1 is written in the 4th and 5th rows of the 3rd column again; afterwards the above-mentioned method is applied to form the 2nd column. So a number of columns and rows of figures can be formed.

If the method used to form the columns at the right side of the table1is applied symmetrically with respect to column 0, columns can be written at the left side too.

The compiled table 1 is composed of right and left triangles. The numbers in the rows of the right and left triangles specify the coefficients of the polynomial expressions to which $a^n + b^n$ and $a^n - b^n$ expressions equal respectively. They were put in sequence from right to left in the right triangle and from left to right in the left triangle. One of the interesting properties of this table is that the sum of the numbers at the right and at the left rows forms Lucas and Fibonacci sequences correspondingly.

3. IDENTITIES

Let's accept the following replacements in order to simplify the method of writing polynomial expressions: $a+b \equiv x$ and $ab \equiv y$. Let's show the identities of even and odd powers. If n is an even power,

$$a^n + b^n = E_n^1 x^n - E_n^2 x^{n-2} y + E_n^3 x^{n-4} y^2 - ...$$

$$... - E_n^{\frac{n}{2}} x^2 y^{\frac{n-2}{2}} + E_n^{\frac{n+2}{2}} y^{\frac{n}{2}} \quad (2)$$

here

$$E_n^{k+1} = \tfrac{n}{k}\binom{n-k-1}{k-1}; \quad k = 0,1,2,3,...,\tfrac{n}{2}; \quad (3)$$

$E_n^1 = \tfrac{n}{0}\binom{n-1}{-1} \equiv 1$ is accepted.

If n is an odd power,

TABLE I
ABIYEV'S TRIANGLES (LEFT AND RIGHT)

	7	6	5	4	3	2	1	0	1	2	3	4	5	6	7	
F								**n**								**L**
1							1	**1**	1							1
1						1	0	**2**	2	1						3
2						1	1	**3**	3	1						4
3					1	2	0	**4**	2	4	1					7
5					1	3	1	**5**	5	5	1					11
8				1	4	3	0	**6**	2	9	6	1				18
13				1	5	6	1	**7**	7	14	7	1				29
21			1	6	10	4	0	**8**	2	16	20	8	1			47
34			1	7	15	10	1	**9**	9	30	27	9	1			76
55		1	8	21	20	5	0	**10**	2	25	50	35	10	1		123

$$a^n + b^n = x\Big(T_n^1 x^{n-1} - T_n^2 x^{n-3} y + T_n^3 x^{n-5} y^2 - \dots$$
$$\dots + T_n^{\frac{n-1}{2}} x^2 y^{\frac{n-3}{2}} - T_n^{\frac{n+1}{2}} y^{\frac{n-1}{2}}\Big) \qquad (4)$$

here

$$T_n^{k+1} = \tfrac{n}{k}\binom{n-k-1}{k-1}; \quad k = 0,1,2,3,\dots,\tfrac{n-1}{2} \qquad (5)$$

and $T_n^1 = \frac{n}{0}\binom{n-1}{-1} \equiv 1$ is accepted.

The written E_n^{k+1} and T_n^{k+1} coefficients can be described in another way too:

$$E_n^{k+1} = \binom{n-k}{k} + \binom{n-1-k}{k-1}; \; k = 0,1,2,\dots,\tfrac{n}{2} \qquad (n\text{-even})$$

$$T_n^{k+1} = \binom{n-k}{k} + \binom{n-1-k}{k-1}; \; k = 0,1,2,\dots,\tfrac{n-1}{2} \qquad (n\text{-odd})$$

Let's write the polynomial expressions for $a^n - b^n$ expression. If n is an even power,

$$\frac{a^n - b^n}{a-b} = x\Big(M_n^1 x^{n-2} - M_n^2 x^{n-4} y + M_n^3 x^{n-6} y^2 - \dots$$
$$\dots - M_n^{\frac{n-2}{2}} x^2 y^{\frac{n-4}{2}} + M_n^{\frac{n}{2}} y^{\frac{n-2}{2}}\Big) \qquad (6)$$

here $M_n^{k+1} = \binom{n-k-1}{k}; \quad k = 0,1,2,3,\dots,\frac{n}{2}-1 \qquad (7)$

If n is an odd power,

$$\frac{a^n - b^n}{a-b} = N_n^1 x^{n-1} - N_n^2 x^{n-3} y + N_n^3 x^{n-5} y^2 - \dots$$
$$\dots + N_n^{\frac{n-1}{2}} x^2 y^{\frac{n-3}{2}} - N_n^{\frac{n+1}{2}} y^{\frac{n-1}{2}}\Big) \qquad (8)$$

here $N_n^{k+1} = \binom{n-k-1}{k}; \quad k = 0,1,2,3,\dots,\frac{n-1}{2}. \qquad (9)$

The written M_n^{k+1} and N_n^{k+1} coefficients can be described in another way too:

$$M_n^{k+1} = \binom{n-k}{k} - \binom{n-1-k}{k-1}; \; k = 0,1,2,\dots,\tfrac{n}{2}-1 \qquad (n\text{-even})$$

$N_n^{k+1} = \binom{n-k}{k} - \binom{n-1-k}{k-1};\; k = 0,1,2,...,\frac{n-1}{2};\; \binom{n-1}{-1} \equiv 0$
(n-odd).

Note: As the signs in (2), (4), (6) and (8) polynomial expressions always change in sequence, $(-1)^n$ symbol hasn't been used in them.

Let's show 2 examples for the identities:

If $n = 6$, $a^6 + b^6 = x^6 - 6x^4y + 9x^2y^2 - 2y^3$;

If $n = 7$, $\dfrac{a^7 - b^7}{a-b} = x^6 - 5x^4y + 6x^2y^2 - y^3$.

See the 6th row in the right triangle and the 7th row in the left triangle of the table 1.

IV PROPERTIES OF COEFFICIENTS

It's known that the sum of the numbers in each row of Pascal triangle equals to 2^{n-1}. But in Abiyev triangles the sum of the numbers in the rows forms Lucas (at the right) and Fibonacci (at the left) sequences. Though in the literature [4] it's shown that the sum of the numbers in a diagonal line in Pascal triangle forms Fibonacci sequence, what values these numbers have is unknown.

Let's display that in Abiyev triangles these singularities are not random.

For this purpose the coefficients of (2), (4), (6) and (8) polynomial expressions and Binet formula $L_n = \alpha^n + \beta^n; F_n = \frac{\alpha^n + \beta^n}{\alpha - \beta}; \alpha = \frac{1+\sqrt{5}}{2};\; \beta = \frac{1-\sqrt{5}}{2}$ of Lucas and Fibonacci sequences are used [3]. Here $x = \alpha + \beta = 1;\; y = \alpha\beta = -1$. If these values are put into (2), (4), (6) and (8) polynomial expressions, the following equations can be obtained:

$$a^n + b^n = \alpha^n + \beta^n = E_n^1 + E_n^2 + E_n^3 + ... + E_n^{\frac{n}{2}} + E_n^{\frac{n+2}{2}} = L_n$$
n –even; (Lucas term).

$$a^n + b^n = \alpha^n + \beta^n = T_n^1 + T_n^2 + T_n^3 + ... + T_n^{\frac{n-1}{2}} + T_n^{\frac{n+1}{2}} = L_n;$$
n –odd; (Lucas term).

$$\frac{a^n - b^n}{a-b} = \frac{\alpha^n - \beta^n}{\alpha - \beta} = M_n^1 + M_n^2 + M_n^3 + ... + M_n^{\frac{n-2}{2}} + M_n^{\frac{n}{2}} = F_n;$$
n –even; (Fibonacci term).

$$\frac{a^n - b^n}{a-b} = \frac{\alpha^n - \beta^n}{\alpha - \beta} = N_n^1 + N_n^2 + N_n^3 + ... + N_n^{\frac{n-1}{2}} + N_n^{\frac{n+1}{2}} = F_n;$$
n –odd; (Fibonacci term).

So taking into account (3), (5), (7) and (9) polynomial expressions new formulae can be written for Lucas and Fibonacci triangles:

$$L_n = \sum_{k=0}^{\frac{n}{2}} E_n^{k+1} = \sum_{k=0}^{\frac{n}{2}} \frac{n}{k}\binom{n-1-k}{k}; \qquad (10a)$$

$$L_n = \sum_{k=0}^{\frac{n-1}{2}} T_n^{k+1} = \sum_{k=0}^{\frac{n-1}{2}} \frac{n}{k}\binom{n-1-k}{k}; \qquad (10b)$$

$$F_n = \sum_{k=0}^{\frac{n}{2}-1} M_n^{k+1} = \sum_{k=0}^{\frac{n}{2}-1} \binom{n-1-k}{k}; \qquad (11a)$$

$$F_n = \sum_{k=0}^{\frac{n-1}{2}} N_n^{k+1} = \sum_{k=0}^{\frac{n-1}{2}} \binom{n-1-k}{k}; \qquad (11b)$$

Let's give examples for L_6 and F_7 formulas:

$$L_6 = \sum_{k=0}^{3} E_6^{k+1} = \sum_{k=0}^{3} \frac{6}{k}\binom{5-k}{k-1} \Rightarrow E_6^1 + E_6^2 + E_6^3 + E_6^4 =$$
$$= 1 + \frac{6}{1}\binom{4}{0} + \frac{6}{2}\binom{3}{1} + \frac{6}{3}\binom{2}{2} = 1 + 6 + 9 + 2 = 18.$$

$$F_7 = \sum_{k=0}^{3} N_7^{k+1} = \sum_{k=0}^{3} \binom{6-k}{k} \Rightarrow F_7 = N_7^1 + N_7^2 + N_7^3 + N_7^4 =$$
$$= \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = 1 + 5 + 6 + 1 = 13.$$

The above-formed formulae enable us to express the following theorem.

Theorem. There are Lucas and Fibonacci numbers expressed by $L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{k}\binom{n-1-k}{k-1}$ and $F_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{k}\binom{n-1-k}{k}$ formulas for any positive whole number.

Proof. Taking into consideration the peculiarities of the algorithm used for formulating Abiyev triangles, let's prove the theorem applying a mathematical induction method. According to (3) and (5) formulae let's form right Abiyev triangle for $(n-1), n, (n+1), (n+2)$ rows:

In accordance with the singularities of the right and left triangles the following expressions can be written:

$$\text{I} - \; l_{ij} + l_{i+1,j-1} = l_{i+2,j} \;\; ; \;\; \text{II} - \; l_{i-1,j-1} + l_{i,j-1} = l_{i+1,j-1};$$
$$2 \le i \le 2p; \;\; 2 \le j \le p+1; \;\; p \ge 1. \;\; \text{(right)}$$

$$\text{I} - \; f_{ij} + f_{i+1,j-1} = f_{i+2,j} \;\; ; \text{II} - \; f_{i-1,j-1} + f_{i,j-1} = f_{i+1,j-1};$$
$$2 \le i \le 2p; \;\; 2 \le j \le p+1; \;\; p \ge 1. \;\; \text{(left)}$$

here l and f are elements of the right and left triangles respectively.

Let's conduct the following algebraic operations using this triangle:

$$\text{I} - \; l_{n,3} + l_{n+1,2} = l_{n+2,3}$$

$$\frac{n}{\frac{n-4}{2}}\binom{\frac{n+2}{2}}{\frac{n-6}{2}} + \frac{n+1}{\frac{n-2}{2}}\binom{\frac{n+2}{2}}{\frac{n-4}{2}} = \frac{n+2}{\frac{n-2}{2}}\binom{\frac{n+4}{2}}{\frac{n-4}{2}} \Rightarrow$$

$$\Rightarrow \frac{n}{\frac{n-4}{2}}\binom{\frac{n+2}{2}}{4} + \frac{n+1}{\frac{n-2}{2}}\binom{\frac{n+2}{2}}{3} = \frac{n+2}{\frac{n-2}{2}}\binom{\frac{n+4}{2}}{4} \Rightarrow$$

$$\Rightarrow \frac{n}{\frac{n-4}{2}} \frac{\frac{n+2}{2}\frac{n}{2}\frac{n-2}{2}\frac{n-4}{2}}{1.2.3.4} + \frac{n+1}{\frac{n-2}{2}} \frac{\frac{n+2}{2}\frac{n}{2}\frac{n-2}{2}}{1.2.3} =$$

$$= \frac{n+2}{\frac{n-2}{2}} \frac{\frac{n+4}{2}\frac{n+2}{2}\frac{n}{2}\frac{n-2}{2}}{1.2.3.4} \Rightarrow n^2 + 6n + 8 = n^2 + 6n + 8.$$

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II − $l_{n-1,2} + l_{n,2} = l_{n+1,2}$

$$\frac{n-1}{\frac{n-4}{2}}\binom{\frac{n}{2}}{\frac{n-6}{2}} + \frac{n}{\frac{n-2}{2}}\binom{\frac{n}{2}}{\frac{n-4}{2}} = \frac{n+1}{\frac{n-2}{2}}\binom{\frac{n+2}{2}}{\frac{n-4}{2}} \Rightarrow$$

$$\Rightarrow \frac{n-1}{\frac{n-4}{2}}\binom{\frac{n}{2}}{3} + \frac{n}{\frac{n-2}{2}}\binom{\frac{n}{2}}{2} = \frac{n+1}{\frac{n-2}{2}}\binom{\frac{n+2}{2}}{3} \Rightarrow$$

$$\Rightarrow \frac{n-1}{\frac{n-4}{2}}\frac{\frac{n}{2}\,\frac{n-2}{2}\,\frac{n-4}{2}}{1.2.3} + \frac{n}{\frac{n-2}{2}}\frac{\frac{n}{2}\,\frac{n-2}{2}}{1.2} =$$

$$= \frac{n+1}{\frac{n-2}{2}}\frac{\frac{n+2}{2}\,\frac{n}{2}\,\frac{n-2}{2}}{1.2.3} \Rightarrow n^2 + 3n + 2 = n^2 + 3n + 2.$$

These operations specify that the algorithm of formation in the right triangle is defined for any *n*. The same assumption concerns the left triangle too. By this way, the theorem is proved.

V APPLICATIONS

The solution of

$$\begin{cases} X^n \pm Y^n = a \\ X + Y = b \end{cases}$$

equation system is simplified, i.e. the power of the system is reduced to a half and brought to the solution of $\frac{n}{2}$ equivalent equation system.

The calculation of $(X+iY)^n \pm (X-iY)^n = Z^n \pm \overline{Z}^n$ expression mostly used in linear algebra is rather facilitated

For instance, let's show $C = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^7 + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^7$.

Here taking into account

$x = a + b = 1$; $y = ab = 1$ and

$$a^7 + b^7 = x\left(x^6 - 7x^4y + 14x^2y^2 - 7y^3\right),$$

$C = 1 - 7 + 14 - 7 = 1$ is easily calculated.

VI CONCLUSION

The content of the article is absolutely new. The coefficients of the polynomial expressions can especially be accepted as a spectrum of Lucas and Fibonacci sequences. As these sequences have a significant importance in mathematics, their spectrum can also be applied in different fields. The identities proposed in the article will allow simplifying mathematical calculations in various fields of science and technology.

TABLE 2 PART OF THE ABIYEV'S RIGHT TRIANGLE

$\frac{col.\rightarrow}{\downarrow row}$	1	2	3	...	$\frac{n-4}{2}$	$\frac{n-2}{2}$	$\frac{n}{2}$	$\frac{n+2}{2}$	$\frac{n+4}{2}$
$n-1$:	$\frac{n-1}{\frac{n-2}{2}}\binom{\frac{n-2}{2}}{\frac{n-4}{2}}$	$\frac{n-1}{\frac{n-4}{2}}\binom{\frac{n}{2}}{\frac{n-6}{2}}$	$\frac{n-1}{\frac{n-6}{2}}\binom{\frac{n+2}{2}}{\frac{n-8}{2}}$	...	$\frac{n-1}{2}\binom{n-4}{1}$	$\frac{n-1}{1}\binom{n-3}{0}$	1		
n:	$\frac{n}{\frac{n}{2}}\binom{\frac{n-2}{2}}{\frac{n-2}{2}}$	$\frac{n}{\frac{n-2}{2}}\binom{\frac{n}{2}}{\frac{n-4}{2}}$	$\frac{n}{\frac{n-4}{2}}\binom{\frac{n+2}{2}}{\frac{n-6}{2}}$	...	$\frac{n}{3}\binom{n-4}{2}$	$\frac{n}{2}\binom{n-3}{1}$	$\frac{n}{1}\binom{n-2}{0}$	1	
$n+1$:	$\frac{n+1}{\frac{n}{2}}\binom{\frac{n}{2}}{\frac{n-2}{2}}$	$\frac{n+1}{\frac{n-2}{2}}\binom{\frac{n+2}{2}}{\frac{n-4}{2}}$	$\frac{n+1}{\frac{n-4}{2}}\binom{\frac{n+4}{2}}{\frac{n-6}{2}}$	...	$\frac{n+1}{3}\binom{n-3}{2}$	$\frac{n+1}{2}\binom{n-2}{1}$	$\frac{n+1}{1}\binom{n-1}{0}$	1	
$n+2$:	$\frac{n+2}{\frac{n+2}{2}}\binom{\frac{n}{2}}{\frac{n}{2}}$	$\frac{n+2}{\frac{n}{2}}\binom{\frac{n+2}{2}}{\frac{n-2}{2}}$	$\frac{n+2}{\frac{n-2}{2}}\binom{\frac{n+4}{2}}{\frac{n-4}{2}}$	...	$\frac{n+2}{4}\binom{n-3}{3}$	$\frac{n+2}{3}\binom{n-2}{2}$	$\frac{n+2}{2}\binom{n-1}{1}$	$\frac{n+2}{1}\binom{n}{0}$	1

$$A_p^{p+1} + B_p^{p+1} = F\left[\left(A_p + B_p\right), A_pB_p\right]$$

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