Application of the Method of Additional Conditions to the Neumann Problem for the Laplace Equation in the Wedge

IGOR NEYGEBAUER
University of Dodoma
Department of Mathematics and Statistics
P.O.Box 259, Dodoma
TANZANIA
newigor52r@yahoo.com

Abstract: The method of additional conditions (MAC) gives a well characteristic of the finite stresses near the tip of a crack of the Griffith’s problem in fracture mechanics, where is supposed of a finite elastic potential which entails the zero value of the integral. MAC uses special additional conditions to correct the given traditional problems. It could be considered as a method introducing a special variation to smooth the contradiction between the given problem and introduced additional condition. In this paper we apply this method to the probably most frequently occurring partial differential equation governing the behavior of certain physical quantities. The Laplace equation is considered in the wedge. For the given Neumann problem the derivative of the solution has a singularity at the origin for an obtuse angle. The solution is unique. The singularity can be considered as impossible from the physical point of view. There is suggested an additional condition which follows from the classical Green’s formula and from the supposed to be bounded solution and its derivatives, and that can characterize the nonsingular solution at the origin. The first approximation is considered.

Key–Words: Singularity, Laplace Equation, Method of Additional Conditions, Neumann Problem

1 Introduction

If we consider some physical problem and the corresponding boundary value problem then the different classes of functions could be considered [13, 22]. There are a lot of problems where the singularities of solutions of boundary value problems can not be avoided [2, 3, 4, 6, 8, 19]. For example in fracture mechanics there were introduced some additional criteria to analyze the solutions with singularities like stress concentration factor, energy release rate and others [1, 20]. The methods to find solutions with singularities are developed [7, 10, 11, 12, 18, 23]. To choose the needed solution we use some physical criteria or some our ideas about the behavior of solutions [21]. To get an approximate solution sometimes are useful the following conditions which could follow or not from the stated mathematical problem: conservation laws, conservation of the initial states [14, 15], invariant integrals, and boundedness of functions. The MAC solution of the Dirichlet problem for the Laplace equation in the angle domain was considered in [16]. The principle of superposition to create the MAC model of the membrane was presented in [17]. There is not known a regular theory which can give the needed additional conditions for the given boundary value problem. Therefore it is necessary to consider the method of additional conditions from the different points of view and specially to analyze the well known classical problems applying to them some physically motivated additional conditions [1, 20]. In this paper we consider the classical Neumann problem for the Laplace equation in the wedge [9]. We apply an additional condition which follows from the Green’s formula [5, 21] and from finiteness of functions. The received approximate solution includes two additional unknown parameters and is determined satisfying the equation, boundary values and the nonsingularity of the solution at the origin, but as a first approximation it does not satisfy the additional condition. This will be explained below.
2 The Statement of the Problem

We usually represent the physical problem as a mathematical problem, for example, as a boundary value problem such as the Neumann problem consisting of the Laplace equation

\[ \Delta u = 0 \]  
\[ \text{(1)} \]

or

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \]  
\[ \text{(2)} \]

and the given boundary values

\[ \left. \frac{\partial u}{\partial \nu} \right|_{L} = f, \]  
\[ \text{(3)} \]

where \( L \) is a closed contour containing the domain \( \Omega \). We can consider a lot of properties of this problem [5]. For example, if our solution

\[ u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \]  
\[ \text{(4)} \]

and the function \( f \) satisfies the necessary condition of existence of the solution:

\[ \int_{L} f \, ds = 0, \]  
\[ \text{(5)} \]

then the solution of the problem \( (2), (3) \) is unique. Let us consider the case \( n = 2 \), then the equation \( (2) \) becomes:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \]  
\[ \text{(6)} \]

or in polar coordinates:

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} = 0. \]  
\[ \text{(7)} \]

Let us consider a wedge \( 0 < r < R, 0 < \phi < \alpha \) as a domain \( \Omega \) with the following boundary conditions:

\[ \frac{\partial u}{\partial \phi} = 0, \quad \text{for} \ 0 \leq r \leq R, \ \phi = \alpha, \]  
\[ \text{(8)} \]

The exact solution of the Neumann problem \( (7), (8) \) is given by

\[ u = r^{\frac{\pi}{\alpha}} \cos \left( \frac{\pi}{\alpha} \cdot \phi \right). \]  
\[ \text{(9)} \]

Let us consider the partial derivative of the solution \( (9) \)

\[ \frac{\partial u}{\partial r} = \frac{\pi}{\alpha} \cdot r^{\left( \frac{\pi}{\alpha} - 1 \right)} \cdot \cos \left( \frac{\pi}{\alpha} \cdot \phi \right). \]  
\[ \text{(10)} \]

If the angle is obtuse and \( \alpha > \pi \) then we obtain

\[ \frac{\partial u}{\partial r}(0) = \lim_{r \to 0} \frac{\pi}{\alpha} \cdot r^{\left( \frac{\pi}{\alpha} - 1 \right)} \cdot \cos \left( \frac{\pi}{\alpha} \cdot \phi \right) = \infty \]  
\[ \text{(11)} \]

and it means the singularity or the infinite value of \( \frac{\partial u}{\partial r} \) at \( r = 0 \).

What does it mean from the physical point of view? The Laplace equation describes usually different physical problems. Let us consider some of these representations:

- \( u \) is the deflection of the stretched membrane. The Laplace equation is obtained in this problem under an assumption that the deflections of the membrane and the angles of inclinations are small. It means that the partial derivative \( \frac{\partial u}{\partial r} \) must be small and in any way bounded.

- \( u \) is the gravitational potential according to the Newton’s law of gravitation and satisfies the Laplace equation. The value

\[ \frac{\partial u}{\partial r}(0) = \infty \]  
\[ \text{(12)} \]

in this problem means the infinite force in the gravitational field. It is nonsense from the physical point of view. The modern theories of interaction between particles consider more and more the bounded potentials that could give the bounded forces of interaction between particles.

- \( u \) is the electrostatic potential according to Coulomb law and satisfies the Laplace equation. The value \( (12) \) means that there exists an infinite force in the electrostatic field. We do not observe the infinite forces in the nature. The modern theories consider also the bounded laws of interaction between the charged particles.

- \( u \) is the temperature in the steady-state heat problem. According to the physical observations the gradient of the temperature or \( \frac{\partial u}{\partial \phi} \) can not be infinite also. \( u \) is the velocity potential in the ideal incompressible fluid flow and \( \frac{\partial u}{\partial r} \) is a component of the velocity of some point in the domain occupied by fluid. This velocity can not be infinite if we observe this problem from the physical point of view.

We see that the question is now how to estimate the finite value \( \frac{\partial u}{\partial r}(0) \) of some real physical problems?

Some of the physical problems have the nonlinear generalizations of the stated Neumann prob-
lem for Laplace equation, like a problem of membrane. But other problems like about gravitational or electrostatic fields don’t have that. We will try to find some possibilities to estimate the finite value of the derivative at the origin using only the linear statement of the problem and also some additional conditions.

4 Additional Conditions

To obtain the additional conditions let us consider the classical Green’s identity:

$$\int \int_{\Omega} (v \Delta u - u \Delta v) \, d\Omega = \oint_{\partial \Omega} \left( v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) \, ds,$$

(13)

where the functions

$$u \quad \text{and} \quad v \in C^2(\Omega).$$

(14)

Let us take the solution of our Neumann problem (7), (8)

$$u = r^\gamma \cos \left( \frac{\pi}{\alpha} \cdot \varphi \right)$$

and the correspondent conjugate function

$$v = r^{-\gamma} \cos \left( \frac{\pi}{\alpha} \cdot \varphi \right).$$

(15)

(16)

Then we obtain

$$\Delta u = 0$$

and

$$\Delta v = 0$$

(17)

(18)

and than we have for any closed contour $L$ which is piecewise smooth and belongs to $\Omega$

$$\oint_{L} \left( v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) \, ds = 0.$$  

(19)

Let $L = L_1 + L_2 + L_3 + L_4$, where the contours $L_1, L_2, L_3, L_4$ are

$$L_1 : r = R_1, 0 \leq \varphi \leq \alpha;$$  

(20)

$$L_2 : r = R_1 < \varphi < R_2, \varphi = 0;$$  

(21)

$$L_3 : r = R_2, 0 \leq \varphi \leq \alpha;$$  

(22)

$$L_4 : r = R_1 \leq \varphi \leq R_2, \varphi = \alpha;$$  

(23)

where

$$0 \leq R_1 \leq R_2 \leq R;$$  

(24)

and the contour follows in the opposite clockwise direction. We can split our integral (19) in four parts

$$\oint_{L} f \, ds = \oint_{L_1} f \, ds + \oint_{L_2} f \, ds + \oint_{L_3} f \, ds + \oint_{L_4} f \, ds = 0,$$

(25)

where $f$ is the following function

$$f = v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}.$$  

(26)

Since

$$\oint_{L_2} f \, ds = \oint_{L_4} f \, ds = 0$$

(27)

according to $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on $L_2$ and $L_4$, that follows from the boundary conditions (8), then we obtain from (25)

$$\oint_{L_1} f \, ds + \oint_{L_3} f \, ds = 0$$

(28)
or
\[ \int_{L} f \, ds = - \int_{L_1} f \, ds. \tag{29} \]

If we change the direction of integration in the integral along \( L_1 \), then we obtain an invariant integral along any curve which begins on the boundary \( \varphi = \alpha \) and finishes on the boundary \( \varphi = 0 \). Let us calculate the value \( I \) of this invariant integral
\[
I = \int_{L_3} \left( v \cdot \frac{\partial u}{\partial \nu} - u \cdot \frac{\partial v}{\partial \nu} \right) \, ds =
\]
\[
\int_{0}^{\alpha} \left[ r^{-\frac{n}{2}} \cos \left( \frac{\pi}{\alpha} \varphi \right) \cdot \left( \frac{\pi}{\alpha} \right) \cdot r^{-(\frac{n}{2} - 1)} \cdot \cos \left( \frac{\pi}{\alpha} \varphi \right) \right] \, r \, d\varphi =
\]
\[
= \frac{2 \cdot \pi}{\alpha} \cdot \int_{0}^{\alpha} \cos^{2} \left( \frac{\pi}{\alpha} \varphi \right) \, d\varphi =
\]
\[
= \frac{\pi}{\alpha} \cdot \int_{0}^{\alpha} \left[ 1 + \cos \left( \frac{2 \cdot \pi}{\alpha} \varphi \right) \right] \, d\varphi =
\]
\[
= \frac{\pi}{\alpha} \cdot \left[ \varphi - \frac{\alpha}{2 \cdot \pi} \cdot \sin \left( \frac{2 \cdot \pi}{\alpha} \varphi \right) \right] |_{0}^{\alpha} =
\]
\[
= \frac{\pi}{\alpha} \cdot \alpha = \pi. \tag{30} \]

Let us consider this integral for our physical problem when the function \( u \) and its derivatives are bounded. We will take the same conjugate solution (16) and we suppose that the functions \( \tilde{u} \) and \( \frac{\partial \tilde{u}}{\partial \nu} \) are smooth and bounded. It means that near the origin \( r = 0 \) we have
\[
\left| \frac{\partial \tilde{u}}{\partial r} \right| \leq C \tag{31} \]

and
\[ |\tilde{u}| \leq C \cdot r, \tag{32} \]

there \( C \) is a constant. Let us consider a curve \( L = AB \), which connects two arbitrary points \( A \) and \( B \) on the both sides of the angle \( \Omega \). Then we obtain
\[
|I| = \left| \int_{L=AB} \left( v \cdot \frac{\partial \tilde{u}}{\partial r} - \tilde{u} \cdot \frac{\partial v}{\partial r} \right) \, ds \right| \leq
\]
\[
\leq \int_{L} \left| v \cdot \frac{\partial \tilde{u}}{\partial r} - \tilde{u} \cdot \frac{\partial v}{\partial r} \right| \, ds \leq
\]
\[
\leq \int_{L} \left| v \cdot \frac{\partial \tilde{u}}{\partial r} + \tilde{u} \cdot \frac{\partial v}{\partial r} \right| \, ds \leq
\]
\[
\leq \int_{0}^{\alpha} r^{-\frac{n}{2}} \cos \left( \frac{\pi}{\alpha} \varphi \right) \cdot C \cdot r \, d\varphi +
\]
\[
+ \int_{0}^{\alpha} C \cdot r \cdot \left( \frac{\pi}{\alpha} \right) \cdot r^{-(\frac{n}{2} - 1)} \cdot \cos \left( \frac{\pi}{\alpha} \varphi \right) \cdot r \, d\varphi =
\]
\[
= r^{(\frac{n}{2} + 1)} \cdot \left( 1 + \frac{\pi}{\alpha} \right) \cdot C \cdot \int_{0}^{\alpha} \cos \left( \frac{\pi}{\alpha} \varphi \right) \, d\varphi =
\]
\[
= C \cdot \left( 1 + \frac{\pi}{\alpha} \right) \cdot r \cdot \left( 1 - \frac{\pi}{\alpha} \right) \cdot \left[ 2 \cdot \frac{\alpha}{\pi} \cdot \sin \left( \frac{\pi}{\alpha} \varphi \right) \right] |_{0}^{\alpha} =
\]
\[
= 2 \cdot \frac{\alpha}{\pi} \cdot \left( 1 + \frac{\pi}{\alpha} \right) \cdot r \cdot \left( 1 - \frac{\pi}{\alpha} \right) \cdot C. \tag{33} \]

If the angle \( \alpha > \pi \) and so it is obtuse then we have not a singularity of the derivative \( \frac{\partial \tilde{u}}{\partial r} \) at \( r = 0 \) in the physical case and the integral \( I \to 0 \) as \( r \to 0 \). Our conclusion is: we have a contradiction between the linear case where
\[
I = \pi
\]
and the physical case, where we obtained
\[
I = 0
\]
for \( r = 0 \).

This condition \( I(0) = 0 \) we can take as additional condition to the Neumann problem. To satisfy this condition we can introduce near \( r = 0 \) some changes of the stated Neumann problem. For example we can consider the unknown boundary function or there could be taken the Poisson equation with unknown function on the right side of the equation. It is possible as we consider the Laplace equation as approximation to the more general nonlinear problem which sometimes is not known exactly.

5 Solution of the problem

Let us take additionally the Poisson equation
\[
\Delta u = F(r, \varphi) \tag{34} \]

with the function
\[
F(r, \varphi) = F_0 r^{\beta} \cos \left( \frac{\pi}{\alpha} \varphi \right), \quad if \ : 0 < r < a, 0 < \varphi < \alpha,
\]
\[
F = 0, \quad if \ : a < r < R, \ 0 < \varphi < \alpha, \tag{35} \]
where $F_0, \alpha, \beta$ are some unknown constants. Then we have that the right side of the Poisson equation is not zero in the domain $\Omega_1$:

$$0 < r < a, \ 0 < \varphi < \alpha.$$ 

The following conditions could be used to find the unknown constants:
- the nonsingularity of the partial derivative $\frac{\partial u}{\partial \varphi}(0)$,
- continuity of the solution $u$ and its derivative $\frac{\partial u}{\partial r}$ at $r = a$,
- also that the integral $I(r) = \pi$ at $a \leq r \leq R$
- and that $I(r)$ is continuous at $r = a$.

The stated problem could be written in the following form:
- partial differential equation:

$$\Delta u = F_0 r^\beta \cos \left( \frac{\pi}{\alpha} \varphi \right), \text{ if } 0 < r < a, \ 0 < \varphi < \alpha,$$

$$\Delta u = 0, \text{ if } a < r < R, \ 0 < \varphi < \alpha,$$  

- boundary conditions:

$$\frac{\partial u}{\partial \varphi} = 0, \text{ if } 0 \leq r \leq R, \ \varphi = 0,$$  

$$\frac{\partial u}{\partial \varphi} = 0, \text{ if } 0 \leq r \leq R, \ \varphi = \alpha,$$  

$$\frac{\partial u}{\partial r} = \frac{\pi}{\alpha} R(\frac{\pi}{\alpha} - 1) \cos \left( \frac{\pi}{\alpha} \varphi \right), \text{ if } r = R, \ 0 \leq \varphi \leq \alpha,$$  

- the continuity conditions at $r = a$:

$$u(a-0, \varphi) = u(a+0, \varphi), \text{ if } 0 < \varphi < \alpha,$$  

$$\frac{\partial u}{\partial r}(a-0, \varphi) = \frac{\partial u}{\partial r}(a+0, \varphi), \text{ if } 0 < \varphi < \alpha,$$  

$$I(a-0) = I(a+0) = \pi,$$  

- the finiteness condition:

$$\frac{\partial u}{\partial r}(0, \varphi) < \infty, \text{ if } 0 \leq \varphi \leq \alpha.)$$

The solution of the problem (36)-(44) does not exist for $a < R$. If we take $a = R$ and then we obtain the solution in the form

$$u = \frac{4 \cdot F_0}{\alpha \cdot ((\beta + 2)^2 - \left( \frac{\pi}{\alpha} \right)^2)} \cdot r^{\beta+2} \cdot \cos \left( \frac{\pi}{\alpha} \cdot \varphi \right) =$$

$$= \frac{\pi}{\alpha} \cdot R(\frac{\pi}{\alpha} - 2 - \beta) \cdot r \cdot \cos \left( \frac{\pi}{\alpha} \cdot \varphi \right),$$  

where $F_0$ is determined if $u$ satisfies the boundary condition (40) and is equal to

$$F_0 = \frac{\pi}{4} \cdot \left( \beta + 2 \right)^2 - \left( \frac{\pi}{\alpha} \right)^2 \cdot R(\frac{\pi}{\alpha} - 2 - \beta).$$

The condition (44) will be satisfied if

$$\beta \geq -1.$$  

Since (45) is the first approximation then we take the minimum value $\beta = -1$. The obtained first approximation of the solution is

$$u = \frac{\alpha}{\pi} \cdot R(\frac{\pi}{\alpha} - 1) \cdot r \cdot \cos \left( \frac{\pi}{\alpha} \cdot \varphi \right)$$

in the domain $\Omega : 0 \leq r \leq R, 0 \leq \varphi \leq \alpha$.

The solution (48) satisfies the boundary conditions (38), (39), (40), but it does not satisfy the condition (44). This solution is valid at $\pi < \alpha \leq 2 \cdot \pi$ and at $\alpha = \pi$ coincides with the exact solution (9).

### 6 Conclusion

The method of additional conditions was applied to the Neumann problem for Laplace equation in the wedge. The first approximation of nonsingular solution is obtained for the angles which are not obtuse. The obtained approximate solution coincides with the exact solution at the angle $\alpha = \pi$.

Therefore we can see that the method gives an interesting result for the classical Neumann problem and could be applied to the other classical problems such as Neumann problem for Helmholtz equation and to the similar equations. We see also that in this nonsingular case we need more information near the origin to create the nonsingular solution more precisely.

**References:**


