MAC-solutions for the circular elastic plate and membrane

IGOR NEYGEBAUER
University of Dodoma
Department of Mathematics and Statistics
P.O.Box 259, Dodoma
TANZANIA
newigor52r@yahoo.com

Abstract: The method of additional conditions or MAC is applied to the boundary value problems of mathematical physics, where the classical solution does not exist or a nonphysical generalized solution is obtained. The Dirichlet problem for a circular elastic membrane is considered as the simplest example of the problem with nonexistent solution. The constant boundary conditions are given in the center of the membrane and on the finite radius. The scheme of Dugdale-Barrenblat in Griffith’s linear elastic crack problem is used to obtain the MAC-solution. The additional pressure on membrane is applied near the center, the conditions of smoothness are used, the line-integral, which follows from the second Green’s identity is considered and it is taken as a measure of the introduced correction of the initial Dirichlet problem for Laplace equation. The applied conditions create an unique solution, which at least corresponds to the physical situation and easily can be proofed experimentally. The similar approach is applied to the circular elastic plate under force in the center of a plate.

Key–Words: Laplace and Biharmonic Equations, Membrane, Plate, MAC-solutions

20 March 2011

1 Introduction

Some classical boundary value problems from elasticity will be considered [3]. Let us apply the method of additional conditions (MAC) to consider the displacements of an elastic membrane and the plate. MAC model for an elastic membrane is suggested in [7] using the principle of superposition. MAC solution for an angle domain and the Laplace equation is presented in [6]. The method of additional conditions (MAC) is considered in this paper to obtain the displacements of an elastic membrane and the plate using additional condition in form of an invariant integral.

2 Circular elastic membrane

2.1 Statement of the problem

Let us consider a circular elastic membrane with symmetric Dirichlet boundary conditions. Then it is reasonable to consider the symmetric solution. The Laplace’s equation in this 2D case is:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = 0; \]

We consider domain \( \Omega: 0 \leq r \leq R \);

The boundary conditions are:

\[ u(0) = 0; u(R) = u_1 \neq 0; \]

2.2 MAC-solution

2.2.1 Algorithm to obtain the MAC-solution

To create the MAC-solution we use the following algorithm:

- The second Green’s identity gives the invariant integral like \( J \)-integral in fracture mechanics.
- The generalized solution is used.
- The applied additional external pressure near the center of the membrane in the domain \( \Omega_1: 0 \leq r \leq a \) is one term of the form: \( \beta \cdot p \cdot r^{\beta-2} \), where \( p \) and \( \beta \) are some constants.
- Solution has not singularity in gradient at \( r = 0 \).
- The domain \( \Omega \) is divided in two parts:
  - Domain \( \Omega_1: 0 \leq r \leq a \);
  - Domain \( \Omega_2: a < r \leq R \).
Proceedings of the 4th WSEAS International Conference on Finite Differences - Finite Elements - Finite Volumes - Boundary Elements

- Solution $u$ is from the class $C^1$.

- The maximum of the $J$- integral in the first domain must be equal to that constant value, which is obtained on the classical generalized solution.

### 2.2.2 Additional condition in the form of the line integral

To obtain the additional condition in the form of line integral let us consider the classical second Green’s identity [4]:

$$
\int_{\Omega} (v \Delta u - u \Delta v) \, d\Omega = \int_{\partial\Omega} \left( v \frac{\partial u}{\partial v} - u \frac{\partial v}{\partial v} \right) \, ds,
$$

where the functions $u$ and $v \in C^2(\Omega)$. Consider any two solutions of Laplace equation $u, v$ then we have:

$$
\Delta u = 0; \Delta v = 0. \quad (4)
$$

Then we obtain an invariant integral along any curve $L : r = R_1$, where the radius $R_1$ is arbitrary and satisfies the condition $0 < R_1 \leq R$. We have

$$
J = \int_{L} \left( v \cdot \frac{\partial u}{\partial v} - u \cdot \frac{\partial v}{\partial v} \right) \, ds = C, \quad (5)
$$

where $C$ is some constant.

The formula (5) is right for any closed piecewise smooth curve $L$ around the origin $O$. It takes the following form in case of circumference $L : r = R_1$:

$$
J = 2 \cdot \pi \cdot r \cdot \left( v \cdot \frac{\partial u}{\partial r} - u \cdot \frac{\partial v}{\partial r} \right) = C \quad (6)
$$

### 2.2.3 The value of the $J$-integral

Let us calculate this constant, taking two solutions of the Laplace equation (1). The first solution is $u = u_1$ and the second solution is the general solution of the Laplace equation:

$$
v = A + B \cdot \ln(r), \quad (7)
$$

where $A, B$ are arbitrary constants and they will be determined below. The invariant integral $J$ is equal in this case

$$
J = -2 \cdot \pi \cdot u_1 \cdot B = C. \quad (8)
$$

### 2.2.4 The value $J(0)$ for physical solution

The experiments with membrane show that it is reasonable to suggest that the real physical solution has the following properties: the deflection is smooth and its derivative is bounded at the origin and in its vicinity. If we denote the MAC-solution as $\tilde{u}$, then we obtain in the small enough vicinity of the origin $r = 0$:

$$
\left| \frac{\partial \tilde{u}}{\partial r} \right| \leq C_1, \quad |\tilde{u}| \leq C_1 \cdot r, \quad (9)
$$

where $C_1$ is some positive constant. It is easy now to estimate the value of function $J(r)$ according to (5) or (6) at $r \to 0$, where the generalized solution $u$ is replaced by the physical MAC-solution $\tilde{u}$. We obtain in the vicinity of the origin

$$
|J(r)| \leq 2 \cdot \pi \cdot r \cdot C_1 \cdot (|A + B \cdot \ln (r)| + |B|). \quad (10)
$$

It follows from the inequality (10) that we can define the value of $J(0)$ as

$$
J(0) = \lim_{r \to 0} J(r) = 0. \quad (11)
$$

### 2.2.5 MAC-solution in the domain $\Omega_1$

Consider the domain $\Omega_1 : 0 \leq r \leq a$ and the Poisson equation:

$$
\frac{1}{r} \cdot \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = \beta^2 \cdot p \cdot r^\beta - 2, \quad (12)
$$

where the constant $p$ and $\beta$ are unknown parameters, that must help to transform the nonphysical generalized solution to the physical MAC-solution. The parameters $p$ and $\beta$ will be determined some later. We have introduced some additional pressure on the right side of the equation (12) near the origin.

We have to avoid singularity at $r = 0$. Then the MAC-solution in the domain $\Omega_1$ is

$$
u = p \cdot r^\beta. \quad (13)
$$

The parameters $p$ and $\beta$ will be determined later.

### 2.2.6 MAC-solution in the domain $\Omega_2$

Consider the domain $\Omega_2 : a \leq r \leq R$ and Laplace equation:

$$
\frac{1}{r} \cdot \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = 0. \quad (14)
$$

The general solution of the equation (14) is

$$
u = A_1 + B_1 \cdot \ln(r), \quad (15)$$
where $A_1, B_1$ are arbitrary constants. The boundary condition gives the relation between the constants $A_1$ and $B_1$:

$$u_1 = A_1 + B_1 \cdot \ln(R).$$ (16)

Then the MAC-solution (15) can be written in the form:

$$u = u_1 + B_1 \cdot \ln \left( \frac{r}{R} \right).$$ (17)

There are four parameters to be determined: $p, \beta, B_1$ and $a$. We will use two smoothness conditions and the properties of the function $J(r)$ (6) to obtain the unique values of these parameters.

2.2.7 Smoothness conditions

Consider the smoothness conditions at $r = a$ on the boundary between domains $\Omega_1$ and $\Omega_2$. We require that the MAC-solution is smooth, therefore

$$u(a - 0) = u(a + 0), \quad \frac{\partial u}{\partial r}(a - 0) = \frac{\partial u}{\partial r}(a + 0).$$ (18) (19)

If we use the MAC-solutions (13) and (17), then the smoothness conditions (18) and (19) give the relations:

$$B_1 = \frac{\beta \cdot u_1}{1 + \beta \cdot \ln \left( \frac{R}{a} \right)},$$ (20)

$$p = \frac{u_1}{a^\beta \cdot (1 + \beta \cdot \ln \left( \frac{R}{a} \right))}.$$ (21)

The remaining parameters $a$ and $\beta$ will be found using the function $J(r)$.

2.2.8 Extremum of $J(r)$

Consider the function $J(r)$ according to (6), where the function $u$ is the MAC-solution: (13) in $\Omega_1$ and (17) in $\Omega_2$ and the function $v$ in the whole domain $\Omega$ equals to the function (17). Then it is easy to see that $J(0) = 0$ and the property (11) and conditions (9) are satisfied if we have

$$\beta \geq 1.$$ (22)

The point of extremum is

$$r_0 = a \cdot e^{-\frac{1}{\beta}}.$$ (23)

The extremum of the function $J(r)$ inside the domain $\Omega_1$ is:

$$J(r_0) = -\frac{2 \cdot p \cdot u_1 \cdot \pi \cdot \beta \cdot a^\beta}{e \cdot (1 + \beta \cdot \ln \left( \frac{R}{a} \right))}.$$ (24)

2.2.9 Radius $a$ and parameter $\beta$

Consider the condition.

$$-\frac{2 \cdot p \cdot u_1 \cdot \pi \cdot \beta \cdot a^\beta}{e \cdot (1 + \beta \cdot \ln \left( \frac{R}{a} \right))} = -2 \cdot \pi \cdot u_1 \cdot p \cdot \beta \cdot a^\beta,$$ (25)

where the expression for $B_1$ is applied. The nonzero solutions of the equation (25) are

$$a = R \cdot e^{\left( \frac{1}{\beta} - \frac{1}{2} \right)},$$ (26)

or

$$\beta = \frac{1}{\ln \left( \frac{R}{a} \right)}.$$ (27)

As $0 < a < R$ then it follows from (27) that

$$\beta < 0.$$ (28)

The inequality (28) contradicts the inequality (22). It means that we can not reach the equality (25) for any $\beta \geq 1$ and for any $0 < a < R$. Then we can try to choose the parameters $\beta, a$ so that the ratio of the expressions on the left and on the right sides of the equality (25)

$$e \cdot (1 + \beta \cdot \ln \left( \frac{R}{a} \right))$$ (29)

has a minimum value. It will be if $\beta = 1$ for any fixed $a \to R$. Then it is reasonable to accept the values $\beta = 1$ and $a = R$. It means that we have only the domain $\Omega_1$ and the domain $\Omega_2$ does not exist in the considered problem. The MAC-solution of the problem (1) and (2) is given in the form

$$u = u_1 \cdot \left( \frac{r}{R} \right).$$ (30)

2.2.10 Remarks to the MAC-solution

The obtained MAC-solution (30) shows that if we measure the function and its derivative on the boundary then we can determine the inclusion with a given potential of any size. It is evident if we consider an experiment with membrane: any nonzero displacement in the center of a membrane can be determined through the measurements on the boundary. If the generalized solution is used then we have to conclude that it is impossible to determine the point inclusion in the middle of a membrane. It could be interesting to consider this effect of two theories in experiment with an electrostatic field. The classical approach and the theory is presented in [1].
3 Circular plate

3.1 Nonexistent solution

In this section we will consider the same problem as for a circular membrane but we will use the equation for a circular elastic plate. The solution of this problem can be taken from [9].

Consider the bending of a circular elastic plate. The differential equation of our problem is

\[ \Delta \Delta u = 0, \quad (31) \]

where \( u \) is the transverse displacements of the points of a circular plate, \( \Delta \) is the Laplace operator. We consider an axis symmetric case, then the equation (31) will take the form

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \cdot \frac{\partial}{\partial r} \left( 1 \cdot \frac{\partial}{\partial r} (r \cdot \frac{\partial u}{\partial r}) \right) \right) = 0, \quad (32) \]

where the plate is in the domain \( \Omega : 0 \leq r \leq R \). The general solution of the equation (32) is

\[ u = C_1 \cdot \ln r + C_2 \cdot r^2 \cdot \ln r + C_3 + C_4 \cdot r^2, \quad (33) \]

where the arbitrary constants can be found from the boundary conditions. Suppose the following boundary conditions

\[ u(0) = u_0, u(R) = 0. \quad (34) \]

The function (33) creates

\[ \frac{\partial u}{\partial r} = \frac{C_1}{r} + C_2 \cdot r \cdot (2 \cdot \ln r + 1) + 2 \cdot C_4 \cdot r. \quad (35) \]

Then we have at \( r = 0 \)

\[ \frac{\partial u}{\partial r} (0) = 0 \quad (36) \]

and the constant \( C_1 = 0 \) that follows from the finiteness of that derivative in the experiments with the plate. It follows from (36) that if this condition is not zero then the solution of the given problem does not exist.

The condition gives

\[ C_1 = u(0) = u_0. \quad (37) \]

Then the condition gives

\[ C_2 \cdot R^2 \cdot \ln R + u_0 + C_4 \cdot R^2 = 0. \quad (38) \]

If the force \( P \) is applied at \( r = 0 \) then we have the condition from the equilibrium equation

\[ Q(R) = \frac{P}{2 \cdot \pi \cdot R}. \quad (39) \]

Then we obtain from (39) and (38) The solution (33) will take the form

\[ u(r) = \frac{P}{8 \cdot \pi \cdot D} \cdot r^2 \cdot \ln r + u_0 + \left( \frac{P \cdot \ln R}{8 \cdot \pi \cdot D} - \frac{u_0}{R^2} \right) \cdot r^2. \quad (40) \]

\[ \frac{\partial u}{\partial r} (r) = \frac{P}{8 \cdot \pi \cdot D} \cdot r \cdot (2 \cdot \ln r + 1) + 2 \cdot \frac{P \cdot \ln R}{8 \cdot \pi \cdot D} - \frac{u_0}{R^2} \cdot r. \quad (41) \]

The bending moments are [9]:

\[ M_r = -2 \cdot D \cdot (C_2(1 + \nu) \cdot (2 \cdot \ln r + 1) + C_2 + C_4(1 + \nu)), \quad (42) \]

\[ M_t = -2 \cdot D \cdot (C_2(1 + \nu) \cdot (2 \cdot \ln r + 1) + \nu \cdot C_2 + C_4(1 + \nu)). \quad (43) \]

We can see in the experiment with plate under force \( P \) in the middle of a circular plate, that there exist only the finite bending moments and corresponding stresses. It means that the logarithmic terms in the expression (42) must be avoided. Then we have

\[ C_2 = \frac{P}{8 \cdot \pi \cdot D} = 0. \quad (44) \]

Therefore we obtain from (44) that

\[ P = 0. \quad (45) \]

The force \( P \neq 0 \) according to the stated problem. This contradicts to the value (45). We can conclude from the obtained contradiction that the solution of the stated problem does not exist.

As we can see that the situation with the plate is similar to the situation with the membrane: the classical solution of the problem does not exist but the physical solution of the problem exists evidently.

3.2 Generalized solution

We can obtain the generalized solution of the problem using the similar way as in the membrane problem. It means that can suppose the distribution of the force \( P \) near the origin in the small circle. Then we can find the solution of the stated problem. And after that the radius of the small circle must tend to zero. This approach is presented in [9]. For instance consider the case there is

\[ \frac{d u}{dr} (R) = 0 \quad (46) \]

on the boundary. Then the generalized solution of the problem is [9]:

\[ u(r) = \frac{P \cdot r^2}{8 \cdot \pi \cdot D} \cdot \ln \frac{r}{R} + \frac{P}{16 \cdot \pi \cdot D} \cdot (R^2 - r^2). \quad (47) \]
The bending moments are [9]:

\[ M_r = \frac{P}{4 \cdot \pi} \cdot \left( (1 + \nu) \cdot \ln \frac{R}{r} - 1 \right), \quad (48) \]

\[ M_t = \frac{P}{4 \cdot \pi} \cdot \left( (1 + \nu) \cdot \ln \frac{R}{r} - \nu \right). \quad (49) \]

We can see that the expression for the displacement \( u \) according to (47) is well enough. But the expressions (48), (49) for the bending moments have singularities at the point of application of the force. They can not be used to determine the stresses near the origin. Another models have to be considered to determine the real stresses. We will use for that the MAC-solution of the problem.

### 3.3 MAC-solution

We can use the method of Marcus [9] to obtain the MAC-solution for our plate problem. Consider the differential equation of the plate (31) in the form

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0. \quad (50) \]

The bending moments are

\[ M_x = -D \cdot \left( \frac{\partial^2 u}{\partial x^2} + \nu \cdot \frac{\partial^2 u}{\partial y^2} \right), \quad (51) \]

\[ M_y = -D \cdot \left( \frac{\partial^2 u}{\partial y^2} + \nu \cdot \frac{\partial^2 u}{\partial x^2} \right), \quad (52) \]

If we introduce a new notation

\[ M = \frac{M_x + M_y}{1 + \nu} = -D \cdot \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (53) \]

the equations (50) and (??) can be written in the form:

\[ \frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} = 0, \quad (54) \]

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{M}{D}. \quad (55) \]

We consider the symmetric case and therefore the equations (54), (55) in polar coordinates will take the form

\[ \frac{1}{r} \cdot \frac{\partial}{\partial r} \left( r \cdot \frac{\partial M}{\partial r} \right) = 0, \quad (56) \]

\[ \frac{1}{r} \cdot \frac{\partial}{\partial r} \left( r \cdot \frac{\partial u}{\partial r} \right) = -\frac{M}{D}. \quad (57) \]

The equation (56) is the membrane equation (1) and its general MAC-solution can be written in the form:

\[ M(r) = C_1 + C_2 \cdot r, \quad (58) \]

where \( C_1 \) and \( C_2 \) are arbitrary constants. The MAC-solution (58) corresponds only to one-term approximation, which introduces only one additional term into differential equation. It could be also interesting to introduce some additional terms into equations for \( M_r, M_t, Q, \frac{\partial u}{\partial r}, w \). Then we will have the multiple-term approximation. We need then more computations to analyze these case and it can be done on the future. It must be mentioned, that the values of \( M_r, M_t, Q, \frac{\partial u}{\partial r}, w \), obtained in the domain \( \Omega_1 \), are used only to calculate the stresses and displacements. For example, the function \( Q \) in the domain \( \Omega_1 \) does not satisfy the equilibrium equation. Moreover the functions \( M_r, M_t, Q \), which satisfy the equilibrium equations exactly, have singularities in the solution.

Then the equation (57) will be

\[ -\frac{1}{r} \cdot \frac{\partial}{\partial r} \left( r \cdot \frac{\partial u}{\partial r} \right) = -\frac{1}{D} \cdot (C_1 + C_2 \cdot r). \quad (59) \]

Consider the particular solution of the equation (59)

\[ w(r) = -\frac{1}{D} \cdot \left( C_1 \cdot \frac{r^2}{4} + C_2 \cdot \frac{r^3}{9} \right). \quad (60) \]

Consider the function

\[ v = u - w. \quad (61) \]

Then this function \( v \) (61) satisfies the membrane equation:

\[ \frac{1}{r} \cdot \frac{\partial}{\partial r} \left( r \cdot \frac{\partial v}{\partial r} \right) = 0. \quad (62) \]

The general MAC-solution of the equation (62) is

\[ v(r) = C_3 + C_4 \cdot r, \quad (63) \]

where \( C_3 \) and \( C_4 \) are arbitrary constants. Therefore we obtain the general MAC-solution, using (60), (61), (63), in the following form

\[ u(r) = -\frac{1}{D} \cdot \left( C_1 \cdot \frac{r^2}{4} + C_2 \cdot \frac{r^3}{9} \right) + C_3 + C_4 \cdot r, \quad (64) \]

where \( C_1, C_2, C_3 \) and \( C_4 \) are arbitrary constants. Then we can obtain

\[ \frac{\partial u}{\partial r}(r) = -\frac{1}{D} \cdot \left( C_1 \cdot \frac{r}{2} + C_2 \cdot \frac{r^2}{3} \right) + C_4. \quad (65) \]

The bending moments are

\[ M_r = \frac{1 + \nu}{2} \cdot C_1 + \frac{2 + \nu}{3} \cdot C_2 \cdot r - \frac{\nu \cdot D}{r} \cdot C_4, \quad (66) \]
\[ M_i = \frac{1 + \nu}{2} \cdot C_1 + \frac{1 + 2 \cdot \nu}{3} \cdot C_2 \cdot r - \frac{D \cdot C_4}{r}, \quad (67) \]

\[ M = \frac{M_x + M_y}{1 + \nu} = C_1 + C_2 \cdot r - \frac{D \cdot C_4}{r}. \quad (68) \]

The force \( Q \) is

\[ Q(r) = \frac{1}{r} \cdot \frac{\partial M}{\partial r} = -C_2 + \frac{D \cdot C_4}{r^2}. \quad (69) \]

Then the bending moments and the force \( Q \), using equations (66), (67), (69) are

\[ M_r(r) = \frac{6 \cdot D \cdot u_0}{R^2} \cdot (1 + \nu - (2 + \nu) \cdot \frac{r}{R}), \quad (70) \]

\[ M_t(r) = \frac{6 \cdot D \cdot u_0}{R^2} \cdot (1 + \nu - (1 + 2 \cdot \nu) \cdot \frac{r}{R}), \quad (71) \]

\[ Q(r) = \frac{18 \cdot D \cdot u_0}{R^3}. \quad (72) \]

We obtain the following values of bending moments in the center \( r = 0 \) and on the boundary \( r = R \) of the plate:

\[ M_r(0) = \frac{6 \cdot D \cdot u_0}{R^2} \cdot (1 + \nu), \quad (73) \]

\[ M_r(R) = -\frac{6 \cdot D \cdot u_0}{R^2}, \quad (74) \]

\[ M_t(0) = \frac{6 \cdot D \cdot u_0}{R^2} \cdot (1 + \nu), \quad (75) \]

\[ M_t(R) = -\frac{6 \cdot D \cdot u_0}{R^2} \cdot \nu. \quad (76) \]

The values of bending moments (73) - (76) show that the fracture must start under the applied force \( P \) in the center of the plate. The condition (39) with the expression (72) give the following connection between the applied force \( P \) and the displacement in the center of the plate \( u_0 \):

\[ u_0 = \frac{P \cdot R^2}{36 \cdot \pi \cdot D}. \quad (77) \]

The value of the displacement (77) is equal to the \( \frac{1}{9} \) of the value, which gives the generalized solution at \( r = 0 \) (47).

4 Conclusion

The nonexistent, generalized and MAC solutions of the circular membrane and elastic plate were considered.

References:


