On a Scale Invariant Model of Statistical Mechanics and the Kinetic Theory of Ideal Gas

SIAVASH H. SOHRAB
Robert McCormick School of Engineering and Applied Science
Department of Mechanical Engineering
Northwestern University, Evanston, Illinois 60208
s-sohrab@northwestern.edu
http://www.mech.northwestern.edu/web/people/faculty/sohrab.php

Abstract: - A scale invariant model of statistical mechanics is applied to derive invariant Maxwell-Boltzmann speed and Planck energy spectrum of equilibrium statistical fields including that of isotropic stationary turbulence. The latter is shown to lead to the definitions of (electron, photon, neutrino) respectively as the most-probable equilibrium sizes of (photon, neutrino, tachyon) clusters. Also, invariant modified form of the first law of thermodynamics is derived and a modified definition of entropy, invariant forms of transport coefficients, and hierarchies of absolute zero temperatures and vacua are described.

The physical basis for the coincidence of normalized spacings between zeros of Riemann zeta function and the normalized Maxwell-Boltzmann distribution and its connections to Riemann Hypothesis are examined. New paradigms for hydrodynamic foundations of both Schrödinger as well as Dirac wave equations are discussed.

Key-Words: - Kinetic theory of ideal gas; Thermodynamics; Statistical mechanics; Riemann Hypothesis; TOE.

1 Introduction

It is well known that the methods of statistical mechanics can be applied to describe physical phenomena over a broad range of scales of space and time from the exceedingly large scale of cosmology to the minute scale of quantum optics (Fig.1). All that is needed is that the system should contain a large number of weakly coupled particles. The similarities between stochastic quantum fields [1-17] and classical hydrodynamic fields [18-29] resulted in recent introduction of a scale-invariant model of statistical mechanics [30], and its application to thermodynamics [31], fluid mechanics [32-34], and quantum mechanics [35].

In the present study the invariant model of statistical mechanics and its implications to the physical foundations of thermodynamics, kinetic theory of ideal gas [36-42], and quantum mechanics are examined. Invariant Planck energy and Normalized Maxwell-Boltzmann speed distribution functions are derived and connection of the latter to Montgomery-Odlyzko law and Riemann Hypothesis is examined. The impact of Poincaré recurrence theory [43] on the problem of irreversibility in thermodynamics is discussed. The invariant forms of conservation equations based on linearization of Boltzmann equation and in harmony with Enskog [44] method are presented. Finally, hydrodynamic foundations of the invariant forms of both Schrödinger as well as Dirac wave equations are discussed.

2 A Scale Invariant Model of Statistical Mechanics

Following the classical methods [44-49] the invariant definitions of density $\rho_\beta$ and velocity of atom $u_\beta$, element $v_\beta$, and system $w_\beta$ at the scale $\beta$ are given as

$$\rho_\beta = n_p m_p = m_p \int f_p u_p \, du_p, \quad u_\beta = v_{mp\beta-1}$$

$$v_\beta = \rho_\beta m_p \int u_p f_p du_p, \quad w_\beta = v_{mp\beta+1}$$

Similarly, the invariant definition of the peculiar and diffusion velocities are introduced as

$$v'_\beta = u_\beta - v_\beta, \quad v_\beta = v_\beta - w_\beta$$

such that

$$v_\beta = v'_{\beta+1}$$

For each statistical field, one defines particles that form the background fluid and are viewed as point-mass or "atom" of the field. Next, the elements of the field are defined as finite-sized composite entities each composed of an ensemble of "atoms" as shown in Fig.1. According to (1)-(2) the atomic and system velocities of scale $\beta$ ($u_\beta, w_\beta$) are the most-probable speeds of the lower and upper adjacent scales ($v_{mp\beta-1}, v_{mp\beta+1}$) as shown in Fig.12. Finally, the ensemble of a large number of "elements" is defined as the statistical "system" at that particular scale.
rate, is the absolute enthalpy \[ h^i_{\beta} \] for chemical species \( i \) [32]
\[
\bar{h}_i^\beta = \int_0^T \tilde{c}_i p_i^\beta dT^\beta
\]  
(12)
where \( T^\beta \) is the standard temperature. The definition (12) helps to avoid the conventional practice of arbitrarily setting the standard heat of formation of naturally occurring species equal to zero. Furthermore, following \textit{Nernst-Planck} statement of the third law of thermodynamics one has \( \bar{h}_\beta \to 0 \) in the limit \( T^\beta \to 0 \) as expected.

The classical definition of vorticity involves the curl of linear velocity \( \nabla \times \mathbf{v} = \mathbf{\omega} \) thus giving rotational velocity of particle a secondary status in that it depends on its translational velocity \( \mathbf{v}_{\beta} \). However, it is known that particle’s rotation about its center of mass is independent of the translational motion of its center of mass. In other words, translational, rotational, and vibrational (pulsational) motions of particle are independent degrees of freedom that should not be necessarily coupled. To resolve this paradox, the iso-spin of particle at scale \( \beta \) is defined as the curl of the velocity at the next lower scale of \( \beta - 1 \) [51]
\[
\mathbf{\omega}_{\beta} = \nabla \times \mathbf{v}_{\beta-1} = \mathbf{\omega}_{\beta-1} = \nabla \times \mathbf{u}_\beta
\]  
(13)
such that the rotational velocity, while having a connection to some type of translational motion at internal scale \( \beta - 1 \), retains its independent degree of freedom at the external scale \( \beta \) as desired. A schematic description of iso-spin and vorticity fields is shown in Fig.2. The nature of galactic vortices in cosmology and the associated dissipation have been discussed [25, 52].

The local velocity \( \mathbf{v}_\beta \) in (5)-(8) is expressed in terms of the convective \( \mathbf{w}_\beta \) and the diffusive \( \mathbf{V}_\beta \) velocities [32]
where \((V_{\beta\beta}, V_{\beta\phi}, V_{\phi\beta}, V_{\phi\phi})\) are respectively the diffusive, the thermo-diffusive, the translational and rotational hydro-diffusive velocities.

Because by definition fluids can only support compressive normal forces, following Cauchy the total stress tensor for fluids is expressed as [32]

\[ P_{ij\beta} = p_{i\beta} \delta_{ij} - \lambda_{i\beta} \nabla_{i} \cdot \delta_{ij} - 2\mu_{i\beta} e_{i\beta} \delta_{ij} \]

Making the conventional Stokes assumption, i.e. setting the bulk viscosity \(b_{\beta}\) to zero, the two Lame constants will be related by [53]

\[ b_{\beta} = \lambda_{\beta} + \frac{2}{3} \mu_{\beta} = 0 \]

and (15) reduces to [32]

\[ P_{ij\beta} = p_{i\beta} \delta_{ij} - \frac{1}{3} \mu_{i\beta} \nabla_{i} \cdot \delta_{ij} = (p_{i\beta} + p_{\phi\phi}) \delta_{ij} \]

that involves thermodynamic \(p_{i\beta}\) and hydrodynamic \(p_{\phi\phi}\) pressures.

The expression for hydrodynamic pressure in (17) could also be arrived at directly by first noting that classically hydrodynamic pressure is defined as the mean normal stress

\[ p_{i\beta} = \frac{1}{3}(\tau_{xix} + \tau_{yiy} + \tau_{ziz}) \]

since shear stresses in liquids vanish by definition. Next, normal stresses are expressed as diffusional flux of the corresponding momenta by (14c) as

\[ \tau_{ij} = \rho v_{i} \nabla_{j\beta} = \frac{1}{2} \mu_{i\beta} \nabla_{i} \cdot \delta_{ij} \]

Substituting from (19) into (18) results in

\[ p_{i\beta} = \frac{1}{3}(\tau_{xix} + \tau_{yiy} + \tau_{ziz}) = \frac{1}{3} \mu_{i\beta} \nabla_{i} \cdot \delta_{ij} \]

that is in accordance with (17). The occurrence of a single rather than two Lame constants in (17) is also in harmony with the perceptions of Cauchy and Poisson who both assumed the limit of zero for the expression [54]

\[ \lambda_{\beta} + \mu_{\beta} = \lim_{R \to 0} R^4 f(R) \]
The central question concerning Cauchy equation of motion (7) say at $\beta = m$ is how many “molecules” are included in the definition of the mean molecular velocity $v_m = \langle u_m \rangle = u_m$. One can identify three distinguishable cases: (1) When $v_m = u_m$ is itself random then all three velocities $(u_m, v_m, w_m)$ in (3) are random in a stochastically stationary field made of ensembles of clusters and molecules with Brownian motions and hence Gaussian velocity distribution, Planck energy distribution, and Maxwell-Boltzmann speed distribution. If in addition both the vorticity $\omega_m = \nabla \times v_m = 0$ as well as the iso-spin (13) $\omega_m = \nabla \times u_m = 0$ are zero, then for an incompressible flow the continuity equation (5) and Cauchy equation of motion (7) lead to Bernoulli equation. In Section 11 it will be shown that under the above mentioned conditions Schrödinger equation (206) can be directly derived from Bernoulli equation (201) such that the energy spectrum of the equilibrium field will be governed by quantum mechanics and hence by Planck law. (2) When $v_m = u_m$ is not random but the vorticity vanishes $\omega_m = 0$ the flow is irrotational and ideal, inviscid $\mu_m = 0$, and once again one obtains Bernoulli equation from (5) and (7) with the solution given by the classical potential flow. (3) When $v_m = u_m$ is not random and vorticity does not vanish $\nabla \times v_m \neq 0$, the rotational non-ideal viscous $\mu_m \neq 0$ flow will be governed by the equation of motion (24) with the convection velocity $w_m = v_m$, obtained from the solution of potential flow at the next larger scale of $\beta+1$. In Section 11 it will be shown that the viscous equation of motion (24) is associated with Dirac relativistic wave equation. In the sequel, amongst the three cases of flow conditions discussed in [35] only cases (1) and (3) will be examined.

4 Hierarchies of Embedded Statistical Fields

The invariant model of statistical mechanics (1)-(4) suggests that all statistical fields shown in Fig.1 are turbulent and governed by (5)-(8) [33, 34]. First, let us start with the field of laminar molecular dynamics LMD when molecules, clusters of molecules (cluster), and cluster of clusters of molecules (eddy) form the “atom”, the “element”, and the “system” with the velocities $(u_m, v_m, w_m)$. Similarly, the fields of laminar cluster-dynamics LCD and eddy-dynamics LED will have the velocities $(u_e, v_e, w_e)$, and $(u_s, v_s, w_s)$ in accordance with (1)-(2). For the fields of LED, LCD, and LMD, typical characteristic “atom”, element, and system lengths are [50]

$$EED \quad (\ell_\gamma, \lambda_e, L_e) = (10^{-2}, 10^{-3}, 10^{-1}) \, m \quad (27a)$$
$$ECD \quad (\ell_\gamma, \lambda_e, L_e) = (10^{-2}, 10^{-3}, 10^{-2}) \, m \quad (27b)$$
$$EMD \quad (\ell_m, \lambda_m, L_m) = (10^{-9}, 10^{-7}, 10^{-5}) \, m \quad (27c)$$

If one applies the same (atom, element, system) = $(\ell_\gamma, \lambda_p, L_p)$ relative sizes in (27) to the entire spatial scale of Fig.1, then the resulting cascades or hierarchy of overlapping statistical fields will appear as schematically shown in Fig.3.

![Fig.3 Hierarchy of statistical fields with $(\ell_\gamma, \lambda_p, L_p)$ from cosmic to Planck scales [34].](image)

According to Fig.3, starting from the hydrodynamic scale $(10^3, 10^3, 10^{-1})$ after seven generations of statistical fields one reaches the electro-dynamic scale with the element size $10^{-17}$, and exactly after seven more generations one reaches Planck length scale $(\hbar G/c^2)^{1/2} \approx 10^{-35} \, m$, where $G$ is the gravitational constant. Similarly, seven generations of statistical fields separate the hydrodynamic scale $(10^3, 10^3, 10^{-1})$ from the scale of galactic-dynamics (cosmology) $10^{23} \, m$.

The left hand side of Fig.1 corresponds to equilibrium statistical fields when the velocities of elements of the field are random since at thermodynamic equilibrium particles i.e. oscillators
of such statistical fields will have normal or Gaussian velocity distribution. For example, for stationary homogeneous isotropic turbulence at EED scale, the experimental data of Townsend [58] confirms the Gaussian velocity distribution of eddies as shown in Fig.4.

Fig.4 Measured velocity distribution in isotropic turbulent flow [58].

The evidence for the existence of the statistical field of equilibrium cluster-dynamics ECD (Fig.1) is the phenomena of Brownian motions [25, 59-65]. Modern theory of Brownian motion starts with the Langevin equation [25]

$$\frac{du_p}{dt} = -\beta u_p + A(t)$$ (28)

where \(u_p\) is the particle velocity. The drastic nature of the assumptions inherent in the division of forces in (28) was emphasized by Chandrasekhar [25].

To account for the stationary nature of Brownian motions fluid fluctuations at scales much larger than molecular scales are needed as noted by Gouy [59]. Observations have shown that as the size of the particles decrease their movement become faster [59]. According to classical arguments Brownian motions are induced by multiple collisions of a large number of molecules with individual suspended particle. However, since the typical size of particle is about 100 times larger than that of individual molecules, such collisions preferentially from one side of the particle could not occur in view of the assumed Maxwell-Boltzmann distribution of molecular motions. On the other hand, if one assumes that Brownian motions are induced by collisions of particles with groups, i.e. clusters, of molecules then in view of the stationary nature of Brownian motions, the motions of such clusters themselves must also be governed by the Maxwell-Boltzmann distribution. But this would mean the existence of the statistical field of equilibrium cluster dynamics.

Because at thermodynamic equilibrium the mean velocity of each particle, Heisenberg-Kramers virtual oscillator [66], vanishes \(<u_p> = 0\) the translational kinetic energy of particle oscillating in two directions (x+, x−) is expressed as

$$\varepsilon_p = m_p \langle u_{px+}^2 \rangle / 2 + m_p \langle u_{px-}^2 \rangle / 2$$

$$= m_p \langle u_{px+}^2 \rangle = \langle p_p \rangle \langle \lambda_{p+}^2 \rangle / 2 \langle v_p^2 \rangle / 2$$ (29)

where \(m_p <u_{px+}^2>^1/2 = \langle p_p\rangle\) is the root-mean-square momentum of particle and \(<u_{px-}^2> = \langle p_p\rangle\) by Boltzmann equipartition principle. At any scale \(\beta\), the result (29) can be expressed in terms of either frequency or wavelength

$$\varepsilon_p = m_p \langle u_{p+}^2 \rangle = \langle p_p \rangle \langle \lambda_{p+}^2 \rangle / 2 \langle v_{p+}^2 \rangle / 2 = h_p \lambda_p$$ (30a)

$$\varepsilon_p = m_p \langle u_{p-}^2 \rangle = \langle p_p \rangle \langle \lambda_{p-}^2 \rangle / 2 \langle v_{p-}^2 \rangle / 2 = k_p \lambda_p$$ (30b)

when the definition of stochastic Planck and Boltzmann factors are introduced as [33]

$$h_p = \langle p_p \rangle \langle \lambda_{p+} \rangle / 2$$ (31a)

$$k_p = \langle p_p \rangle \langle v_{p+} \rangle / 2$$ (31b)

At the important scale of EKD (Fig.1) corresponding to Casimir vacuum [67] composed of photon gas, the universal constants of Planck [68, 69] and Boltzmann [31] are identified from (30)-(31) as

$$h = h = m_c c \langle \lambda_{k+} \rangle / 2 = 6.626 \times 10^{-34} \text{ J-s}$$ (32)

$$k = k = m_c c \langle v_{k+} \rangle / 2 = 1.381 \times 10^{-23} \text{ J/K}$$ (33)

Next, following de Broglie hypothesis for the wavelength of matter waves [2]

$$\lambda_p = h / p_p$$ (34)

the frequency of matter waves is defined as [31]

$$v_p = k / p_p$$ (35)

When matter and radiation are in the state of thermodynamic equilibrium (35) and (36) can be expressed as

$$h_p = h = h, \quad k_p = k = k$$ (36)

The definitions (34) and (35) result in the gravitational mass of photon [31]

$$m_k = (h k / c^3) / 2 = 1.84278 \times 10^{-41} \text{ kg}$$ (37)

that is much larger than the reported value of \(4 \times 10^{-51} \text{ kg}\) [70]. The finite gravitational mass of photons was anticipated by Newton [71] and is in accordance with Einstein-de Broglie theory of light [72-76]. Avogardo-Loschmidt number was predicted as [31]
\[ N^o = 1/(m_c c^2) = 6.0376 \times 10^{-23} \]  
(38)

leading to the modified value of the universal gas constant

\[ R^o = N^o k = 8.338 \ \text{kJ/(kmol-K)} \]  
(39)

Also, the atomic mass unit is obtained from (39) as

\[ \text{amu} = m_c c^2 = (hkc)^{1/2} = 1.6563 \times 10^{-27} \ \text{kg/kmol} \]  
(40)

Since all baryonic matter is known to be composed of atoms, the results in (38) and (40) suggest that all matter in the universe is composed of light [77].

From (32)-(33) the wavelength and frequency of photon in vacuum \( \langle \lambda_k^2 \rangle^{1/2} \langle \nu_k^2 \rangle^{1/2} = c \) are

\[ \lambda_k = \langle \lambda_k^2 \rangle^{1/2} = 1/R^o = 0.119935 \ \text{m} \],  
\[ \nu_k = \langle \nu_k^2 \rangle^{1/2} = 2.49969 \times 10^9 \ \text{Hz} \]  
(41)

The classical definition of thermodynamic temperature based on two degrees of freedom

\[ 2kT = m_v^2_{\text{mp}} = 2m_v^2_{\text{mpx}} = 4m_v^2_{\text{mpx}} \]  
(42)

was recently modified to a new definition based on a single degree of freedom [77]

\[ 2kT = m_v^2_{\text{mpx}} = 2m_v^2_{\text{mpx}} + \]  
(43)

such that

\[ T' = 2T \]  
(44)

The factor 2 in (44) results in the predicted speed of sound in air [78]

\[ a = v_{\text{rmsx}} = \sqrt{p/(2\rho)} \]

\[ = \sqrt{3kT'/2m} = \sqrt{3kT/m} = 357 \text{ m/s} \]  
(45)

in close agreement with observations. Also, (45) leads to calculated r.m.s molecular speeds (1346, 336, 360, 300, 952, 287) m/s that are in reasonable agreement with the observed velocities of sound (1286, 332, 337, 308, 972, 268) m/s in (H₂, O₂, N₂, Ar, He, CO₂) [78].

The square root of 2 in (45) resolves the classical problem of Newton concerning his prediction of velocity of sound as

\[ a = \sqrt{p/\rho} \]  
(46)

discussed by Chandrasekhar [79]

“Newton must have been baffled, not to say disappointed. Search as he might, he could find no flaw in his theoretical framework—neither could Euler, Lagrange, and Laplace; nor, indeed, anyone down to the present”

Indeed, the expression introduced by Euler \[ p = \rho v^2/3 \], Lagrange \[ p \propto \rho^{4/3} \] as well as Laplace’s assumption of isentropic relation \( p = \chi \rho^\gamma \), where \( \chi \) is a constant and \( \gamma = c_p/c_v \), that leads to the conventional expression for the speed of sound in ideal gas

\[ a = \sqrt{\gamma RT} \]  
(47)

are all found to deviate from the experimental data [39].

The factor of 2 in (44) also leads to the modified value of Joules-Mayer mechanical equivalent of heat J [77]

\[ J = 2J_c = 2 \times 4.169 = 8338 \ \text{Joules/(kcal)} \]  
(48)

where the value \( J_c = 4.169 = 4.17 \ [kJ/kcal] \) is the average of the two values \( J_c = (4.15, 4.19) \) reported by Pauli [80]. The number in (48) is thus identified as the universal gas constant (39) when expressed in appropriate MKS system of units

\[ R^o = kN^o = J = 8338 \ \text{Joules/(kmol-K)} \]  
(49)

The modified value of the universal gas constant (49) was recently identified [81] as De Pretto number 8338 that appeared in the mass–energy equivalence equation of De Pretto [82]

\[ E = mc^2 \ \text{Joules} = mc^2/8338 \ \text{kcal} \]  
(50)

Unfortunately, the name of Olinto De Pretto in the history of evolution of mass energy equivalence is little known. The relativistic form of (50) was first introduced in 1900 by Poincaré [83]

\[ E = m_v c^2 \]  
(51)

where \( m_v = m_o / \sqrt{1-v^2/c^2} \). Since the expression (50) is the only equation in the paper by De Pretto [82], the exact method by which he arrived at the number 8338 is not known even though one possible method was recently suggested [81]. The important contributions by Hasenöhrl [84] and Einstein [85] as well as the equivalence principle, equivalence of the rest or gravitational mass and the inertial mass were discussed in a recent study [77].

### 5 Invariant Boltzmann Distribution Function

The kinetic theory of gas as introduced by Maxwell [36] and generalized by Boltzmann [37, 38] is based on the nature of the molecular velocity distribution function that satisfies certain conditions of space isotropy and homogeneity and being stationary in time. However, in his work on generalization of Maxwell’s result Boltzmann introduced the important
concept of “complexions” and the associated combinatorics [41] that was subsequently used by Planck in his derivation of the equilibrium radiation spectrum [68, 69]. In the following the invariant model of statistical mechanics and Boltzmann’s combinatorics will be employed to arrive at the Maxwell-Boltzmann distribution function.

To better reveal the generality of the concepts, rather than the usual scale of molecular-dynamics, we consider the statistical field of equilibrium eddy-dynamics EED at the scale $\beta = e$. According to Fig.1, the homogenous isotropic turbulent field of EED is a hydrodynamic system composed of an ensemble of fluid elements

$$\text{System}_{\text{EED}} = \text{Hydro System} = h = \sum_k f_{jk}$$  \hspace{1cm} (52)

Next, each fluid element $f_k$ is an ensemble of a spectrum of eddies

$$\text{Element}_{\text{EED}} = \text{Fluid Element} = f_k = \sum_j e_{jk}$$  \hspace{1cm} (53)

Finally, an eddy or the “atom” of EED field is by definition (1) the most probable size of an ensemble of molecular clusters (Fig.1)

$$\text{Atom}_{\text{EED}} = \text{Eddy} = e_j = \sum c_{kj}$$  \hspace{1cm} (54)

At the lower scale of ECD each eddy of type $j$ will correspond to the energy level $j$ and is composed of ensemble of clusters or quantum states $c_{ij}$ within the energy level $j$. Cluster of type $c_{ij}$ does not refer to different cluster “specie” but rather to its different energy. The above procedure could then be extended to higher and lower scales within the hierarchy shown in Fig.1

$$\text{System}_p = \text{Element}_{p-1} = \sum_j \text{Elements}_j$$  \hspace{1cm} (55)

$$\text{Element}_p = \text{Atom}_{p-1} = \sum_j \text{Atom}_{pj}$$  \hspace{1cm} (56)

It is noted again that by (27) typical system size of (EED, ECD, EMD) scales are ($L_x, L_y, L_z$) = $(10^{-1}, 10^{-3}, 10^{-5})$ m.

Following Boltzmann [37, 38] and Planck [68] the number of complexions for distributing $N_j$ indistinguishable eddies among $g_j$ distinguishable cells or “quantum states” eddy-clusters of ECD scale is

$$W_j = \prod_j \frac{(N_j + g_j - 1)!}{N_j! (g_j - 1)!}$$  \hspace{1cm} (57)

The total number of complexions for independent energy levels $W_j$ is obtained from (57) as

$$W = \prod_j \frac{(N_j + g_j - 1)!}{N_j! (g_j - 1)!}$$  \hspace{1cm} (58)

As was discussed above, the hydrodynamic system is composed of $g$ distinguishable fluid elements that are identified as energy levels of EED system. Each fluid element is considered to be composed of eddy clusters made of indistinguishable eddies. However, the smallest cluster contains only a single eddy and is therefore considered to be full since no other eddy can be added to this smallest cluster. Because an empty cluster has no physical significance, the total number of available cells or quantum states will be $(g_j - 1)$. Therefore, Planck-Boltzmann formula (57) is the exact probability of distribution of $N_j$ indistinguishable oscillators (eddies) amongst $(g_j - 1)$ distinguishable available clusters. The invariant model of statistical mechanics (Fig.1) provides new perspectives on the probabilistic nature of (57) and the problem of distinguishability discussed by Darrigol [41].

Under the realistic assumptions

$$g_j \gg N_j, \quad N_j \gg 1$$  \hspace{1cm} (59)

it is known that the number of complexions for Bose-Einstein statistics (57) simplifies such that all three types namely Corrected Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac statistics will have [86]

$$W_j = g_j^{N_j} / N_j!$$  \hspace{1cm} (60)

The most probable distribution is obtained by maximization of (60) that by Stirling’s formula results in

$$\ln W_j = N_j \ln g_j - N_j \ln N_j + N_j$$  \hspace{1cm} (61)

and hence

$$d(\ln W_j) = dN_j \ln(g_j / N_j) = 0$$  \hspace{1cm} (62)

In the sequel it will be argued that at thermodynamic equilibrium because of the equipartition principle of Boltzmann the energy of all levels $U_j$ should be the same and equal to the most probable energy that defines the thermodynamic temperature such that

$$dU_j = \varepsilon_j dN_j + N_j d\varepsilon_j = 0$$  \hspace{1cm} (63)

or

$$dU_j = \varepsilon_j dN_j + \frac{N_j d\varepsilon_j}{dN_j} dN_j = \varepsilon_j dN_j + \frac{d(N_j \varepsilon_j)}{dN_j} dN_j = \varepsilon_j dN_j + \hat{\mu} dN_j$$  \hspace{1cm} (64)
where Gibbs chemical potential is defined as

$$\hat{\mu}_j = \left( \frac{\partial U_j}{\partial N_j} \right)_{S,V,N_{ij}} = \hat{g}_j = G_j / N_j$$  

(65)

Introducing the Lagrange multipliers $-\beta$ and $\alpha$ one obtains from (62) and (64)

$$\mathrm{d}N_j \{ \ln \left( g_j / N_j \right) - \beta (\epsilon_j - \alpha \hat{\mu}_j) \} = 0$$  

(66)

that leads to Boltzmann distribution

$$N_j = g_j e^{-\beta (\epsilon_j - \alpha \hat{\mu}_j)} = g_j e^{-(\epsilon_j - \alpha \hat{\mu}_j) / kT}$$  

(67)

It is noted that in the above derivation the constant Lagrange multiplier $\alpha$ is separate and distinct from the chemical potential $\hat{\mu}_j$ that is a variable as required. Hence, for bosons $\hat{\mu}_j = 0$ one can have $\alpha \hat{\mu}_j = 0$ without requiring $\alpha = 0$ corresponding to none conservation of number of photons. Following the classical methods [86-88] the first Lagrange multiplier becomes $\beta = 1 / kT$. In Section 10, it is shown that the second Lagrange multiplier will be $\alpha = 1$.

### 6 Invariant Planck Energy Distribution Function

In this section, the invariant Planck energy distribution law will be derived from the invariant Boltzmann statistics (67) introduced in the previous section. To obtain a correspondence between photon gas at EKD scale and the kinetic theory of ideal gas in statistical fields of other scales, by (29)-(30) particles with the energy $E_p = h_p \nu_p = \nu_\beta = m_p v_p^2$ are viewed as virtual oscillators [66] that act as composite bosons [89] and hence follow Bose-Einstein statistics. It is well known that the maximization of the thermodynamic probability given by Planck-Boltzmann formula (57) leads directly to Bose-Einstein distribution [86-88]

$$N_j = \frac{g_j}{e^{\epsilon_j / kT} - 1}$$  

(68)

However, since Boltzmann distribution (67) itself was derived by maximization of (57) as discussed in the previous section, it should also be possible to arrive at (68) directly from (67).

The analysis is first illustrated for the two consecutive equilibrium statistical fields of EED scale $\beta = e$ when (“atom”, cluster, system) are (eddy, fluid element, hydrodynamic system) identified by $(e, f, h)$ with indices $(j, k, \ell)$ and ECD at scale $\beta = c$ when (“atom”, element, system) are (molecule, cluster, eddy) identified by $(m, c, e)$ with indices $(m, i, j)$.

The statistical field of EED is a hydrodynamic system composed of a spectrum of fluid elements (energy levels) that are eddy clusters of various sizes as shown in Fig.1. For an ideal gas at constant equilibrium temperature internal energy $U_f$ will be constant and by (65) one sets $\hat{\mu}_j = 0$ and the number of fluid element of type $f$ (energy level $j$) in the hydrodynamic system from (67) becomes

$$N_{fh} = g_{fh} e^{-\epsilon_j / kT}$$  

(69)

Assuming that the degeneracy of all levels $f$ is identical to a constant average value $g_{fh} = \overline{g}_{fh}$, the average number of fluid elements in the hydrodynamic system $h$ from (69) becomes

$$\overline{N}_h = \sum_f g_{fh} e^{-\epsilon_j / kT} = \overline{g}_{fh} \sum_j e^{-\epsilon_j / kT} = \overline{g}_{fh} \sum_j \left( e^{-\epsilon_j / kT} \right)^{N_j} = \frac{\overline{g}_{fh}}{1 - e^{-\epsilon_j / kT}}$$  

(70)

In the derivation of (70) the relation

$$\epsilon_f = \sum_j \epsilon_{jf} = N_j \epsilon_{jf} = U_j$$  

(71)

for the internal energy of the fluid element $f$ has been employed that is based on the assumption that all eddies of energy level $f$ are at equilibrium and therefore stochastically stationary indistinguishable eddies with stationary size and energy.

At the next lower scale of ECD, the system is a fluid element composed of a spectrum of eddies that are energy levels of ECD field. Eddies themselves are composed of a spectrum of molecular clusters i.e. cluster of molecular-clusters hence super-cluster. Again, following the classical methods of Boltzmann [86-88], for an ideal gas at constant temperature hence $U_j$ by (65) $\hat{\mu}_j = 0$ and from (67) the number of eddies in the energy level $j$ within the fluid element $f$ becomes

$$N_{jf} = g_{jf} e^{-\epsilon_j / kT}$$  

(72)

The result (72) is based on the fact that all eddies of element $jf$ are considered to be indistinguishable with identical energy.
\[ \varepsilon_j = \sum_i \varepsilon_{ij} = N_i \varepsilon_{ij} = U_i \]  
(73)

that is in harmony with (71).

It is now possible to determine the distribution of eddies as Planck oscillators (Heisenberg-Kramers virtual oscillators) among various energy levels (fluid elements) with degeneracy under the constraint of a constant energy of all levels (63). From (70) and (72), the average number of eddies in the energy level \( f \) of hydrodynamic system can be expressed as

\[ N_j = \bar{N}_{jh} = \bar{N}_{hf} \bar{N}_{jf} = \frac{g_j}{e^{\varepsilon_j/kT} - 1} \]  
(74)

that is the Bose-Einstein distribution (68) when the total degeneracy is defined as \( g_j = \bar{g}_{jh} = \bar{g}_{jf} \bar{g}_{hf} \).

In the sequel it will be shown that the relevant degeneracy \( g_j \) for ideal gas at equilibrium is similar to the classical Rayleigh-Jeans expression for degeneracy of equilibrium radiation here expressed as [90, 91]

\[ dg_j = \frac{8\pi V}{u_j^3} v_j^2 dv_j \]  
(75)

At thermal equilibrium (75) denotes the number of eddies (oscillators) at constant mean “atomic” velocity \( u_j \) in a hydrodynamic system with volume \( V \) within the frequency interval \( v_j \) to \( v_j + dv_j \). The results (74) and (75) lead to Planck energy distribution function [68, 35] for isotropic turbulence at EED scale

\[ \varepsilon_j dN_j = \frac{8\pi h}{V} \frac{v_j^3}{u_j^3} e^{\frac{h v_j}{kT} - 1} dv_j \]  
(76)

when the energy of each eddy is \( \varepsilon_j = h v_j \). The calculated energy distribution (76) at \( T = 300 \) K is shown in Fig.5.

The three-dimensional energy spectrum \( E(k) \) for isotropic turbulence measured by Van Atta and Chen [92, 93] is found to be in qualitative agreement with Planck energy spectrum shown in Fig.5 [35]. In fact, it is expected that to maintain stationary isotropic turbulence both energy supply as well as energy dissipation spectrum should follow Planck law (76). The experimental data [94] obtained for one dimensional dissipation spectrum along with Planck energy distribution as well as this same distribution shifted by a constant amount of energy are shown in Fig.6.

![Fig.5 Planck energy distribution law governing the energy spectrum of eddies at the temperature \( T = 300 \) K.](image)

![Fig.6 One-dimensional dissipation spectrum [94] compared with (1) Planck energy distribution (2) Planck energy distribution with constant displacement.](image)

Similar comparison with Planck energy distribution as shown in Fig.6 is obtained with the experimental data for one-dimensional dissipation spectrum of isotropic turbulence from the study of Saddoughi and Veeravalli [95].

In a more recent experimental investigation the energy spectrum of turbulent flow within the boundary layer in close vicinity of rigid wall was measured by Marusic et al., [96] and the reported energy spectrum appears to have profile quite similar to Planck distribution law. Also, the normalized three-dimensional energy spectrum for homogeneous isotropic turbulent field was obtained from the transformation of one-dimensional energy spectrum of Lin [97] by Ling and Huang [98] as

\[ E^* = \frac{\alpha^2}{3} (K^* + \alpha K^{**}) \exp(-\alpha K^*) \]  
(77)

with the distribution comparable with Fig.5.

A most important aspect of Planck law (76) is that at a given fixed temperature the energy spectrum of equilibrium field is time invariant. Since one may view Planck distribution as spectrum of eddy cluster sizes this means that cluster sizes are stationary. Therefore, even though the number of eddies \( N_{j'} \) and their energy \( \varepsilon_{j'} \) in different fluid elements
(energy levels) are different their product that is the total energy of all energy levels is the same

\[ U_j = \sum_j \varepsilon_j = N_j \varepsilon_j = U_{\text{total}} = \ldots = U_{\text{total}} = \bar{U} \]  

(78)

in accordance with (63). Thus Boltzmann’s equipartition principle is satisfied in order to maintain time independent spectrum (Fig.5) and avoid Maxwell’s demon paradox [33]. Therefore, in stationary isotropic turbulence, energy flux occurs between fluid elements by transition of eddies of diverse sizes while leaving the fluid elements stochastically stationary in time. A schematic diagram of energy flux across hierarchies of eddies from large to small size is shown in Fig.7 from the study by Lumley et al [99].

![Fig.7 A realistic view of spectral energy flux [99].](image)

In Section 11, it will be suggested that the exchange of eddies between various size fluid elements (energy levels) is governed by quantum mechanics through an invariant Schrödinger equation (206). Therefore, transition of an eddy from a small rapidly oscillating fluid element to a large slowly oscillating fluid element results in energy emission by “subatomic particle” that for EED will be a molecular cluster \( \varepsilon_{ji} \) as schematically shown in Fig.8.

![Fig.8 Transition of eddy \( \varepsilon_{ij} \) from fluid element-j to fluid element-i leading to emission of cluster \( \varepsilon_{ij} \).](image)

Hence, the stochastically stationary states of fluid elements are due to energy exchange through transitions of eddies according to

\[ \Delta \varepsilon_{ijl} = \varepsilon_{jl} - \varepsilon_{il} = \hbar (v_{jl} - v_{il}) \]  

(79)

parallel to Bohr’s stationary states in atomic theory [66] to be further discussed in Sec.11.

The above procedures (68)-(76) could be applied to other pairs of adjacent statistical fields (ECD-EMD), (EMD-EAD), (EAD-ESD), (ESD-EKD), (EKD-ETD), … shown in Fig.1 leading to Planck energy distribution function (76) for the energy spectrum of respectively molecular-clusters, molecules, atoms, sub-particles (electrons), photons, tachyons, . . . at thermodynamic equilibrium. Therefore, (76) is the invariant Planck energy distribution law and can be written in invariant form for any scale \( \beta \) as [35]

\[ \frac{e^\beta dN_\beta}{V} = \frac{8\pi \hbar}{u_\beta} \frac{v^3_\beta}{e^{\hbar v_\beta/kT} - 1} dv_\beta \]  

(80)

with the spectrum shown in Fig.5.

The invariant Planck energy distribution (80) is a universal law giving energy spectra of all equilibrium statistical fields from cosmic to sub-photic scales shown in Fig.1. Such universality is evidenced by the fact that the measured deviation of Penzia-Wilson cosmic background radiation temperature of about 2.73 K from Planck law is about \( 5 \times 10^{-5} \) K. In view of the finite gravitational mass of photon (37), it is expected that as the temperature of the radiation field is sufficiently lowered photon condensation should occur parallel to superconductivity, BEC, and superfluidity at the scales of electo-dynamics, atomic-dynamics, and molecular-dynamics [51]. Such phenomena have indeed been observed in recent experiments [100] reporting on light condensation and formation of photon droplets. Furthermore, one expects a hierarchy of condensation phenomena to continue to tachyonic [101], or sub-tachyonic fields … ad infinitum.

The important scales ESD \( \beta = s \) and EKD \( \beta = k \) are respectively associated with the fields of stochastic electrodynamics SED and stochastic chromodynamics SCD [1-17]. For EKD scale of photon gas \( \beta = k \), also identified as Casimir vacuum [67] or the physical space with the most probable thermal speed of photon in vacuum \( u_k = v_{mpt} = c \) [77], the result (80) corresponds to a spectrum of photon clusters with energy distribution given by the classical Planck energy distribution law [68]

\[ \frac{e^\beta dN_\beta}{V} = \frac{8\pi \hbar}{c^3} \frac{v^3}{e^{\hbar v_\beta/kT} - 1} dv \]  

(81)
The notion of “molecules of light” as clusters of photons is in accordance with the perceptions of de Broglie [41, 102, 103]. It is emphasized that the velocity of light is therefore a function of the temperature of Casimir vacuum [67], i.e. the tachyonic fluid [77] that is Dirac stochastic ether [104] or de Broglie “hidden thermostat” [3]. However since such vacuum temperature changes by expansion of the cosmos through cons [35], one may assume that c is nearly a constant for the time durations relevant to human civilization.

The historical evolution of Planck law of equilibrium radiation, his spectral energy distribution function (81), and the central role of energy quanta $\varepsilon = \hbar \nu$ are all intimately related to the statistical mechanics of Boltzmann discussed in the previous section. This is most evident from the following quotation taken from the important 1872 paper of Boltzmann [37]

“We wish to replace the continuous variable $x$ by a series of discrete values $\varepsilon, 2\varepsilon, 3\varepsilon \ldots p\varepsilon$. Hence we must assume that our molecules are not able to take up a continuous series of kinetic energy values, but rather only values that are multiples of a certain quantity $\varepsilon$. Otherwise we shall treat exactly the same problem as before. We have many gas molecules in a space $R$. They are able to have only the following kinetic energies:

$\varepsilon, 2\varepsilon, 3\varepsilon, 4\varepsilon, \ldots p\varepsilon$.

No molecule may have an intermediate or greater energy. When two molecules collide, they can change their kinetic energies in many different ways. However, after the collision the kinetic energy of each molecule must always be a multiple of $\varepsilon$. I certainly do not need to remark that for the moment we are not concerned with a real physical problem. It would be difficult to imagine an apparatus that could regulate the collisions of two bodies in such a way that their kinetic energies after a collision are always multiples of $\varepsilon$. That is not a question here.”

The quotation given above and the introduction of the statistical mechanics of complexities discussed in the previous section are testimony to the significant role played by Boltzmann in the development of the foundation of quantum mechanics as was also emphasized by Planck in his Nobel lecture [60, 105].

Similarly, Boltzmann gas theory had a strong influence on Einstein in the development of the theory of Brownian motion even though Boltzmann himself made only a brief passing remark about the phenomena [60]

“\ldots likewise, it is observed that very small particles in a gas execute motions which result from the fact that the pressure on the surface of the particles may fluctuate.”

Although Einstein did not mention the importance of Boltzmann’s gas theory in his autobiographical sketch [60]

“Not acquainted with the earlier investigations of Boltzmann and Gibbs which appeared earlier and which actually exhausted the subject, I developed the statistical mechanics and the molecular kinetic theory of thermodynamics which was based on the former. My major aim in this was to find facts which would guarantee as much as possible the existence of atoms of definite finite size. In the midst of this I discovered that, according to atomic theory, there would have to be a movement of suspended microscopic particles open to observation, without knowing that observations concerning Brownian motion were long familiar”

much earlier in September of 1900 Einstein did praise Boltzmann’s work in a letter to Mileva [60, 106]

“The Boltzmann is magnificent. I have almost finished it. He is a masterly expounder. I am firmly convinced that the principles of the theory are right, which means that I am convinced that in the case of gases we are really dealing with discrete mass points of definite size, which are moving according to certain conditions. Boltzmann very correctly emphasizes that the hypothetical forces between the molecules are not an essential component of the theory, as the whole energy is of the kinetic kind. This is a step forward in the dynamical explanation of physical phenomena”

Similar high praise of Boltzmann’s theory appeared in April 1901 letter of Einstein to Mileva [60]

“I am presently studying Boltzmann’s gas theory again. It is all very good, but not enough emphasis is placed on a comparison with reality. But I think that there is enough empirical material for our investigation in the O. E. Meyer. You can check it the next time you are in the library. But this can wait until I get back from Switzerland. In general, I think this book deserves to be studied more carefully.”

The central role of Boltzmann in Einstein’s work on statistical mechanics has also been recently emphasized by Renn [107]

“In this work I argue that statistical mechanics, at least in the version published by Einstein in 1902 (Einstein 1902b), was the result of a reinterpretation of already existing results by Boltzmann.”
In order to better reveal the nature of particles versus the background fields at (ESD-EKD) and (EKD-ETD) scales, we examine the normalized Maxwell-Boltzmann speed distribution (119) from Section 8 shown in Fig.9.

According to Fig.9, in ETD field one starts with tachyon [101] “atom” to form a spectrum of tachyon clusters. Next, photon or de Broglie “atom of light” [103] is defined as the most probable size tachyon cluster of the stationary ETD field (Fig.9). Moving to the next larger scale of EKD, one forms a spectrum of photon clusters representing ideal photon gas of equilibrium radiation field. Finally, one identifies the “electron” as the most probable size photon cluster (Fig.9) of stationary EKD field. From ratio of the masses of electron and photon (37) the number of tachyons in a photon is estimated as

\[
N_t = \frac{9.1086 \times 10^{-37}}{1.84278 \times 10^{-36}} = 4.9428 \times 10^{10} \text{ tachyons (84)}
\]

The above definition of electron suggests that not all electrons may be exactly identical since by (82) a change of few hundred photons may not be experimentally detectable due to small photon mass (37). With electron defined as the “atom” of electrodynamics, one construct a spectrum of electron clusters to form the statistical field of equilibrium sub-particle dynamics ESD (SED) as ideal electron gas in harmony with the perceptions of Lorentz [108]

“Now, if within an electron there is ether, there can also be an electromagnetic field, and all we have got to do is to establish a system of equations that may be applied as well to the parts of the ether where there is an electric charge, i.e. to the electrons, as to those where there is none.”

The most probable electron cluster of ESD field is next identified as the “atom” of EAD field. As shown in Fig.9, the most probable element of scale \( \beta \) becomes the “atom” of the higher scale \( \beta+1 \) and the “system” of lower scale \( \beta-1 \)

\[
W_{\beta} = V_{\beta+1} = U_{\beta+2}
\]

in accordance with (1)-(2).

At EKD scale Planck law (81) gives energy spectrum of photon conglomerates, Sackur’s “clusters”, or Planck’s “quantum sphere of action” as described by Darrigol [41] with sizes given by Maxwell-Boltzmann distribution (Fig.9) in harmony with the perceptions of de Broglie [103]

“Existence of conglomerations of atoms of light whose movements are not independent but coherent”

Thus photon is identified as the most probable size tachyon cluster (Fig.9) of stationary ETD field. From ratio of the masses of photon (37) and tachyon \( m_t = m_s = 3.08 \times 10^{-45} \text{ kg} \) the number of photons in a photon is estimated as

\[
N_p = \frac{1.84278 \times 10^{-36}}{3.08 \times 10^{-45}} = 5.983 \times 10^{17} \text{ photons (84)}
\]

Comparison of (82) and (84) suggests that there may be another particle (perhaps Pauli’s neutrino) with the approximate mass of \( m_{\nu} = 10^{-3} \) kg between photon and tachyon scales. Also, as stated earlier, the “atoms” of all statistical fields shown in Figs.1, 3, 6, and 9 are considered to be “composite bosons” [89] made of “Cooper pairs” of the most probable size cluster of the statistical field of the adjacent lower scale (Fig.9). Indeed, according to de Broglie as emphasized by Lochak [103],

“Photon cannot be an elementary particle and must be composed of a pair of particles with small mass, maybe “neutrinos”.

Fig.9 Maxwell-Boltzmann speed distribution for ESD, EKD, and ETD fields.
Therefore, one expects another statistical field called equilibrium neutrino-dynamics END to separate EKD and ETD fields shown in Figs.1 & 9.

The invariant Planck law (80) leads to the invariant Wien displacement law [87]

\[ \lambda_{\text{mP}} T = 0.2014 c_2 \]  
(85)

For \( \beta = k \) by (29) and (30) the second radiation constant \( c_2 \) is identified as the inverse square of the universal gas constant (39)

\[ c_2 = \frac{h c}{k} = \frac{m_k c^4}{k^2} = \frac{1}{k^2 N^{\text{ul}}^2} = \frac{1}{R^{\text{ul}}^2} \]  
(86)

such that one may also express (85) as

\[ \lambda_{\text{mP}} T = \frac{0.2014}{R^{\text{ul}}^2} = 0.002897 \text{ m-deg} \]  
(87)

It is also possible to express (87) in terms of the root mean square wavelength of photons in vacuum

\[ c_2 = \frac{h c}{k} = \frac{m_k \lambda_k c c}{m_{\text{vK}} c} = \frac{\lambda_k c}{\lambda_{\text{vK}}} = \lambda_{\text{vK}}^2 \]  
(88)

from (41) [77]

\[ \lambda_k = 0.119933 \text{ m} \]  
(89)

By the definition of Boltzmann constant (31b), (33), and (36) the absolute thermodynamic temperature becomes the root mean square wavelength of the most probable state

\[ T = \langle \lambda_{\text{mp}}^2 \rangle^{1/2} = \lambda_{\text{mP}} \]  
(90)

Therefore, by (87), and (89) Wien displacement law (85) may be also expressed as

\[ \lambda_{\text{mP}} T = \lambda_{\text{mP}}^2 = 0.2014 \lambda_k^2 \]  
(91)

relating the most probable and the root mean square wavelengths of photons in the radiation field at thermodynamic equilibrium.

It is possible to introduce a displacement law for most probable frequency parallel to Wien's displacement law for most probable wavelength (85). By setting the derivative of Planck energy density to zero one arrives at the transcendental equation for maximum frequency as

\[ e^{c_2 \nu_m / k c T} + \frac{c_2 \nu_m}{3 k c T} - 1 = 0 \]  
(92)

From the numerical solution of (92) one obtains the frequency displacement law

\[ \frac{T}{\nu_m} = 0.354428 \frac{c_2}{c} = 0.354428 \frac{h}{k} \]  
(93)

or

\[ \nu_m = 5.8807375 \times 10^5 T \]  
(94)

From (94) one obtains the frequency at the maxima of Planck energy distribution at temperature \( T \) such as \( \nu_m = 1.764 \times 10^{13} \text{ Hz} \) at \( T = 300 \text{ K} \) in accordance with Fig.5 and \( \nu_m = 3.5284 \times 10^{14} \text{ Hz} \) at \( T = 6000 \text{ K} \) in agreement with Fig.6.1 of Baierlein [88].

From division of the Wien displacement law for wavelength (85) and frequency (93) one obtains

\[ \lambda_{\text{mP}} \nu_{\text{mP}} = 0.5682 \text{ c} \]  
(95)

Because by (41) the speed of light in vacuum is

\[ \lambda_k \nu_k = c \]  
(96)

one can express (95) as

\[ \frac{\lambda_{\text{mP}} \nu_{\text{mP}}}{\nu_{\text{r.m.s}}} = \frac{\nu_{\text{mp},k}}{c} = 0.57 \]  
(97)

The result (97) may be compared with the ratio of the most probable speed \( \nu_{\text{mp}} = \sqrt{k T / m} \) and the root mean square speed \( \nu_{\text{r.m.s}} = \sqrt{3 k T / m} \) that is

\[ \nu_{\text{mp}} / \nu_{\text{r.m.s}} = 1 / \sqrt{3} \approx 0.577 \]  
(98)

The reason for the difference between (97) and (98) requires further examination.

At thermodynamic equilibrium each system of Fig.9 will be stationary at a given constant temperature. The total energy of such equilibrium field will be the sum of the potential and internal energy expressed by the modified form of the first law of thermodynamics to be further discussed in Section 10 [31]

\[ Q = H + p V - U \]  
(99)

In a recent investigation [77] it was shown that for monatomic ideal gas with \( \tilde{c}_V = 3 R^\circ \) and \( \tilde{c}_p = 4 R^\circ \) one may express (99) as

\[ Q = U + p V = H = \frac{3}{4} H + \frac{1}{4} H = E_{\text{d}p} + E_{\text{d}m} \]  
(100)

Therefore, the total energy (mass) of the atom of \( \beta \) scale is the sum of the internal energy (dark
energy \( DE_{\beta^{-1}} \) and potential energy (dark matter \( DM_{\beta^{-1}} \)) at the lower scale \( \beta^{-1} \) \[ \varepsilon_{p} = E_{\beta^{-1}} = U_{\beta^{-1}} + p_{\beta^{-1}} V_{\beta^{-1}} = DE_{\beta^{-1}} + DM_{\beta^{-1}} \] (101)

To better reveal the origin of the potential energy \( p_{\beta^{-1}} V_{\beta^{-1}} \) in (101) one notes that by (30) the dimensionless particle energy in Maxwell-Boltzmann distribution (111) could be expressed as

\[
\frac{m v_j^2}{k T} = \frac{m v_{mp}^2}{k \lambda_{mp}^2} = \frac{\lambda_j}{\lambda_{mp}^2}
\]

hence

\[
\frac{v_j}{v_{mp}} = \left( \frac{\lambda_j}{\lambda_{mp}} \right)^{1/2}
\]

(102)

Therefore Maxwell-Boltzmann distribution (111) may be expressed as a function of the square root of dimensionless wavelength by (103) thus revealing the relative (atomic, element, and system) lengths \( (\ell, \lambda, L)_{\beta} = (0, 1, \infty)_{\beta} \) of the adjacent scales \( \beta \) and \( \beta^{-1} \) as shown in Fig.10.

**Fig.10 Maxwell-Boltzmann speed distribution as a function of oscillator wavelengths \( (\lambda_j/\lambda_{mp})^{1/2} \).**

According to Fig.10, the interval \( (0, 1)_{\beta} \) of scale \( \beta \) becomes \( (1, \infty)_{\beta^{-1}} \) of \( \beta^{-1} \) scale. However, the interval \( (0, 1)_{\beta^{-1}} \) is only revealed at \( \beta^{-1} \) scale and is unobservable at the larger scale \( \beta \) (Figs.10, 14). Therefore, in three dimensions such coordinate extensions results in volume generation leading to release of potential energy as schematically shown in Fig.11. Hence by (102)-(103) as one decompactifies the atom of scale \( \beta \), ¼ of the total mass (energy) of the atom “evaporates” into energy due to *internal* translational, rotational, and vibrational (pulsational) motions and is therefore none-baryonic and defined as dark energy (electromagnetic mass) \[ [77] \]. The remaining ¼ of the total mass that appears as potential energy (dark matter) \[ [77] \] in part “evaporates” as new volume generation (Figs.10, 11) and in part forms the gravitational mass (dark matter) of the next lower scale.

\[
DM_{\beta^{-1}} = E_{\beta^{-2}} = DE_{\beta^{-2}} + DM_{\beta^{-2}}
\]

(104)

The concepts of internal versus external potential energy are further discussed in the following section.

**Fig.11 Effects of internal versus external potential energy as the system volume is increased.**

As an example, to determine the total energy of a photon one starts from the thermodynamic relation for an ideal photon gas

\[
\tilde{h} = \tilde{u} + \tilde{p} v
\]

with specific molar enthalpy, internal energy and volume \( (\tilde{h}, \tilde{u}, \tilde{v}) = (H, U, V) / \tilde{N} \) that can also be expressed as

\[
\tilde{c}_p = \tilde{c}_v + R^o
\]

(106)

where \( \tilde{N} = N / N^o \), and \( \tilde{W} = N^o m \) is the molecular weight. Since Poisson coefficient \( \gamma \) of photon gas is

\[
\gamma = \tilde{c}_p / \tilde{c}_v = 4 / 3
\]

one arrives at \( \tilde{c}_p = 4R^o \) and \( \tilde{c}_v = 3R^o \) such that by (105) the total energy of the photon could be expressed as
\[ E_γ = (3/4)mc^2 + (1/4)mc^2 = E_{de} + E_{dey} \]  

(107)

From (107) one concludes that of the total energy constituting a photon, \( \frac{3}{4} \) is associated with the electromagnetic field (dark energy \( E_d \)) and \( \frac{1}{4} \) with the gravitational field (dark matter \( E_{DM} \)) [77]. Hence, as one decompactifies atoms of smaller and smaller scales by (38), (40), and (100)-(107), ultimately all matter will be composed of dark energy or electromagnetic mass as was anticipated by both Lorentz [110] and Poincaré [111-113].

It is known that exactly \( \frac{3}{4} \) and \( \frac{1}{4} \) of the total energy of Planck black body equilibrium radiation falls on \( \lambda > \lambda_m \) and \( \lambda < \lambda_m \) side of \( \lambda_m \) given by the Wien displacement law (85). Indeed, the first part of (107) confirms the apparent mass \( \mu = 4E/3c^2 \) that was measured for the black body radiation pressure in the pioneering experiments by Hasenöhrl in 1905 [84]. According to (107), the finite gravitational mass of the photon (37) that is associated with Poincaré stress [111-113] accounts for the remaining \( \frac{1}{4} \) of the total mass as dark matter. This longitudinal component would be absent if photon gravitational mass (37) was zero in matter. This longitudinal component would be absent if photon gravitational mass (37) was zero in harmony with the perceptions of Higgs [114]. The result (107) is also consistent with the general theory of relativity [115] according to which of the total energy constituting matter \( \frac{1}{4} \) is to be ascribed to the electromagnetic field and \( \frac{3}{4} \) to the gravitational field.

7 Invariant Maxwell-Boltzmann Speed Distribution Function

Because of its definition, the energy spectrum of particles in an equilibrium statistical field is expected to be closely connected to the spectrum of speeds of particles. Indeed, it is possible to obtain the invariant Maxwell-Boltzmann distribution function directly from the invariant Planck distribution function (80) that in view of (34) and (37) can be written as

\[
\frac{dN_\beta}{N_\beta} = \frac{\lambda_\beta^3}{8\pi\beta \pi^2} \frac{\lambda_\beta^3 \beta^3}{e^{\frac{\lambda_\beta^3 \beta^3}{\pi^2}} - 1} \frac{d\beta}{\beta^2 \pi^2} = \frac{\lambda_\beta^3}{8\pi\beta \pi^2} \frac{\lambda_\beta^3 \beta^3}{e^{\frac{\lambda_\beta^3 \beta^3}{\pi^2}} - 1} \frac{d\beta}{\beta^2 \pi^2} \]  

(110)

By (111), one arrives at a hierarchy of embedded Maxwell-Boltzmann distribution functions for EED, ECD, and EMD scales shown in Fig.12.

As stated earlier, the invariant results (80) and (111) suggest that particles of all statistical fields (Fig.1) will have Gaussian velocity distribution, Planck energy distribution, and Maxwell-Boltzmann speed distribution.

It is possible to express the number of degeneracy commonly obtained from field quantization [86-88] for particles, Heisenberg-Kramers virtual oscillators [66] in a spherical volume \( V_s \) as

\[
g_\beta = 2V_s / \lambda_\beta^3 \]  

(112)

where \( \lambda_\beta^3 \) is the rectangular volume occupied by each oscillator \( V_o = \lambda_\beta^3 \) when due to isotropy \( \lambda_\beta = \lambda_\lambda^3 \). When due to isotropy \( \lambda_\beta = \lambda_\lambda^3 \) and the factor 2 comes from allowing particles to have two modes either (up) or (down) iso-spin (polarization). The system spherical \( V_s \) and rectangular \( V \) volumes are related as

\[
V_s = \frac{4\pi}{3} R^3 = \frac{4\pi}{3} V \]  

(113)
For systems in thermodynamic equilibrium the temperature \(3kT_\beta = m_\beta \langle u_{\beta}^2 \rangle\) will be constant and hence \(\langle u_{\beta}^2 \rangle^{1/2} = \langle \lambda_{\beta}^2 \rangle^{1/2} \langle v_{\beta}^2 \rangle^{1/2}\) or

\[
\lambda_{\beta} = u_{\beta}/v_{\beta} \tag{114}
\]

Substituting from (113)-(114) into (112) results in

\[
g_{\beta} = \frac{8\pi V}{3u_{\beta}^3} v_{\beta}^3 \tag{115}
\]

that leads to the number of oscillators between frequencies \(v_{\beta}\) and \(v_{\beta}+dv_{\beta}\)

\[
dg_{\beta} = \frac{8\pi V}{u_{\beta}^3} v_{\beta}^3 dv_{\beta} \tag{116}
\]

in accordance with Rayleigh-Jeans expression (75).

The expression (116) for degeneracy is for application to Planck law involving frequency as the variable for energy quanta \(\epsilon = hv\) (30a). It is also possible to arrive at the degeneracy (109) for Maxwell-Boltzmann speed distribution from (112). However, because only positive values of speeds \((u_x, u_y, u_z)\) are allowed one must take 1/8 of the total volume of the velocity space and (112) in terms of the relevant volume gives

\[
g_{\beta} = 2V / \lambda_{\beta}^3 = \frac{2(L')^3}{\lambda_{\beta}^3} = \frac{2(2L'/\sqrt{\pi})^3 \pi^{3/2}}{8\lambda_{\beta}^3}
\]

\[
= \frac{2(2L'/\sqrt{\pi})^3 (\pi m_{\beta}^2 u_{\beta}^2)^{3/2}}{8\lambda_{\beta}^3 m_{\beta}^2 u_{\beta}^3} = \frac{2L'(\pi m_{\beta}^2 u_{\beta}^2)^{3/2}}{8h_{\beta}^3}
\]

\[
= 2V (\frac{m_{\beta} kT_{\beta}}{h_{\beta}^2})^{3/2} = 2V (\frac{2m_{\beta} kT_{\beta}}{h_{\beta}^2})^{3/2} \tag{117}
\]

that is in accordance with (109). In (117) the correct relevant volume of the speed space is \(V_{\epsilon} = (2L'/\sqrt{\pi})^3 / 8\), while \(V = (L'/\sqrt{2\pi})^3\), and \(h_{\beta} = h\) by (36). The coordinate \(L'\) in (117) was first normalized as \(L = 2L'/\sqrt{\pi}\) with a measure based on Gauss’s error function as discussed in [116] and shown in Fig.14. The result (117) is twice the classical translational degeneracy [86]

\[
g_{t} = V\frac{2\pi mkT_{\beta}}{h_{\beta}^2}^{3/2} \tag{118}
\]

The additional factor of two arises from the fact that similar to Boltzmann factor \(e^{-\beta \epsilon/kT}\) in Planck distribution law (81) by (42)-(44) the modified Maxwell-Boltzmann distribution (111) will also involve \(e^{-\beta u_{\beta}^2/kT}\) rather than the classical expression \(e^{-m_{\beta} \epsilon/kT}\).

### 8 Connections between Riemann Hypothesis and Normalized Maxwell-Boltzmann Distribution Function

Because Maxwell-Boltzmann speed distribution (111) may be also viewed as distribution of sizes of particle clusters, if expressed in dimensionless form it can also be viewed as the sizes of “clusters of numbers” or Hilbert “condensations”. Therefore, a recent study [117] was focused on exploration of possible connections between the result (111) and the theoretical findings of Montgomery [118] and Odlyzko [119] on analytical number theory that has resulted in what is known as Montgomery-Odlyzko law [120, 121]

“The distribution of the spacings between successive non-trivial zeros of the Riemann zeta function (suitably normalized) is statistically identical with the distribution of eigenvalue spacings in a GUE operator”

The pair correlation of Montgomery [118] was subsequently recognized by Dyson to correspond to that between the energy levels of heavy elements [120-121] and thus to the pair correlations between eigenvalues of Hermitian matrices [122]. Hence, a connection was established between quantum mechanics on the one hand and quantum chaos [123] on the other hand. However, the exact nature of the connections between these seemingly diverse fields of quantum mechanics, random matrices, and Riemann hypothesis [120, 121] is yet to be understood.

When the oscillator speeds (cluster sizes) in (111) are normalized through division by the most probable speed (the most probable cluster size) one arrives at Normalized Maxwell-Boltzmann NMB distribution function [117]

\[
\rho_{\beta} = (8 / \pi_{\beta}) \left[ (2 / \sqrt{\pi_{\beta}}) x_{\beta} \right]^2 e^{-\frac{(2\sqrt{\pi_{\beta}} x_{\beta})^2}{2}} \tag{119}
\]

The additional division by the “measure” \(\sqrt{\pi_{\beta}} / 2\) in (119) is for coordinate normalization as discussed in [116] and shown in Fig.14. Direct comparisons between (119) and the normalized spacings between the zeros of Riemann zeta function and the eigenvalues of GUE calculated by Odlyzko [119] are shown in Fig.13. Therefore, a definite connection has been established between analytic number theory, the kinetic theory of ideal gas NMB (119), and the
normalized spacings between energy levels in quantum mechanics [117].

\begin{equation}
\gamma_n = 10^{12} \leq n \leq 10^{12} + 10^{4} \quad [119], \text{ normalized spacings between eigenvalues of GUE [119], and the NMB distribution function (119) [117].}
\end{equation}

To further examine the connection between Riemann hypothesis and Maxwell-Boltzmann kinetic theory of ideal gas the speed distribution is first related to distribution of sizes or wavelengths of number clusters (Fig.10). According to (29)-(30) and (36) particle energy and frequency are related by

\begin{equation}
\epsilon_j = \text{mv}_j^2 = k\lambda_j = \nu_j
\end{equation}

Therefore, the normalized spacings between energy levels can be expressed in terms of the normalized spacing between frequencies of virtual oscillators as

\begin{equation}
\frac{\epsilon_{ij} - \epsilon_{il}}{\epsilon_{mp}} = \frac{(\nu_j - \nu_i)}{\nu_{mp}}
\end{equation}

Because of Boltzmann’s equipartition principle the particles’ random rotational and vibrational (pulsational) kinetic energy in two directions (θ+,θ−) and (r+,r−) will be equal to their translational kinetic energy (120) and follow Planck law (80). Also, the corresponding momenta of all three degrees of freedom will be randomly distributed and once properly normalized should follow NMB distribution (119). Therefore, parallel to (30) the rotational counterpart of (120) is expressed as

\begin{align*}
\epsilon_{ij} &= I\langle \omega_{j\beta}^2 \rangle / 2 + 2 = I\langle \omega_{j\beta}^2 \rangle \\
&= \text{m} < r^2 \omega_{j\beta}^2 > ^{1/2} < (2\pi)^2 \nu_{j\beta}^2 > ^{1/2} \\
&= (\text{m} < u_{j\beta}^2 > ^{1/2} < \lambda_{j\beta}^2 > ^{1/2} ) < \nu_{j\beta}^2 > ^{1/2} = \nu_{j\beta}
\end{align*}

where I is the moment of inertia and by equipartition principle $<\omega_{j\beta}^2>$ = $<\omega_{j\phi}^2>$. By (122) the normalized spacings between rotational energy levels will also be related to the normalized spacings between frequencies of oscillators (121).

Following the classical methods [86], for the vibrational degree of freedom the potential energy of harmonic oscillator is expressed as

\begin{align*}
\epsilon_{ij} &= \kappa \langle x_{j\beta}^2 \rangle / 2 + \kappa \langle x_{j\beta}^2 \rangle / 2 = \kappa \langle x_{j\phi}^2 \rangle \\
&= \text{m} \langle \omega_{j\beta}^2 \rangle \langle x_{j\beta}^2 \rangle = \text{m} \langle 2\pi \nu_{j\beta} \rangle ^2 \langle \nu_{j\beta}^2 \rangle \\
&= (\text{m} < u_{j\beta}^2 > ^{1/2} < \lambda_{j\beta}^2 > ^{1/2} ) < \nu_{j\beta}^2 > ^{1/2} = \nu_{j\beta}
\end{align*}

where $\kappa$ is the spring constant $\omega = \sqrt{\kappa/m}$ [86]. Similar to (120) and (122), by (123) the normalized spacings between potential energy levels of harmonic oscillator are also related to the normalized spacings between frequencies of virtual oscillators (121). In summary, the normalized spacings between energy levels for translational, rotational, and vibrational motions are related to their corresponding normalized frequencies by

\begin{equation}
\frac{(\epsilon_{ij} - \epsilon_{il})}{\epsilon_{mp}} = \frac{(\nu_j - \nu_i)}{\nu_{mp}} \quad q = t, r, v
\end{equation}

Therefore, the classical model of diatomic molecule with rigid-body rotation and harmonic vibration [86] is herein considered to also posses an internal translational harmonic motion (124). The internal translational degree of freedom is associated with thermodynamic pressure and may therefore be called internal potential energy. In addition to this internal harmonic translation there will be an external harmonic motion due to the peculiar translational velocity (3) and a corresponding external potential energy (Fig.11). In Section 11, it will be shown that the external potential energy appears in Schrödinger equation (206) and acts as Poincaré stress [111-113] that is responsible for “particle” stability. The particle trajectory under all four degrees of freedom namely the three internal translational, rotational and vibrational motions and the external peculiar motion will be quite complicated. Clearly, addition of radial oscillations about the center of mass to the rigid rotator will result in particle motion on a radial wave, de Broglie wave, along the circumference of the otherwise circular particle trajectory.

Now that the normalized spacings between energy levels have been related to the normalized spacings between oscillator frequencies $\nu_j$ (124), the latter should be connected to the zeros of Riemann zeta function. The zeros of Riemann zeta function are related to prime numbers through Euler’s Golden Key [120]

\begin{equation}
\zeta(s) = \frac{1}{\pi^s} = \Pi(1 - p_i^{-s})^{-1}
\end{equation}
where $s$ is a complex number. Clearly, the zeros of zeta function (125) will coincide with the zeros of the powers of primes

$$p_j^s = p_j^{a+ib} = 0$$  \hspace{1cm} (126)

It is most interesting that according to his Nachlass [121] Riemann was working on the problem of Riemann Hypothesis and the hydrodynamic problem of stability of rotating liquid droplets simultaneously. In view of (119) and the connections between normalized energy and frequency spacings shown in Fig.13, it is natural to consider the situation when particle frequencies are given by prime numbers as

$$\nu_j = p_j^1, p_j^2, p_j^3, ... = p_j^N = \mathbb{Q}_{p_j}$$  \hspace{1cm} (127)

in harmony with Gauss’s clock calculator [121] and Hensel’s $p_{\beta}$-adic numbers $\mathbb{Q}_{p_j}$ [120].

Now, since the atomic energy of particles must be quantized according to Planck formula $\varepsilon = h \nu$ by (127) one writes

$$m \nu_j^2 = h \nu_j = h p_j^N = h \mathbb{Q}_{p_j}$$  \hspace{1cm} (128)

that by (30)-(31) and (36) results in the particle velocity

$$\nu_{j\pm} = \sqrt{h/m} p_j^{x_{\pm}} = \sqrt{h/m} p_j^{a+ib}$$  \hspace{1cm} (129)

such that by (128)

$$\varepsilon_j = m(\nu_j, \nu_j^*) = h p_j^{2a} = h \mathbb{Q}_{p_j}$$  \hspace{1cm} (130)

It is clear that since by (128) Planck quantum of energy relates to prime number $p_{\beta}$ and is therefore an indivisible quantum of energy one must require the coefficient in (130) to be $a = \frac{1}{2}$ such that

$$\nu_{j\pm} = \sqrt{h/m} p_j^{1/2+ib} = \sqrt{h/m} \mathbb{Q}_{p_j}^{1/2}$$  \hspace{1cm} (131)

and by $\nu = \lambda \nu_j$ the wavelength becomes

$$\lambda_{j\pm} = \sqrt{h/m} p_j^{1-2ib} = \sqrt{h/m} \mathbb{Q}_{p_j}^{-1/2}$$  \hspace{1cm} (132)

According to (127)-(132), the reason for $a = \frac{1}{2}$ as the position of the critical line in Riemann Hypothesis [120, 121] is the fact that the quantized energy of particles (128) cannot be fractioned because prime numbers are being employed as “atomic” species.

At thermodynamic equilibrium Maxwell-Boltzmann speed distribution (119) corresponds to translational, rotational, and vibrational particle momenta that must follow Gaussian distribution like Fig.4. Also, space isotropy requires that the translational, rotational, and vibrational momenta of particles in two directions $(x, x)$, $(\theta, \theta)$, $(\tau, \tau)$ be equal in magnitude and opposite in direction

$$v_{j\pm} = \sqrt{h/m} p_j^{1/2} e^{\pm ib (p_j^\beta)} = \sqrt{h/m} p_j^{1/2} [\cos(ln p_j^\beta) \pm i \sin(ln p_j^\beta)]$$  \hspace{1cm} (133)

such that the corresponding average particle momenta become identically zero. Therefore, one expects “stationary states” at mean translational position $(\bar{x} = 0)$, mean angular position $(\bar{\theta} = 0)$, and mean radial position $(\bar{\tau} = 0)$ at which particle velocities $(v_x, v_\theta, v_\tau)$ and hence energies $(\varepsilon_x, \varepsilon_\theta, \varepsilon_\tau)$ vanish. Because by (124) all three forms of energy could be expressed as products of Planck constant and frequency, one identifies the zeros of velocities (131) as the “stationary states” of particle’s translational, rotational, and vibrational momenta.

To summarize, first the normalized spacings between energy levels were related to the normalized spacings between oscillator frequencies (124). Next, it was shown that when oscillator frequencies are taken as $p_{\beta}$-adic numbers (127) the zeros of frequencies, hence those of energy (130) and velocity (131), could be related to the zeros of Riemann zeta function by (126) through Euler’s golden key (125). The model therefore provides a physical explanation of Montgomery-Odlyzko law [120-121] and the connection between normalized spacings of energy levels in quantum mechanics and the zeros of Riemann zeta function. The close agreement shown in Fig.13 is because at equilibrium by (124) the distribution of normalized particle energy (130) and speed (131) are related by Maxwell-Boltzmann distribution (119). For complete resolution of Riemann Hypothesis one must now identify a method of obtaining the zeros of Riemann zeta function (125) and hence (126), coinciding with the zeros of (127), (130), and (131) at the “stationary states”, that is simpler than Riemann-Siegel formula.

It is interesting to examine the analytical number theory and Riemann hypothesis discussed above in terms of quantized spatial coordinate i.e wavelengths (132) rather than frequencies (127). Recently a logarithmic system of coordinates was introduced as [116]

$$X_{p} = \ln N_{a\beta}$$  \hspace{1cm} (134)

whereby the spatial distance of each statistical field (Fig1) is measured on the basis of the number of “atoms” of that particular statistical field $N_{a\beta}$. With definition (134) the counting of numbers must begin with the number zero naturally since it corresponds to
a single atom. The number of atoms in the system is expressed as [116]
\[ N_{AS\beta} = (N_{AE\beta})^{\frac{N_{ES\beta}}{2}} \]  
where \((N_{ES\beta}, N_{x\beta}, N_{AE\beta})\) respectively refer to the number of atoms in the system, number of elements in the system, and number of atoms in elements. The hierarchy of the resulting normalized coordinates is shown below [116]
\[ L_{x\beta} \rightarrow l_{\beta} \rightarrow 0_{\beta} \]
\[ L_{x_{\beta-1}} \rightarrow l_{\beta-1} \rightarrow 0_{\beta-1} \]
\[ \ldots \]

(136)
The exact connections between spatial coordinates of hierarchies of statistical fields (Fig.1) will involve the important concept of re-normalization [124]. Normalization in (136) is based on the concept of "dimensionless" or "measureless" numbers [116]
\[ \frac{E_{x\beta}}{N_{x\beta}} = \delta_{\beta} = \frac{\pi}{2} \]  
where the "measure" \(\delta_{\beta}\) is defined by Gauss’s error function as
\[ \delta_{\beta} = \int_{0_{\beta}}^{l_{\beta}} e^{-x_{\beta}^2} dx_{\beta} = \sqrt{\pi} \beta^{-1}/2 \]  
(138)
In view of the relation (137)-(138), the range \((-1,1)\) of the outer coordinate \(x_{\beta}\) will correspond to the range \((-\infty, \infty)\) of the inner coordinate \(x'_{\beta-1}\) leading to the coordinate hierarchy schematically shown in Fig.14.

Fig.14 Hierarchy of normalized coordinates associated with embedded statistical fields [116].

As discussed above and in [116], one naturally considers the prime numbers \(p_{j\beta} = 2, p_{2\beta} = 3, p_{3\beta} = 5, \ldots\), to be the “atoms” of the statistical field at scale \(\beta\) that may also be viewed as different atomic “species”. However, in view of results (127) and (132) space quantization will be based on the inverse power of \(p_{j\beta}\)-adic numbers such that by (135)
\[ N_{AS\beta}^{1/2} = (N_{AE\beta})^{\frac{N_{ES\beta}^{1/2}}{2}} = (p_{\beta})^{-\frac{N_{ES\beta}}{2}} \]  
(139)
that parallel to (132) is expressed as
\[ \lambda_{\beta\beta} = \sqrt{\hbar/m_{j\beta}} N_{ES\beta}^{-1/2} = \sqrt{\hbar/m_{p_{\beta}}}^{-1/2} \]  
(140)
The reason for the choice of primes is that they represent indivisible “atoms” of arithmetic out of which all natural numbers at any scale could be constructed. The quantized wavelengths in (140) like (132) will correspond to quantized frequencies (127), energies (130), and velocities (131) involving \(p_{j\beta}\)-adic numbers \(Q_{p_{\beta}}\) that were employed in the construction of Adele space of noncommutative geometry of Connes [120, 121, 125].

The wavelength (140) is next normalized with respect to the most probable cluster size
\[ \frac{\lambda_{\beta\beta}}{\lambda_{m\beta}} = \sqrt{\frac{\hbar}{m_{j\beta}}} N_{ES\beta}^{-1/2} = \sqrt{\frac{\hbar}{m_{Q_{p_{\beta}}}}}^{-1/2} \]  
(141)
to obtain
\[ x_{j\beta} = \frac{\lambda_{\beta\beta}}{\lambda_{m\beta}} = \frac{p_{j\beta}}{p_{m\beta}} = Q_{p_{\beta}}^{-1/2} \]  
(142)
With \(p_{j\beta}\)-adic numbers \(Q_{p_{\beta}}\) incorporated into the quantized wavelengths (142) one arrives at dimensionless velocities (131) and hence the Normalized Maxwell-Boltzmann NMB distribution (119) for prime “specie” \(p_{j\beta}\).

Since each partial density in (119) corresponds to single “prime” specie, one next constructs a mixture density by summing all the partial densities of all “prime” species
\[ \rho_{\beta} \approx \sum_{j} \rho_{j\beta} \]  
(143)
to arrive at
\[ \rho_{\beta} = \frac{8}{\pi_{p_{\beta}}} [(2/\sqrt{\pi_{p_{\beta}}})x_{j\beta}]^2 e^{-(2/\sqrt{\pi_{p_{\beta}}})x_{j\beta}} \]  
(144)
where
\[ \sum_{j} x_{j\beta}^2 = \sum_{j} \frac{v_{j\beta}^2}{v_{m\beta}} = \sum_{j} \frac{m_{j\beta} v_{j\beta}^2}{m_{m\beta} v_{m\beta}} = \frac{1}{v_{m\beta}} \sum_{j} Y_{j\beta} v_{j\beta}^2 \]
\[ = \frac{v_{\beta}^2}{v_{m\beta}^2} = x_{\beta}^2 \]  
(145)
because the mean energy of all species are identical at thermodynamic equilibrium \( m_j v_{m_j}^2 = m_i v_{m_i}^2 = kT \).

The grand ensemble of NMB \( p_\beta \)-adic statistical fields (144) will have a corresponding GUE that could be identified as Connes’ Adele space [120, 121, 125] \( \mathbb{A}_\beta \) of a particular scale \( \beta \). Therefore, the normalized Adele space \( \mathbb{A}_\beta \) at any particular scale \( \beta \) is constructed from superposition of infinite NMB distribution functions like Fig.13 corresponding to atomic specie \( p_\beta \) and cluster sizes in the form of \( p_\beta \)-adic numbers (131).

Because prime numbers \( p_\beta \) represent atomic species and relate to the wavelength of number clusters or Hilbert condensations [120], their spacing will be related to the normalized spacings between energy levels by (121). The connection to quantum mechanics is further evidenced by direct derivation of Maxwell-Boltzmann speed distribution (111) from Planck distribution (80) discussed in Section 7. Also, because the GUE associated with (144) is based on \( p_\beta \)-adic type numbers (142), the normalized spacings between its eigenvalues should be related to the normalized spacings between the zeros of Riemann zeta function according to the theory of noncommutative geometry of Connes [125]. Although the exact connection between noncommutative geometry and the Riemann hypothesis is yet to be understood according to Connes [121]

“Our spectral series, dominated as they are by integral quantum numbers, correspond, in a sense, to the ancient triad of the lyre, from which the Pythagoreans 2500 years ago inferred the harmony of the natural phenomena; and our quanta remind us of the role which the Pythagorean doctrine seems to have ascribed to the integers, not merely as attributes, but as the real essence of physical phenomena.”

9 Invariant Transport Coefficients and Hierarchies of Absolute Zero Temperatures and Vacua

Following Maxwell [36, 39] one arrives at a scale invariant definition of kinematic viscosity

\[
v_\beta = \frac{1}{3} \ell_\beta u_\beta = \frac{1}{3} \lambda_{\beta-1} v_{\beta-1}
\]

At the scale of \( \beta = e \) corresponding to equilibrium eddy dynamics EED (Fig.1) (146) gives Boussinesq eddy diffusivity [130]

\[
v_e = \frac{1}{3} \ell_e u_e = \frac{1}{3} \lambda_e v_e
\]

On the other hand, the kinematic viscosity at LCD scale involves the “atomic” length \( \ell \) or the molecular mean free path \( \lambda_m \) that appears in Maxwell’s formula for kinematic viscosity [30]

\[
v_c = \frac{1}{3} \ell_c u_c = \frac{1}{3} \lambda_m v_m
\]

associated with viscous dissipation in fluid mechanics. It is important to note that the model predicts a finite kinematic viscosity for the scale \( \beta = s \) i.e. eddy diffusivity scale when energy can be dissipated to Casimir vacuum [67] at the lower scale of chromodynamics

\[
v_s = \frac{1}{3} \ell_s u_s = \frac{1}{3(2\pi)} \lambda_k v_k = \frac{1}{3(2\pi)} \frac{\lambda_k v_k m_k}{m_k} = \frac{1}{3(2\pi)} \frac{\hbar_k}{3m_k}
\]

in exact agreement with the result of de Broglie [3]. Therefore, Ohmic dissipation could occur by transfer of energy from electrons into photon gas constituting Casimir vacuum [67]. In view of Fig.1, it is natural to expect that energy of photon field at chromodynamic scale could be dissipated into a sub-
photonic tachyon field that constitutes a new vacuum called vacuum-vacuum. Therefore, the model suggests that electromagnetic waves propagate as viscous flow due to "viscosity" of radiation [41] in harmony with the well known concept of "tired light".

According to the definition of Boltzmann constant in (30b), (33) and (36), Kelvin absolute temperature is related to the most probable wavelength of oscillations (90) and hence becomes a length scale. Therefore, Kelvin absolute temperature approaches zero through an infinite hierarchy of limits as suggested by (136) and Fig.14. In other words, one arrives at a hierarchy of "zero" temperatures defined as

\[ T_p = 0_p = T_{p-1} = 1_{p-1} \]

\[ T_{p+1} = 0_{p+1} = T_{p+2} = 1_{p+2} \quad (150) \]

As discussed in Section 6, physical space could be identified as a tachyonic fluid [131] that is stochastic ether of Dirac [104] or de Broglie's "hidden thermostat" [3]. The importance of Aristotle's ether to the theory of electrons was emphasized by Lorentz [132, 133]

"I cannot but regard the ether, which is the seat of an electromagnetic field with its energy and its vibrations, as endowed with certain degree of substantiality, however different it may be from all ordinary matter"

Also, in Sec.6 photons were suggested to be composed of a large number of much smaller particles [131] like neutrinos that themselves are composed of large numbers of tachyons [101]. Therefore, following Casimir [67] and in harmony with (150) one expects a hierarchy of vacua

\[ (\text{vacuum-vacuum})_p = (\text{vacuum})_{p-1} \quad (151) \]

as one attempts to resolve the granular structure of physical space and time at ever smaller scales described by the coordinates (136).

The hierarchies of coordinates (136) and vacua (151) will have an impact on the important recurrence theory of Poincaré [43] and its implication to Boltzmann’s expression of thermodynamic entropy

\[ S = k \ln W \quad (152) \]

In particular, the conflict between Poincaré recurrence theory and thermodynamic irreversibility emphasized by Zermelo [134] should be reexamined. As was emphasized by Boltzmann [135], Poincaré recurrence theory cannot be applied to thermodynamic systems because such systems cannot be truly isolated. That is, the unattainable nature of absolute vacuum-vacuum (151) makes isolation of all physical systems impossible. The same limitation will apply to the entire universe when our universe is considered to be just one universe as an open system among others according to Everett’s many universe theory described by DeWitt [136]. The hierarchy of vacua (151) is in harmony with the inflationary theories of cosmology [137-140] and the finite pressure of Casimir vacuum [67] given by the modified van der Waals law of corresponding states [141]

\[ p_r = \frac{1}{Z_c} \left( \frac{T_r}{v_r} - 1/3 \right) \frac{9}{8v_r} + Z_c \frac{3}{8} \quad (153) \]

Clearly, the nature of thermodynamic irreversibility will be impacted by both the hierarchical definition of time discussed in [35] as well as the cascade of embedded statistical fields shown in Fig.1. For example, let us consider at EED scale a hydrodynamic system composed of \(10^5\) fluid elements each of which composed of \(10^5\) clusters of eddies. Next, let eddy, defined as the most probable size ensemble of molecular-clusters at ECD scale, be composed of \(10^2\) mean molecular-clusters each containing \(10^8\) molecules. Let us next assume that only the cluster \(i\) of the eddy \(j\) of the fluid element \(f\) contains molecules of type B and that all other clusters of all other eddies are composed of molecules of type A. The system is then allowed to fully mix at the molecular level. The thermodynamic reversibility will now require that \(10^8\) type B molecules to first become unmixed at hydrodynamic scale by leaving \((10^5-1)\) fluid elements and collecting in the fluid element \(f\). Next, all \(10^8\) type B molecules must leave \((10^5-1)\) clusters of eddy \(j\) and collect in the cluster \(c_{ijf}\). Finally \(10^5\) type B molecules must leave \((10^3-1)\) clusters of eddy \(j\) and collect in the cluster \(c_{ijf}\). Clearly the probability of such preferential motions to lead to immixing against all possible random motions will be exceedingly small. When the above hierarchy of mixing process is extended to yet smaller scales of molecular-dynamics, electrodynamics, and chromodynamics, the probability for reversibility becomes almost zero bordering impossibility in harmony with perceptions of Boltzmann [38]. The broader implications of the hierarchy of coordinate limits (136) to the internal set theory of Nelson [142] and the recurrence theory of Poincaré [43] require further future investigations.
10 Invariant Forms of the First Law of Thermodynamics and Definition of Entropy

In this section, Boltzmann statistical mechanics for ideal monatomic gas discussed in Section 7 will be applied to arrive at invariant forms of Boltzmann equation for entropy and the first law of thermodynamics. The results also suggest a new perspective of the nature of entropy by relating it to more fundamental microscopic parameters of the thermodynamic system.

As stated in Sec. 3, when the three velocities \( \langle u_x \rangle, \langle u_y \rangle, \langle u_z \rangle \) in (3) are all random the system is composed of an ensemble of molecular clusters and single molecules under equilibrium state and one obtains from (3)

\[
m < u^2_{x,y,z} > = m < v^2_{x,y,z} > + m < v^2_{x,y,z} >
\]

\[
\hat{e}_t = \hat{e}_{kek} + \hat{e}_{pe} = \hat{e}_{pek} + p\hat{v}
\]  

(154)

as the sum of the internal translational and potential energies since \( < 2v_{x,y,z}v'_{x,y,z} > = 0 \). One next allows the monatomic particles to also possess both rotational \( \hat{e}_{reo} \) and vibrational (pulsational) \( \hat{e}_{vke} \) kinetic energy [31] and invokes Boltzmann's principle of equipartition of energy such that

\[
m < v^2_{x,y,z} > = \hat{e}_{kek} + \hat{e}_{pek} = \hat{e}_{vke} = \hat{e}_{pe}
\]  

(155)

With the results (154)-(155) the total energy of the particle could be expressed as

\[
\hat{h}_\beta = \hat{u}_\beta + p\hat{v}
\]  

(156)

where \( \hat{h}_\beta \) is the enthalpy, \( \hat{v}_\beta \) is the volume and

\[
\hat{u}_\beta = \hat{e}_{kek} + \hat{e}_{pek} = 3\hat{e}_{tke}
\]  

(157)

is the internal energy per molecule such that by (156)

\[
H_\beta = U_\beta + p\beta V_\beta
\]  

(158)

where \( V_\beta = N_\beta \hat{v}_\beta \), \( H_\beta = N_\beta \hat{h}_\beta \), and \( U_\beta = N_\beta \hat{u}_\beta \) are respectively volume, enthalpy, and internal energy.

In accordance with the perceptions of Helmholtz, one may view (158) as the first law of thermodynamics

\[
Q_\beta = U_\beta + W_\beta
\]  

(159)

when reversible heat and work are defined as

\[
Q_\beta = H_\beta = T_\beta S_\beta
\]  

(160)

and

\[
W_\beta = p_\beta V_\beta
\]  

(161)

and \( S_\beta \) is the entropy [31]. For an ideal gas (159) and (160) lead to

\[
Q_\beta = T_\beta S_\beta = U_\beta + p_\beta V = H_\beta
\]

\[
= 3N_\beta kT_\beta + N_\beta kT_\beta = 4N_\beta kT_\beta
\]  

(162)

When photons are considered as monatomic ideal gas integration of (81) and maximization of (58) are known to lead to internal energy and entropy [86-88]

\[
U_\beta = \frac{8\pi V_\beta (kT)^4}{15(hc)^3}, \quad S_\beta = \frac{32\pi V_\beta k^4 T^3}{45(hc)^3}
\]  

(163)

Since the Poisson coefficient for photon is \( \gamma = \hat{c}_p / \hat{c}_v = 4 / 3 \) the result (162) gives to

\[
T_\beta S_\beta = \frac{4}{3}U_\beta
\]  

(164)

in accordance with (163). Also, since for an ideal gas (157) and (159) give

\[
U_\beta = 3e_{tke} + 3p_\beta V = 3N_\beta kT_\beta
\]  

(165)

by (162) entropy per photon \( \hat{s}_k = S / N \) becomes

\[
\hat{s}_k = 4k
\]  

(166)

as compared to [88]

\[
\hat{s}_k = 3.6k
\]  

(167)

based on the classical model.

The discrepancy between (166) and (167) is due to the classical formulation for the number of photons in a given volume \( v_\beta \) [88]. It is possible to express the total potential energy from the integration of mean energy and number density of oscillators over spherical number density space as

\[
\int_0^N <e> dN = \int_0^N kT dN = NkT
\]

\[
= \int_0^N \bar{v}(4\pi N^2 e) dN = \int_0^N \frac{h v(8\pi N^2 / 3)}{e^{hv/kT} - 1} dN
\]  

(168)

\[
= \int_0^N \frac{h v(8\pi N^2 / 3)}{e^{hv/kT} - 1} dN
\]
One next considers the relation between the number of quantized oscillators in a cube of size $L = L_x = L_y = L_z$ as

$$N_{x+} = \frac{L \nu}{c}$$

and the isotropy condition

$$N^2 = N_{x+}^2 + N_{y+}^2 + N_{z+}^2 = 3N_{x+}$$

Because de Broglie “matter wave packets” [2] or Heisenberg-Kramers virtual oscillators [66] are now considered to represent actual particles of ideal gas according to (169) one requires integral numbers $n$ of the full wavelength $\lambda$, as opposed to the conventional half wavelength $\lambda/2$, to fit within the cavity length $L$. Substituting from (169)-(170) into (168) gives

$$NkT = \frac{8\pi \hbar}{3} \int_0^\infty \frac{v(L/c)^2 v^2}{e^{(hv/kT)/c}} - 1 \, dv = \int_0^\infty \frac{v^3}{e^{(hv/kT)/c} - 1} \, dv$$

Hence, the number of photons in volume $V$ is given by (171) as

$$N = \frac{8\pi \nu}{45} \left( \frac{kT}{\hbar c} \right)^3$$

as compared to the classical result [88]

$$N = 8\pi \nu \left( \frac{kT}{\hbar c} \right)^3 \times 2.404$$

Also, from (172) and (162) one obtains the internal energy

$$U = 3NkT$$

that leads to the expected specific heat at constant volume

$$\tilde{c}_v = 3R^e$$

that is in accordance with (106)

The result (174) only approximately agrees with the classical expression for the internal energy [87, 88]

$$U = \frac{\pi^4 NkT}{30 \zeta(3)}$$

that leads to

$$U = 2.701NkT$$

According to (171) the expression (176) should involve zeta function of 4 rather than 3 such that

$$U = \frac{\pi^4 NkT}{30 \zeta(4)} = 3NkT$$

that agrees with (174) and by (162) leads to the ideal gas law

$$pV = NkT$$

instead of [87]

$$pV = 0.900NkT$$

for equilibrium radiation.

Concerning the expressions (173)-(180) it was stated by Yourgrau et al. [87]

“The reader will not fail to recognize the close resemblance between relations and their counterparts pertaining to a classical ideal gas”

With the modified results (174) and (179) the correspondence between photon gas and classical ideal gas becomes exact thus closing the gap between radiation and gas theory discussed by Darrigol [143]. Also, the important relation between radiation pressure and internal energy [86-88]

$$p = U / 3V = \dot{u} / 3 \dot{v} = u / 3$$

is exactly satisfied by (174) and (179) and closely but approximately satisfied by (173) and (180).

Since Kelvin absolute temperature scale is identified as a length scale (90), thermodynamic temperature relates to spatial and hence temporal “measures” of the physical space [35] or Casimir vacuum [67]. Therefore, the hierarchy of limiting zero temperatures (150) will be related to hierarchy of “measures” applied to renormalize [124] “numbers” (137)-(138) defining coordinates (Fig.14) as discussed earlier [116]. For an ideal monatomic gas one has the relations $\tilde{c}_p = 4R^e$ and $\tilde{c}_v = 3R^e$ such that (162) reduces to the identity

$$4p = 3p + 1$$

Therefore, the mathematical relation (182) always holds for statistical fields of any scale (Fig.1) as the
thermodynamic temperature (150) approaches the “absolute zero” \( T_\beta \rightarrow 0 \) (\( T_\beta \rightarrow 1 \)) associated with coordinates of that particular scale (136).

The result (162) could be also obtained from the Gibbs equation

\[
T_\beta dS_\beta = dU_\beta + p_\beta dV_\beta - \sum_j \mu_{\beta j} dN_j \quad (183)
\]

or

\[
G_{\beta j} = H_{\beta j} - TS_{\beta j} \quad (192)
\]

one obtains from (191) and (185)

\[
\alpha = 1 \quad (193)
\]

The results (191) and (193) lead to the Euler equation

\[
TS_{\beta j} = U_{\beta j} + p_{\beta j} V_\beta - G_{\beta j} \quad (194)
\]

Summation of (194) over all energy levels results in

\[
TS_\beta = U_\beta + p_\beta V_\beta - G_\beta \quad (195)
\]

Since the principle of equipartition of energy of Boltzmann (155) by (162) leads to

\[
H_{\beta j} = Q_{\beta j} = TS_{\beta j} = 4N_{\beta j}kT = 4N_{\beta j}e_{jke} \quad (196)
\]

in view of (30b) the definition of entropy is introduced as Boltzmann factor

\[
\hat{s}_{\beta j} = 4k_{\beta j} = 4m_{\beta j}\langle u_{\beta j}^2 \rangle^{1/2}\langle v_{\beta j}^2 \rangle^{1/2} \quad (197)
\]

that by (36) when multiplied by “temperature” \( T_\beta = \langle x_{\beta j}^2 \rangle^{1/2} \) gives the “atomic” enthalpy of the energy level \( j \) at equilibrium

\[
\hat{h}_{\beta j} = 4k_{\beta j} T_{\beta j} = 4kT_{\beta j} = \hat{s}_{\beta j} T_{\beta j} = \hat{s}_{\beta j} T_{\beta j} \quad (198)
\]

At EKD scale \( \beta = k \), the results (196)-(198) are in accordance with (166) for photon gas. Also, (198) is parallel to the way the universal Boltzmann constant \( k \) times equilibrium temperature \( T \) relates to the most probable atomic internal energy

\[
\hat{u}_{\text{max}} = 3kT_\beta = m_\beta < u_{\text{max}}^2 > \quad (199)
\]

In (198), it is assumed that at thermodynamic equilibrium the temperatures of all energy levels are identical \( T_\alpha = T_\beta \) such that both Planck energy spectrum (Fig.5) and Maxwell-Boltzmann speed spectrum (Fig.13) correspond to an isothermal statistical field at a given thermodynamic temperature \( T_\beta \). Hence, in (199) the most probable atomic internal energy is given as the product of temperature and the universal Boltzmann constant \( k \) (36). In (198) the atomic enthalpy is given as product of temperature and atomic entropy (197) of the energy level \( j \).
The first law of thermodynamics (162) when expressed per molecule, per unit mole, and per unit mass appears as

\[
\begin{align*}
\hat{h}_\beta &= \hat{u}_\beta + \hat{R} T_\beta = \hat{u}_\beta + k T_\beta = \hat{u}_\beta + p_\beta \nabla \beta \\
\hat{h}_\beta &= \hat{u}_\beta + \hat{R} T_\beta = \hat{u}_\beta + R^{-1} T_\beta = \hat{u}_\beta + p_\beta \nabla \beta \\
\vec{L}_{\beta} &= \vec{u}_\beta + \hat{R} T_{\beta j} = \vec{u}_\beta + R^{-1} T_{\beta j} = \vec{u}_\beta + p_\beta \nabla \beta
\end{align*}
\]

(200a, b, c)

Thus one may express the universal gas constant \( R, R, R \) per molecule, per unit mole, and per unit mass as

\[
\begin{align*}
\hat{R} &= R^o / N^o = k \\
\vec{R} &= R^o = k N^o \\
\vec{R}_j &= R^o / \vec{N}_j = R^o / (N^o m_j) = k / m_j
\end{align*}
\]

(201a, b, c)

11 Invariant Schrödinger and Dirac Wave Equations

The fact that the energy spectrum of equilibrium isotropic turbulence is given by Planck distribution (Figs.5, 6) is a strong evidence for quantum mechanical foundation of turbulence [33, 35]. This is further supported by derivation of invariant Schrödinger equation from invariant Bernoulli equation [35]. Hydrodynamic foundation of Schrödinger equation suggests that Bohr’s stationary states [66] will correspond to the statistically stationary sizes of clusters, de Broglie wave packets, which will be governed by Maxwell-Boltzmann distribution function (111) as shown in Fig.9. Different energy levels of quantum mechanics are identified as different size atoms (elements). For example, in ESD field one views the transfer of a sub-particle (electron) from a small rapidly oscillating atom \( j \) to a large slowly oscillating atom \( i \) as transition from the high energy level \( j \) to the low energy level \( i \), see Fig.12, as schematically shown in Fig.15.

The gradient of the action (203) gives volumetric momentum density in harmony with the classical results [3]

\[
\nabla S_{\beta}(x, t) = -\rho_{\beta} \nabla \Phi_{\beta} = \rho_{\beta} v_{\beta} = p_{\beta}
\]

(205)

In a recent study [35] it was shown that one can directly derive from the invariant Bernoulli equation (202) the invariant time-independent Schrödinger equation [35, 145, 146]

\[
\begin{align*}
\nabla^2 \psi_{\beta} + \frac{8 \pi^2 m_{\beta}}{h^2} (\vec{E}_{\beta} - \vec{U}_{\beta}) \psi_{\beta} &= 0 \\
\end{align*}
\]

(206)

as well as the invariant time-dependent Schrödinger equation

\[
\begin{align*}
\hbar \frac{\partial \Psi_{\beta}}{\partial t} + \frac{\hbar^2}{2m_{\beta}} \nabla^2 \Psi_{\beta} - \vec{U}_{\beta} \Psi_{\beta} &= 0
\end{align*}
\]

(207)

that governs the dynamics of particles from cosmic to tachyonic scales (Fig.1) [35, 146]. Since \( \vec{E}_{\beta} = \vec{T}_{\beta} + \vec{U}_{\beta} \) [35] Schrödinger equation (206) gives the stationary states of particles that are trapped within de Broglie wave packet under the potential acting as Poincaré stress. In view of the role of the pressure \( U_{\beta} = p_{\beta} = n_{\beta} \vec{U}_{\beta} \) [35] as the potential in (206)-(207), anticipation of an external pressure or stress as being the cause of particle stability by Poincaré [111-113] is a testimony to the true genius of this great mathematician, physicist, and philosopher.

One may now introduce a new paradigm of the physical foundation of quantum mechanics according to which Bohr’s stationary states [66] will correspond to the statistically stationary sizes of atoms, de Broglie atomic wave packets, which will be governed by Maxwell-Boltzmann distribution function (111) as shown in Fig.9. Different energy levels of quantum mechanics are identified as different size atoms (elements). For example, in ESD field one views the transfer of a sub-particle (electron) from a small rapidly oscillating atom \( j \) to a large slowly oscillating atom \( i \) as transition from the high energy level \( j \) to the low energy level \( i \), see Fig.12, as schematically shown in Fig.15.

Fig.15 Transition of electron \( e_{ij} \) from atom-\( j \) to atom-\( i \) leading to emission of photon \( k_{ij} \) [51].
Such a transition will be accompanied with emission of a “photon” that will carry away the excess energy [34]

\[ \Delta \varepsilon_{ji} = \varepsilon_j - \varepsilon_i = \hbar (\nu_j - \nu_i) \]  

(208)

in harmony with Bohr’s theory of atomic spectra [66]. Therefore, the reason for the quantum nature of “atomic” energy spectra in equilibrium isotropic electrodynamics field is that transitions can only occur between atoms with energy levels that must satisfy the criterion of stationarity imposed by Maxwell-Boltzmann speed distribution function [35, 131]. The results (80) and (111) as well as Figs.8, 12, and 15 suggest a generalized scale invariant description of transitions between energy levels of a statistical field at arbitrary scale \( \beta \) schematically shown in Fig.16.

![Fig.16 Transition of atom \( a_j \) from element-\( j \) to element-\( i \) leading to emission of a sub-atomic particle \( s_{ji} \).](image)

According to Fig.16, transition from high energy level \( j \) to low energy level \( i \) of “atomic” particle will lead to emission of a “sub-particle” at \( \beta-2 \) scale. If such emissions are induced or stimulated rather than spontaneous then one obtains coherent tachyon rays, neutrino rays, photon rays (laser), sub-particles rays (electron rays), atomic rays, molecular rays, . . . as discussed in [51]. For non-stationary relativistic fields, it was recently shown that by relating Schrödinger and Dirac wave functions as \( \Psi_\beta = \Psi_{\beta}(z_j)\Psi_{\beta0}(x_j,t) \) with

\[ \Psi_{\beta0}(z_j) = \exp[(\sqrt{im_{\beta j}w_{\beta j}^2/h})z_j] \]  

(209)

\( \Psi_{\beta0} = \exp(\imath \alpha_{\beta} E_{\beta} t / \hbar) \exp(-\imath \alpha_{\beta} \omega_{\beta} x / \hbar) \), and the total energy defined as \( E_{\beta j} = 2m_{\beta j}w_{\beta j}^2 \), one can derive from the equation of motion (24) the scale invariant relativistic wave equation [35]

\[ i\hbar \left( \frac{1}{c} \frac{\partial \Psi_{\beta0}}{\partial t} + \alpha_j \frac{\partial \Psi_{\beta0}}{\partial x_j} \right) + (\alpha_{n \mu} m_{\nu \mu} w_{\nu \mu}) \Psi_{\beta0} = 0 \]  

(210)

At the scale below Casimir vacuum [67] (ETD in Fig.1), when \( w_{\nu \mu} = \nu = u_0 = c \) is the speed of light, equation (210) becomes Dirac relativistic wave equation for electron [147, 35]

\[ i\hbar \left( \frac{1}{c} \frac{\partial \Psi_{\beta0}}{\partial t} + \alpha_j \frac{\partial \Psi_{\beta0}}{\partial x_j} \right) + (\alpha_{n \mu} m_{\nu \mu} w_{\nu \mu}) \Psi_{\beta0} = 0 \]  

(211)

Therefore, the theory further described in [35] also provides a hydrodynamic foundation of Dirac relativistic wave equation for massive particles in the presence of spin.

12 Concluding Remarks

A scale invariant model of statistical mechanics and its implications to the physical foundations of thermodynamics and kinetic theory of ideal gas were examined. Boltzmann’s method of combinatorics was employed to derive invariant forms of Planck energy and Maxwell-Boltzmann speed distribution functions. The impact of Poincaré recurrence theory on the problem of irreversibility in thermodynamics was discussed. The coincidence of normalized spacings between zeros of Riemann zeta function and normalized Maxwell-Boltzmann distribution function was employed to derive invariant forms of both Schrödinger as well as Dirac wave equations were described. The universal nature of turbulence across broad range of spatio-temporal scales is in harmony with occurrence of fractals in physical science emphasized by Takayasu [148].

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References:
