The general orthogonal projection on a regular surface

Zhi-Teng Zheng  
National Chiayi University  
Department of Applied Mathematics  
Chia-Yi City 600  
Taiwan  
s0932291@mail.nctu.edu.tw

Sheng-Gwo Chen*  
National Chiayi University  
Department of Applied Mathematics  
Chia-Yi City 600  
Taiwan  
Corresponding author: csg@mail.nctu.edu.tw

Abstract: In 2005, Kasap et. al. presented a simple numerical method for estimating the geodesic with boundary conditions on regular surfaces by the system of geodesic equations and the discrete numerical methods. In this paper, we extend Kasap’s approach to improve the general orthogonal projection problem on a regular surface.

Key–Words: general orthogonal projection, regular surface, geodesic, Kasap’s method.

1 Introduction

Finding the shortest path between two points in \( \mathbb{R}^n \) is a traditional problem in geometry. Obviously, the solution of this problem in \( \mathbb{R}^n \) is the straight line between these points. However, it is a difficult problem that the shortest path must lie on a regular surface. In differential geometry and Riemannian geometry, the shortest curve between two points on a regular surface is a geodesic on the regular surface and it always satisfies the system of geodesic equations. From the viewpoint of the method of variation, a curve \( c \) is a geodesic on a surface if it is a critical point of the energy of a proper variation of the curve \( c \). By the theory of intrinsic geometry, the geodesic curvature of a geodesic vanishes at each point. More theories of geodesic can be found in any book of differential geometry (e.g. [1]).

Geodesics on regular surfaces have many important applications in many fields. In 2008, Paluszny[6] presented a method of polynomial path through geodesics, Sánchez-Reyes proposed a method of constrained design from geodesics and Sprynski[12] developed a method of surface reconstruction by geodesic interpolation. Furthermore, the geodesic on a regular surface with general Riemannian metric can be used in computer vision and image etc..

Many discrete methods approach geodesics or the shortest path on tessellated surfaces[14], polygonal surfaces[7] and triangular meshes[5, 13] are proposed. Only a few method can estimate geodesics on regular surfaces directly. In 2002, Pottmann[8] et.al approached the geodesic on a regular surface via the penalty method and the method of variational calculus. In 2005, Kasap[4] et. al. presented a numerical approach for the problem of geodesic with boundary condition by the finite element method. In 2010, Chen[2] proposed a new approach to improve the geodesic problem on parametric surfaces. All of these methods transfer the geodesic problem to a system of nonlinear equations. All of them have some advantages and some shortcomings when we estimate the geodesic on surfaces.

In this note, we shall introduce Kasap’s numerical method for estimating a geodesic on a regular surface. Furthermore, we improve the general orthogonal problem on a regular surface. In section 2, we introduce what geodesic is on regular surfaces and Kasap’s method for approaching geodesics. In section 3, we improve the general orthogonal projection problem, that is finding the shortest path from a point \( p \) to a smooth curves \( c \) on a surface. In fact, the intersection of this shortest path and the curve on the surface is the orthogonal projection point from \( p \) to \( c \). We shows some computations of the general orthogonal projection problem in the last section.

2 Geodesics on regular surfaces and the Kasap’s method

In this section, we shall introduce the Kasap’s method for computing the geodesic on a regular surface. Let \( S \) be a regular surface and let \( x(u, v) \) be a parametrization of \( S \), \( x : U \rightarrow S \). A smooth curve \( \alpha \) lying entirely on \( S \) can be represented by two smooth function

\[ u = u(s) \quad \text{and} \quad v = v(s), \]
where \( s \) is the curve parameter and \( \alpha(s) = x(u(s), v(s)) \). The arc length and the energy of curve \( \alpha \) are

\[
L = \int \sqrt{(g_{11}u'^2 + g_{12}uv' + g_{22}v'^2)} \, ds
\]

and

\[
E = \frac{1}{2} \int (g_{11}u'^2 + g_{12}uv' + g_{22}v'^2) \, ds
\]

where \( \dot{u} = \frac{d}{ds}u(s), \dot{v} = \frac{d}{ds}v(s), g_{ij} = x_u \cdot x_u, g_{ij} = g_{ij} = g_{ij} = g_{ij} \).

Let \( p, q \) be two points on a regular surface \( S \). Suppose that there exists a unique curve \( \alpha : [a, b] \to S \) is represented by \( (u(s), v(s)) \), which is the shortest path between \( \alpha(a) = p \) and \( \alpha(b) = q \) on \( S \). By the standard method of variational calculus, the functions \( u(s), v(s) \) satisfy the Euler equations, that is,

\[
\begin{cases}
\frac{d}{ds}F_{u} - \frac{\partial F}{\partial u} = 0, \\
\frac{d}{ds}F_{v} - \frac{\partial F}{\partial v} = 0,
\end{cases}
\]

where \( F = g_{11}u'^2 + 2g_{12}uv' + g_{22}v'^2 \). After some computations, equations (3) can be rewritten as

\[
\begin{align*}
\ddot{u} + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2 &= 0, \\
\ddot{v} + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2 &= 0.
\end{align*}
\]

The coefficients \( \Gamma_{jk}^i \) in equations (4) are the Christoffel symbols which defined as

\[
\begin{align*}
\Gamma_{11}^1 F + \Gamma_{11}^2 G &= x_{uu} \cdot x_u = \frac{1}{2} E_u, \\
\Gamma_{11}^1 F + \Gamma_{11}^2 G &= x_{uu} \cdot x_u = \frac{1}{2} E_v, \\
\Gamma_{12}^1 F + \Gamma_{12}^2 G &= x_{uv} \cdot x_u = \frac{1}{2} G_u, \\
\Gamma_{12}^1 F + \Gamma_{12}^2 G &= x_{uv} \cdot x_u = \frac{1}{2} G_v, \\
\Gamma_{22}^1 F + \Gamma_{22}^2 G &= x_{vv} \cdot x_u = \frac{1}{2} G_v, \\
\Gamma_{22}^1 F + \Gamma_{22}^2 G &= x_{vv} \cdot x_u = \frac{1}{2} G_v,
\end{align*}
\]

where \( E, F \) and \( G \) are the coefficients of first fundamental form of \( S \). The system of differential equation (4) is called the system of geodesic equations on \( S \) and the solutions of equation (4) are geodesics on \( S \). In other words, the shortest path between two points on a surface is a geodesic between these points.

From the central difference approximation, the 1st and 2nd derivatives of a smooth real function \( f \) can be approach by

\[
\frac{d}{dt} f(t) \approx \frac{f(t + h) - f(t - h)}{2h}
\]

and

\[
\frac{d^2}{dt^2} f(t) \approx \frac{f(t + h) - 2f(t) + f(t + h)}{h^2}
\]

Using the method of finite-difference and the central difference approximation, equation (4) can be approximated by

\[
\begin{align*}
&\frac{1}{h^2} \left[ (u_{i+1} - 2u_i + u_{i-1}) + f(\cdot, \cdot) = 0, \\
&\frac{1}{h^2} \left[ (v_{i+1} - 2v_i + v_{i-1}) + g(\cdot, \cdot) = 0,
\end{align*}
\]

for each \( i \in \{1, 2, \ldots, N\} \), where \( h = \frac{b-a}{N+1} \), \( \alpha(a) = x(u_0, v_0) \) and \( \alpha(b) = x(u_N, v_N) \). The functions \( f(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) are

\[
\begin{align*}
f(\cdot, \cdot) &= \Gamma_{11}^1 (u_i, v_i) \cdot \left( \frac{(u_{i+1} - u_{i-1})^2}{2h} \right) \\
&\quad + \Gamma_{12}^1 (u_i, v_i) \cdot \left( \frac{(u_{i+1} - u_{i-1})}{2h} \cdot \frac{(v_{i+1} - v_{i-1})}{2h} \right) \\
&\quad + \Gamma_{22}^1 (u_i, v_i) \cdot \left( \frac{(v_{i+1} - v_{i-1})^2}{2h} \right),
\end{align*}
\]

and

\[
\begin{align*}
g(\cdot, \cdot) &= \Gamma_{11}^2 (u_i, v_i) \cdot \left( \frac{(u_{i+1} - u_{i-1})^2}{2h} \right) \\
&\quad + \Gamma_{12}^2 (u_i, v_i) \cdot \left( \frac{(u_{i+1} - u_{i-1})}{2h} \cdot \frac{(v_{i+1} - v_{i-1})}{2h} \right) \\
&\quad + \Gamma_{22}^2 (u_i, v_i) \cdot \left( \frac{(v_{i+1} - v_{i-1})^2}{2h} \right).
\end{align*}
\]

And we have the following theorem.

**Theorem 1.** Let \( S \) be a regular surface with a parametrization \( x \) and let \( p, q \) be two points on \( S \). Suppose that \( \gamma : [a, b] \to S \) is a geodesic between \( p \) and \( q \) without any conjugate point. If \( \{(u_i, v_i)\}_{i=0}^{N+1} \) is a solution of equation (8) with \( x(u_0, v_0) = p \) and \( x(u_N, v_N) = q \), then the polyline with vertices \( x(u_i, v_i) \) \( i=0 \) approaches \( \gamma \) when \( n \rightarrow \infty \).

The equations (8) are nonlinear equations of variables \( u_{i-1}, u_i, u_{i+1}, v_{i-1}, v_i, v_{i+1} \). They can be solved by the iterative method, Newton’s method or other numerical methods. Kasap’s[4] numerical method is an elegant method for estimating the geodesic between two given points on a surface. In Kasap’s paper, they have shown that the discritized geodesic equations (8) can be solved very quickly and accuracy.

### 3 The general orthogonal projection on a regular surface

Let us consider the general orthogonal projection on a regular surface. Suppose that \( S \) is a regular surface with a parametrization \( x(u, v) \), \( p \) is a point on \( S \) and \( c \) is a (closed) curve on \( S \) such that \( p \notin c \). The general orthogonal projection problem is finding the shortest path from \( p \) to \( c \) on the surface \( S \). An intuitive traditional method is discretizing the curve \( c \) to a sequence of points and finding the shortest path from \( p \) to each point in that sequence. First, we introduce a traditional method for improving this problem.

**Algorithm 2.** (Traditional method)
1. Digitizing the curve \( c \) to a sequence of points 
   \[ \{p_i\}_{i=1}^{n} = 0. \]

2. Estimating the minimal geodesic \( \gamma_i \) between \( p \) and \( p_i \) for each \( i \).

3. The shortest curve in \( \{\gamma_i\} \) approaches the shortest path from \( p \) to the curve \( c \) when \( n \) is large enough.

Although this traditional algorithm is simple and convergent, it is too expensive to estimate all of these shortest paths. However, Kasap’s method can be applied to improve this problem easily. Since the shortest path from \( p \) to \( c \) is always orthogonal to the curve \( c \) at the intersection point, we have

**Theorem 3.** Let \( S \) be a regular surface with a parametrization \( x(u, v) : U \to S \), \( p \) be a point on \( x(U) \) and let \( c \) be a curve on \( x(U) \). Suppose that 
\[ \gamma : [a, b] \to S \] is the shortest path from \( p \) to \( c \), then 
\[ \gamma'(b) \perp c'(t_0) \] for some \( t_0 \) such that \( c(t_0) \) is the intersection of \( \gamma \) and \( c \).

Assume that the intersection point of \( \gamma \) and \( c \) is \( q \) and \( \gamma(b) = c(t_0) = q \). Because \( \frac{\gamma(b)-\gamma(b-h)}{h} \) approaches \( \gamma'(b) \) when \( h \) tends to 0, theorem 3 implies that \( (\gamma'(b)-\gamma(b-h)) \perp c'(t_0) \) when \( h \) is small enough. Hence, the general orthogonal projection problem can be improve by the extended Kasap’s method in algorithm 4.

**Algorithm 4.** (Extended Kasap’s algorithm) Let \( S \) be a regular surface with a parametrization \( x(u, v) \). Given a point \( p \) and a curve \( c = c(s) \) on the surface \( S \). If the sequence \( \{\bar{x}_0, \bar{u}_1 \cdots \bar{u}_n, \bar{v}_0, \bar{v}_1 \cdots \bar{v}_n, \bar{s}_0\} \) satisfies the equation (10),

\[
\begin{align*}
\mathbf{x}(u_0, v_0) &= p \\
(\mathbf{x}(c(s_0)) - \mathbf{x}(u_n, v_n)) \cdot c'(s_0) &= 0 \\
\frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + f(\cdots) &= 0 \\
\frac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1}) + g(\cdots) &= 0,
\end{align*}
\]

where \( f \) and \( g \) are defined in equation (9). Then the piecewise straight line with vertices \( \{\mathbf{x}(\bar{u}_i, \bar{v}_i)\}_{i=0}^{n} \) and \( \mathbf{x}(c(s_0)) \) approaches the shortest path from \( p \) to \( c \) on \( S \).

In other words, Kasap’ method can be improve the general orthogonal projection problem when it considers the additional condition 
\[ (\mathbf{x}(c(s_0)) - \mathbf{x}(u_n, v_n)) \cdot c'(s_0) = 0. \]

4 Conclusion

We show some simulations of the general projection problem by the extended Kasap’s method on a sphere, a torus and a face model. In figures 1 and 2, the value \( n \) is 20. Obviously, the computed curves are close to the shortest geodesic from \( p \) to \( c \) on the surfaces. In figure 3, the computed curve also approaches the exact solution of the general orthogonal problem on a surface. Of course, the computed curves in all figures are orthogonal to the curve on surfaces. Although the extend Kasap’s method can improve the general orthogonal projection problem easily, it is too expensive when the value \( n \) is large.

![Figure 1: The shortest path between a point and a curve on a sphere.](image1.png)

![Figure 2: The shortest path between a point and a curve on a torus.](image2.png)

![Figure 3: The shortest path between a point and a curve on a NURBS surface.](image3.png)
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