Stress analysis of reinforced axisymmetrical shells using substructures

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Abstract: A computational efficiency method is presented to study the structure of a rocket fuel tank subjected to an internal pressure. The complete structure is divided into a number of substructures that can be analyzed separately by the displacement method. Finally, using the simultaneous relaxation of the substructures boundary, we find the actual displacements and hence, the values of stresses in the tank structure. A numerical example proves the advantages of this proposed method based on the techniques of Finite Element Method (FEM).

Keywords: Finite Elements, FEM, Applications of FEM

1 Introduction

Space research using small satellites involves the design of rockets, which make possible their launch into orbit. Rocket has two tanks: one fuel and one oxygen and together form the mass of rocket propulsion.

In this paper we present a Finite Element Method (MEF) based on substructures technique, [8], for studying the stress state from the fuel tank. First is defined a discrete model equivalent to the actual continuous structure that was considered fixed in the lower ring to the engine support of rocket.

When the structure is complex is much more advantageous for its analysis to use its partitioning into a number of substructures, the boundaries of which may be specified arbitrary. It is preferable that this structural partitioning to correspond to the physical partitioning.

Each substructure is studied separately, assuming that all common boundaries with the adjacent substructures are fixed. Then, these boundaries are relaxed simultaneously and the actual boundary displacements are determined from the equations of equilibrium of forces at the boundary joints. Evidently, the solution of the boundary displacements involves a considerably smaller number of unknowns compared with the solution for the complete structure without partitioning. Also, the shape and composition of the substructures, which in our case are identical, recommend using the technique of structural dividing.

A finite element program was written in MatLab language, with this we determine the stresses in the tank structure and then, the stability of structure is studied.

2 Modelling and formulation

In accordance with the Finite Element Method, the system of equilibrium equations for the tank structure regarded as a free body, may be written in matrix form

\[ KU = P \]  

where \( K \) is the stiffness matrix, \( U \) represents the column matrix of displacements and \( P \) the external forces. If we eliminate the rigid body displacements (zero displacements from section, where the tank is fixed to the engine support), then the matrix \( K \) is nonsingular and (1) can be solved for the unknown displacements \( U \).

In the following discussion, the structure is divided into substructures by introducing interior boundaries. The column matrix of boundary displacements, common to two substructures, is denoted by \( U_b \), and the matrix of interior displacements (each of which appears at an interior point of only one substructure) is \( U_i \). If the corresponding external forces are denoted by matrices \( P_b \) and \( P_i \), then (1) may be written in partitioned form as

\[
\begin{bmatrix}
K_{ii} & K_{ib} \\
K_{bi} & K_{bb}
\end{bmatrix}
\begin{bmatrix}
U_i \\
U_b
\end{bmatrix}
=
\begin{bmatrix}
P_i \\
P_b
\end{bmatrix}
\]  

It will now be assumed that the total displacements of structure may be calculated from the superposition of two matrices such that
\[ U = U^{(\alpha)} + U^{(\beta)} \]  

(3)

where \( U^{(\alpha)} \) denoted the column matrix of displacements due to \( P_i \) with \( U_b = 0 \), while \( U^{(\beta)} \) represents the necessary corrections due to the displacements \( U^{(\alpha)} \) to allow for boundary displacements \( U_b \) when \( P_i = 0 \). Equation (3) may also be written as

\[ U = [U_i]_{\text{boundary}} + [U_i]_{\text{fixed}} + [U_i]_{\text{corrections due to boundary relaxation}} \]  

(4)

where by definition

\[ U^{(\alpha)} = 0 \]  

(5)

Similarly, corresponding to the displacements \( U^{(\alpha)} \) and \( U^{(\beta)} \), the external forces \( P \) can be separated into

\[ P = P^{(\alpha)} + P^{(\beta)} \]  

(6)

or

\[ P = \left[ \begin{array}{c} P_i \\ P_b \end{array} \right] = \left[ \begin{array}{c} P^{(\alpha)}_i \\ P^{(\beta)}_i \end{array} \right] + \left[ \begin{array}{c} P^{(\beta)}_i \\ P^{(\beta)}_b \end{array} \right] \]  

(7)

where by definition

\[ P^{(\alpha)}_i = P_i, \quad P^{(\beta)}_i = 0 \]  

(8)

When the substructure boundaries are fixed, it can readily be shown using (2), that

\[ U^{(\alpha)}_i = K_{ii}^{-1} P_i \]  

(9)

and

\[ P^{(\alpha)}_b = K_{bi} K_{ii}^{-1} P_i = R_b \]  

(10)

It should be noted that \( P^{(\alpha)}_b \) represents boundary reactions necessary to maintain \( U_b = 0 \) when the interior forces \( P_i \) are applied.

When the substructure boundaries are relaxed, the displacements \( U^{(\beta)}_i \) can be determined also from (2), so that

\[ U^{(\beta)}_i = -K_{ii}^{-1} K_{bi} U^{(\beta)}_b \]  

(11)

\[ U^{(\beta)}_b = K_{bb}^{-1} P^{(\beta)}_b \]  

(12)

where

\[ K_b = K_{bb} - K_{bi} K_{ii}^{-1} K_{ib} \]  

(13)

represents the boundary stiffness matrix. The matrix \( P^{(\beta)}_b \) can be determined from (7) and (11), hence

\[ P^{(\beta)}_b = P_b - P^{(\alpha)}_b = P_b - K_{bi} K_{ii}^{-1} P_i = S_b \]  

(14)

When the boundary displacements are set equal to zero, the substructures are completely isolated from one another so that application of an interior force causes displacements in only one substructure. Therefore, it is evident that the interior displacements \( U^{(\alpha)}_i \) with boundary fixed can be calculated for each substructure separately, using (9). Although the determination of boundary displacements \( U^{(\beta)}_b \) involves the complete structure, the boundary stiffness matrix \( K_b \) may be calculated by superposition of component matrices. Considerable computational advantage is derived from the fact that \( K_b \) is of much lower order than the complete stiffness matrix \( K \).

### 4 External forces

During the flight, the rocket structure is subjected to an internal pressure that is the sum of the work pressure, \( p_w \), hydrostatic pressure, \( p_h \) and dynamic pressure, \( p_d \). Work pressure is pumping gas pressure that in our example the value \( p_w = 6 \) atmospheres (1 atm ≈ 1 daN/cm²) and hydrostatic pressure is due to pressure of the liquid column above the section in which it is calculated:

\[ p_h = 0.1 \gamma H \text{ atm} \]  

(15)

where \( H \) (meters) is the height of the column of liquid and \( \gamma = \rho \cdot g \) is its specific weight, \( \rho \) is the density (1Kg/dm³), \( g = 9.8 \text{ m/s}^2 \) – gravitational acceleration. For the period of flight in which the engines are in operation, the dynamic pressure is

\[ p_d = 0.1 \beta \gamma H \]  

(16)

where \( \beta \) is the overload factor, \( \beta = a/g \), \( a \) – rocket acceleration and \( g \) – gravitational acceleration.

For vertical flight of rocket, the total internal pressure is equal to

\[ p = p_w + p_h + p_d \]  

(17)

### 3 Substructure analysis by displacement method (boundaries fixed)

Let us divide the structure of tank into six substructures: AB, BC, CD, DE, EF and FG, where the sections A, B,...,F are chosen so that they to correspond to the stiffening rings. Last section G corresponds to the ring that fixes the tank to the engine support.
According to studies of cylindrical shells, the effect of edge fixing spreads over a portion of the cylinder of length $2.7\sqrt{Rt}$, where $t$ is the shell thickness and $R$ the radius of cylinder, [2].

Each substructure is composed from a thin cylindrical shell that is provided in the front with a stiffening ring. The ring of the section A ensures the junction of the shell with the front lid of tank, Fig. 2a.

To reduce the edge effect for the substructure AB jointed to a spherical lid, the transition at the junction is made gradual: cylinder – torus – sphere. In this case the bending stress is considerably reduced and does not differ greatly from the stress given by the membrane theory:

$$\sigma_x = \frac{pR}{2t}$$

According to the calculations presented in the paper [2], we get now

$$\sigma_x^{\text{torus}} = 0.145 \frac{pR}{t} = 0.145 \frac{24}{6.5} = 0.53 \frac{pR}{t}$$

In our analysis, we will replace the presence of front lid by the force on the ring A (Fig. 2 b):

$$F_x = 2\pi R t \sigma_x^{\text{torus}}$$

Therefore, all substructures will be now of the same form: cylindrical shell and a front ring that carries a part of the hoop stresses (Fig. 1). To keep tank circular section was necessary to choose 12 points along the circumference. First five substructures will be composed of twelve rectangular plate elements and twelve pin-jointed bars. Instead, for the substructure FG we have to add and the elements of the section G ring.

In the first step of the substructure analysis is determined the stiffness matrix for each substructure in the datum coordinate system.

In the second step, using the techniques of Finite Element Method (FEM), these matrices will be assembled to determine the stiffness matrix for complete substructure.

For the substructure AB, the nodes of section A will be considered internal nodes. These are the points of application of forces

$$P_{iA}^{\text{node}} = F_x/12$$

and here we calculate the displacements $U_i$ below.

The nodes that are in section B, which is common with the substructure BC, will be called boundary nodes. The interior displacements of the nodes that are in section A and boundary reactions from the section B due to $P_{iA}$ can be determined from the equations (9) and (10):

$$U_i^A = (K_{ii}^{(1)})^{-1} P_{iA}$$

$$R_b^B = K_{bi}^{(1)} (K_{ii}^{(1)})^{-1} P_{iA}$$

where $K^{(1)}$ is the stiffness matrix of the substructure.
AB in the datum coordinate system. For the next five substructures there are no interior nodes, hence

\[ K_b^{(m)} = K^{(m)}, \quad m = 2,3,4,5,6. \]  

(24)

the notation \((m)\) identifying the substructure according to their sequence \((K^{(2)})\) substructure BC and so on.

5 General solution for displacements  
(substructure relaxation)

Having determined the boundary stiffness matrices \(K_b^{(m)}\) and the reactions matrix \(R_b^{(1)}\) due to \(P_i^A\), we relax all boundaries simultaneously with the exception of the section G, where the complete structure is fixed. When the boundaries are relaxed, the reactions and the external forces applied on the boundaries will not be in balance. Therefore the boundary relaxation will induce boundary displacements of such magnitude as to satisfy equilibrium at each joint on the boundary and the complete structure can be regarded as an assembly of substructures (Fig. 3) subjected to external loading:

\[ \bar{S}_b = \bar{P}_b - \sum_m R_b^{(m)} \]  

(25)

where the summation implies the addition of the corresponding reactions obtained in the first step (for boundaries fixed), while \(\bar{P}_b\) is the loading matrix for external forces applied on the boundaries in the datum system. In our case the value of \(m\) in (25) is equal to 1 (corresponds to the first substructure AB).

The remaining sections are loaded only by the forces due to the total internal pressure \(p\) and the negative sign for \(R_b^{(m)}\) changes the boundary reactions into externally applied forces.

The equations of equilibrium in terms of boundary displacements for the complete structure can now be written as

\[ K_b U_b = \bar{S}_b \]  

(26)

where \(K_b\) is obtained by assembling the matrices \(K_b^{(m)}\).

Since the substructures have common boundaries, the assembly process will lead to a band matrix, whose formation we present below.

Let us consider a boundary stiffness matrix \(K_b^{(m)}\) defined in (24) that we now write as

\[ K_b^{(m)} = \begin{bmatrix} k_{11}^{(m)} & k_{12}^{(m)} \\ k_{21}^{(m)} & k_{22}^{(m)} \end{bmatrix} \]  

(27)

The dimension of each submatrix \(k_{jl}^{(m)}\) is \((36,36)\) and this corresponds with the three displacements of the twelve nodes that are in a border section. Placing the matrices \(K_b^{(m)}\) in their correct position in the larger framework of the boundary stiffness matrix \(K_b\) for complete structure and summing all the overlapping terms, we get

\[ K_b = \begin{bmatrix} k_{bb} + k_{11}^{(2)} & k_{12}^{(2)} & 0 & 0 & 0 \\ k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} & 0 & 0 & 0 \\ 0 & k_{22}^{(3)} + k_{11}^{(4)} & k_{12}^{(4)} & 0 & 0 \\ 0 & 0 & k_{22}^{(4)} + k_{11}^{(5)} & k_{12}^{(5)} & 0 \\ 0 & 0 & 0 & k_{22}^{(5)} + k_{11}^{(6)} & k_{12}^{(6)} \end{bmatrix} \]  

(28)

Amounts of the above depend on the sequence of the individual boundary displacements. We remind that the substructure AB contains the interior nodes and the boundary nodes, so that its stiffness matrix is of the form

\[ K^{(1)} = \begin{bmatrix} K_{ii} & K_{ib} \\ K_{bi} & K_{bb} \end{bmatrix} \]  

(29)

Since some of substructures will not be physically connected, this means that their coupling stiffness matrices will be equal to zero.
Elimination of the nodes displacements that exist in the fixing section G leads to the disappearance of the matrix $k_{22}^{(6)}$ in (28). Therefore the matrix $K_b$ is nonsingular and the boundary displacements $U_b$ can be determined from

$$U_b = K_b^{-1} S_b$$  \hspace{1cm} (30)

With the transformation matrices from the MEF theory will be calculated the nodes displacements for the elements in their local system of coordinates and finally, the stresses in the shell and the stiffening rings. These values are compared with the critical buckling stress and will determine the safety margins for each element.

### 6 Numerical results

The distance between the rings of the tank is presented in the Table 1, where $L$ is the distance in centimeter between the rings. The shell thickness is constant and equal to 0.3 cm, radius of tank is 24 cm and the cross-sectional area of the rings varies from 2.5 cm$^2$ to 5 cm$^2$ depending on the distance between them. The material of the tank is Al 2014-T4.

<table>
<thead>
<tr>
<th>Table 1</th>
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<tbody>
<tr>
<td>$L_{AB}$</td>
</tr>
<tr>
<td>20</td>
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</table>

The load from B, C,..., E sections was calculated for $\beta_{max} = 34$, using the pressure value (17) multiplied by the surface area of shell centered in the considered section. On the ring A we will replaced the presence of the front lid by the force (20).

The values of forces (daN) in the sections are presented in the table:

<table>
<thead>
<tr>
<th>Table 2</th>
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<td>2205</td>
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</table>

Because of geometrical symmetry of the structure and loading, the meridional stress $\sigma_x$, circumferential stress $\sigma_y$ and shear stress $\tau$ will be the same in all twelve panels of a substructure. The figures 4, 5 and 6 show the distribution of these stresses along the tank. The stresses $\sigma_x$, $\sigma_y$ and $\tau$ are noted with sxx, sy and sxy respectively. Their values are presented in the table 3.

<table>
<thead>
<tr>
<th>Table 3</th>
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<tbody>
<tr>
<td>Substr.</td>
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<tr>
<td>$\sigma_x$ (daN/cm$^2$)</td>
</tr>
<tr>
<td>$\sigma_y$ (daN/cm$^2$)</td>
</tr>
<tr>
<td>$\tau$ (daN/cm$^2$)</td>
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</tbody>
</table>
7 Stability study of the shell panels

For aluminum alloy the yield strength \( \sigma_{0.2} = 2900 \text{ daN/cm}^2 \) and the ultimate tensile strength \( \sigma_r = 4250 \text{ daN/cm}^2 \). The stresses presented in table 3 are compared with the appropriate critical buckling stress and then we will determine the safety margins. This study will be done in the area between sections D and E, because there are maximum the meridional and circumferential stresses.

To compare the complex loading state with the tensile state, an equivalent stress is calculated with the formula recommended for the cylindrical panels, [2]:

\[
\sigma_e = \sqrt{\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau^2}
\] (30)

In view of the formula (30) and using the stress values from Table 3, we find for the panels CD and DE

\[
\sigma_e^{CD} = 2950 \text{ daN/cm}^2, \quad \sigma_e^{DE} = 3456 \text{ daN/cm}^2
\]

Therefore, the safety margin for selected panels is respectively equal with

\[
MS_{CD} = \frac{\sigma_{0.2}}{\sigma_e^{CD}} - 1 = \frac{2900}{2950} - 1 = -0.017
\]

\[
MS_{DE} = \frac{\sigma_{0.2}}{\sigma_e^{DE}} - 1 = \frac{2900}{3456} - 1 = -0.17.
\]

These negative values require that the designer to introduce two rings between sections C and E of the tank, which will replace the current ring D.

7 Conclusions

The substructure relaxation method was used to analyze the structure of fuel tank for its critical loading. This modeling technique is very effective if the form of substructures is the same, because in this case a single algorithm is necessary to obtain their stiffness matrix. To obtain such model it was replaced the front lid by the force \( F \) defined by (20).

Substructures are also useful when there is not sufficient computer power to handle the whole model. In our case, the matrix \( K_b \) that has the size (180, 180) replaces the stiffness matrix \( K \) for complete structure that has the size (432, 432). Furthermore, \( K_b \) is a band matrix, such that the inversion program use a reduced time.

This paper proves that finite element analysis for a complex structures, which have large size, can be performed not only with the programs of a software company that involve a large amount of calculations, but also by a program written in Math-Lab language.

References:

[10] Parton V., Perlin P., Mathematical Methods of Theory of Elasticity, Mir Publisher Moscow, 1984