Improved Estimation of the Size of the State Space of Petri Nets for the Analysis of Reachability Problems

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Abstract: - The complete exploration of large system states modeled with Petri net (PN) systems demonstrates state space explosion, limiting reachability analysis. The calculation of the number of states (samples) necessary to conduct a statistical hypothesis test for reachability problems of PN systems is discussed in this paper. The calculation is based on the estimation of the size of the state space. This paper presents two estimators of the size of the state-space for bounded PN systems. The estimators are improved extensions of existing ones and are based on the simple elimination of unreachable markings. The estimations will be used in the calculation of the number of necessary samples to collect in the partial exploration of the state space. The improved estimation obtained in one example presents larger benefits than with the former estimator.

Key-Words: - Bounded Petri nets, State space size, Reachability, Size estimation, Hypothesis test

1 Introduction
Models of system behavior called Petri nets (PN) are used for the analysis of several properties, like reachability, liveness and boundeness, but its main and evident limitation is the state space explosion when working on large systems.

Knowing in advance the limitation in the analysis should put us in a direction where we start searching for estimated solutions instead of struggling for precise answers.

Our research is about determining the appropriateness of using statistical hypothesis tests as the analysis technique for reachability problems of PN systems. For this purpose, an estimation of the size of the state space of the PN system is required.

This paper presents two estimators of the size of the state spaces generated with bounded PN systems. The estimators are improved versions of existing ones and they take into account the elimination of states which are not reached by the PN system.

Works related to the elimination of the number of states to explore exist in the form of partial-order methods [8], which are path-based reductions exploiting redundancy, exploring reduced sequences of actions after identifying linearization of their independent actions; however, their objective is not related to the estimation of the size of the state space. Our estimation of the size of the state space is based on the elimination of unreachable markings and was inspired from those methods, taking in consideration their approaches.

A good estimation is required in order to calculate the number of states necessary to explore (samples) and to conduct a hypothesis test.

With the result of the calculation we perform a partial exploration of the state-space with a heuristic algorithm [4-5]. The algorithm engages in a guided depth-first search of a target marking, and after visiting certain amount of states we decide with a confident margin of probability on a postulated hypothesis test if the target marking exists or not in the entire state-space, giving a margin of certainty and closure to our reachability analysis.

This is the presentation of a work in process, therefore we will limit exclusively to the explanation of the improved estimation of the size of the state space generated with PN systems and briefly present some immediate results generated with the estimations.

This paper is self-contained and arranged in the following order: section two contains a small theoretical background about hypothesis test and PN systems. In the section three the existing estimators are reviewed. The proposed and improved state space size estimators are presented in the section four. Section five contains comparative results of our estimation against the former estimations. At the end are the conclusions of the paper.
2 Hypothesis Tests and PN Systems

A statistical hypothesis test is a method using observed data from a controlled (or uncontrolled) experiment for making decisions about the acceptance or not of specific characteristics in the entire population. In our research we will use them to provide certainty and closure to the results of the partial exploration in reachability problems of PN systems for determining the existence or not of undesired states, and later determine the appropriateness of this analysis technique.

First we have to calculate the number \( n \) of states we need to explore (the number of samples) from the state space of the PN system. The formula to calculate \( n \) uses the size of the entire state space. Since the size of the state space is large and unknown, we use estimations. In theory, the better our estimation is, the lower the number of states we need to explore without compromising the veracity of the hypothesis test will be.

For the calculation of the number of states that we need to explore, we assume that the cardinality in every marking of the state space is a random variable with specific probability distribution.

Taken from [1], a classical formula to calculate the number of samples \( n \) in a large population for a proportion \( p \) is in the formula (1), where \( Z \) is the confidence level (for example \( Z=1.29 \) for 99%), and \( e \) the acceptable error in the interval (for example \( e=0.04 \) means we accept a difference up to 4% in the proportion).

\[
n = \frac{Z^2 \cdot p(1-p)}{e^2}
\]  

(1)

The result from (1) can be improved if the size \( N \) of the entire state space is known (or estimated). The new calculation is in (2).

\[
n = \frac{N}{1 + \frac{(n-1)}{N}}
\]  

(2)

Then, we transform our reachability problem of PN in terms of deciding with an hypothesis test for proportions, based on the sampled states, if there is enough evidence indicating that the states being searched exists or not in the state space.

It is assumed the reader is familiar with the PN theory, but a brief introduction will be given next on the concepts required for the understanding of this paper. For additional reference and proofs of the theorems used in section four we address the reader to [2-3, 6-7, 9-10].

2.1 Petri Nets

A Petri net model is a tuple \( N = (P, T, I, O, Q) \), where \( P \) is a finite non-empty set of \( i \) conditions called places, \( T \) is a finite non-empty set of \( j \) events called transitions, \( I \) is the set of directed arcs connecting places to transitions, \( O \) the set of directed arcs connecting transitions to places and \( Q \) is a capacity function for the places mapping \( P \rightarrow Z^+ \) (for the ease of explanation we sometimes use the nomenclature \( q_1, q_2, ..., q_i \)). Places are graphically represented by circles, transitions by rectangles and all directed arcs by arrows. A PN is called pure when there are no self-loops. The pre-conditions of a transition \( t \) are in the pre-set of input places \( \bullet t \) and the post-conditions in the post-set \( t \bullet \).

The pre-events of a place \( p \) are in the pre-set \( p \bullet \) and the post-events in the post-set \( p \bullet \).

In this paper a forking structure is any transition with \( |p\bullet|>1 \) and a choice structure is any place with \( |p\bullet|>1 \).

The way to represent a state of the system is by putting tokens in the corresponding places. Tokens are black dots that exist only in the places. The function \( m \) called marking maps \( P \rightarrow Z^+ \), and \( m_0 \) is the initial marking. A PN with initial marking is called a PN system and will be denoted by \((N, m_0)\). We say that a place \( p \) is marked iff \( m(p) > 0 \). The finite set of all possible markings (i.e. the state space) of \((N, m_0)\) is denoted by \( R \).

The number of states that a PN system can generate depends on the input and output arcs, the initial marking and the way how the occurrence of transitions is specified. Occurrence of single transition is carried out for the generation of the complete state space of a PN system and mainly used for analysis purposes. Occurrence of concurrent transitions is implemented for generating possibly not-complete state spaces and it is assumed in the rest of this paper except if specified differently.

The occurrence of transitions in PN systems takes no time. A Timed PN (TPN) associates the occurrence of transitions to specific units of times. In this paper the occurrence time of the transitions will be considered zero except if specified differently.

Depending on the modelling requirements, the theory of PN defines different tokens capacity in the places through the capacity function \( Q \). Elementary PN (EPN) have capacity of one token in every place. Place/Transitions Nets (PTN) can assume different
capacities in the places, and general PN (PN) allow infinite capacity in all places.

The sets of arcs I and O can be represented with the pre-incident matrix and the post-incident matrix respectively having both i rows and j columns, with values of I = [I(p, t)] and O = [O(p, t)] and a marking m as a vector \( \mathbf{m} \in \mathbb{Z}^i \) defining the state of the system.

The token game refers to the way how the dynamic behavior of the system is described with the markings evolution (the removal of existing tokens and the creation of new tokens), according to the firing of enable transitions. The enabling rule is defined as: a transition \( t^a \in T \) is said to be enabled at a given marking \( m \) if every of its input places has at least as many tokens as the weight of the arcs joining it and every of its output places has a number of tokens smaller than the sum of their current marking plus the weight of the arc connecting them.

Contact-freeness exists in a PN system when transitions are never refrained from being enabled because the post-conditions were not fulfilled. General PN only stands for fulfillment of pre-conditions for the enabling of transitions.

A transition is called fireable if it is enabled. A fireable transition may fire, eliminating the marking \( m \) and creating the new marking \( m' \); i.e. for all enabled transitions \( t^a, I(p, t^a) \) tokens are removed from all places \( p \in I(t^a) \), and \( O(p, t^a) \) tokens are added to the places \( p \in O(t^a) \).

Using the initial marking vector \( \mathbf{m}_0 \), the next marking vector \( \mathbf{m} \) is mathematically calculated with the State Transition Function \( \mathbf{m} = \mathbf{m}_0 + (O - I) \times \mathbf{\sigma} \). In this formula, \( \mathbf{\sigma} \) is a firing count vector representing the number of times every transition has fired.

A PN system is bounded if the number of tokens in every marking of its state space is equal or less than a constant number. A PN system is conservative if the number of tokens is the same in all the markings of its state space.

3.1 State Space Size by Structural Deduction

From the previous description of EPN, we can easily estimate the number of states with

\[
2^i
\]

which is in exponential relation with the number of places.

On the other hand, the number of states in a PTN depends on the capacity of each place. Let us assume a PTN with three places and the following capacities: \( q_1 = 2 \), \( q_2 = 2 \) and \( q_3 = 3 \). The maximal number of possible states is calculated with \( 3^2 \times 4^1 \).

In general, the estimator of the number of states in a PTN is given by

\[
\prod_{i=1}^{j} (q_{\alpha} + 1)
\]

where \( q_{\alpha} \) is the tokens capacity of the \( \alpha \)th place.

It is important to mention that the number of possible states is undetermined whenever there is at least one place with infinite tokens capacity. In this way, the maximal number of possible states for general PN is undetermined.

3.2 State Space Size by Behavior Deduction

A conservative PN system is such that there is always a constant number \( \tau \) of tokens, i.e. the number of tokens in the initial state remains the same for any evolution of states. The works presented in [11-12] estimates the number of states in conservative PN system with

\[
\frac{(i + \tau - 1)!}{\tau!(i-1)!}
\]

where \( i \) is the number of places and \( \tau \) the number tokens.

From the formula in (5) it is easy to obtain the number of possible states in bounded PN systems. For a bounded PN system with a maximal number \( \beta \) of tokens that can exist in any evolution of states, the estimator is
where \( \tau \) is the number of tokens.

### 4 Improved Estimation

Four simple and general estimators were explained in the last section, all considering a limit in either the tokens capacity of the places or in the entire PN. However, these metrics do not always exist or are unknown to the analyst in real-life problems.

Here we present a more accurate estimation method through the exclusion of some possibly unreachable states, deducted from the structure of the PN system and its contact-free behavior.

#### 4.1 State Space Size by Unreachable Markings Deduction

For the ease of understanding let us declare the function \( m_t : P \rightarrow \mathbb{Z}^+ \) to indicate the number of tokens in the places at a target marking \( m_t \).

A forking-transition is such that \( |t^\ast|>1 \) and \( \{ p \in t^\ast \mid p \notin t' \ast \} \) (to any other transition) and \( |p^\ast|=1 \) (just one posterior-transitions) is a subset of forked-places.

Let us define the case-A; a general PN having the fork structure shown in the Fig.1, where \( t_1 \) is the forking-transition, \( p_2 \) and \( p_3 \) the forked-places, \( t_2 \) and \( t_3 \) the posterior-transitions, \( p_4 \) and \( p_5 \) the posterior-places; tokens capacity in all places is infinite and transition time is zero.

![Fig.1 The Fundamental Fork Structure](image)

**Theorem 1:** “For any initial marking with \( m(p_2)=m(p_3) \), a target marking where \( m(p_2)=1 \) and \( m(p_3)=2 \) is not reachable and in general any target marking where \( m(p_2) \neq m(p_3) \) is also not reachable.”

Tokens are always simultaneously created in the places \( p_2 \) and \( p_3 \) and simultaneously eliminated under the previous conditions. With this fundamental knowledge the estimation of the number of possible states in PN systems can be improved if we modify the previous estimators according to the way how tokens are created and eliminated simultaneously from forked-places.

#### 4.2 Structure-based Improved Estimator

Let us define the case-B; it is the fork structure of the Fig.1 (case-A) with tokens capacity in all places of one and transition time is zero (an EPN).

**Theorem 2:** “For any initial marking with \( m(p_2)=m(p_3) \), a target marking where \( m(p_2)=0 \) and \( m(p_3)=1 \) is not reachable if contact-freeness always exists and in general any target marking where \( m(p_2) \neq m(p_3) \) is also not reachable.”

The estimation of the number of states only for the subnet is \( 2^4 \), but under the theorem 2, if we eliminate unreachable markings our estimation can be adjusted to be \( 2^4 \).

The reason for this adjustment is that the two forked-places \( p_2 \) and \( p_3 \) can be considered as one for the estimation. These two places are always simultaneously marked by the firing of the forking-transition \( t_1 \) since \( t_1^*={p_2,p_3} \) and \( t_1^*={p_2,p_3} \) and their tokens are simultaneously eliminated by their posterior-transitions \( t_2 \) and \( t_3 \) since \( |p_1^*|=1, |p_1^*|=1 \) and contact-freeness exists. Then there will be not four but only two markings coming from the two forked-places: \( \{(1,1), (0,0)\} \).

An improvement to the estimator in (3) of the number of possible states is given when fork structures inside an EPN presenting contact-freeness in their posterior-transitions exist. It is given by

\[
2^\mu \times 2^\varphi \tag{7}
\]

where \( \mu \) is the number of forking-transition and \( \varphi \) is the number of places which are not forked-places.

For a case-C, using the same fork structure of the Fig.1, with the same tokens capacity in all places and larger than one, and transition time of zero (a PTN).

**Theorem 3:** “For any initial marking with \( m(p_2)=m(p_3) \), a target marking where \( m(p_2)=1 \) and \( m(p_3)=2 \) is not reachable if contact-freeness always exists and in general any target marking where \( m(p_2) \neq m(p_3) \) is also not reachable.”

An improvement to the estimator in (4) of the number of possible states is given when fork structures inside a PTN presenting contact-freeness in their posterior-transitions exist. It is given by

\[
\prod_{\tau=1}^{\max\{|\varphi|\}} (\tau+1)^{\mu-\tau+1} \tag{8}
\]
where \( u_\tau \) is the number of forking-transition with \( \tau \) tokens capacity, and \( \phi_\tau \) is the number of places with \( \tau \) tokens capacity which are not forked-places.

5 Example

The bounded PN system in Fig. 2 will be used to explain our improved estimators and their usability. It belongs to a flexible manufacturing system (FMS) with four interconnected lines. The time associated in all transitions is assumed the same because the work stations are balanced in order to achieve a specific synchronic throughput. Initial marking in five places is one, making \( \text{card}(m_0) = 5 \).

![PN System of a FMS](image)

The table 1 shows the results obtained with the original estimators and our improved estimator. Two scenarios were simulated for the initial marking.

<table>
<thead>
<tr>
<th>( \text{card}(m_0) )</th>
<th>Original Estimation</th>
<th>Improved Estimation</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>16,777,216</td>
<td>1,048,576</td>
</tr>
<tr>
<td>10</td>
<td>282,429,536,481</td>
<td>3,486,784,401</td>
</tr>
</tbody>
</table>

A first and evident benefit with this improved estimation is the large reduction of the necessary computational resources (memory) as a justification for the implementation of our analysis technique.

The second benefit of having an improved estimation should be in the calculation of the number of states we need to explore. Although our model is rather small to appreciate this benefit, upon scalability into larger PN systems, this benefit becomes more evident.

Let us explain our method using the following two reachability problems to analyze: do the target markings with a sum of tokens in all its places of 3 and 5 exist? For this we assume that the sum of the number of tokens in every marking of the estimated state space is a random variable with normal distribution \( N(\hat{x}, \hat{s}^2) \) where \( \hat{x} = \beta / 2 \) (the half of the maximal number of tokens that can exist in the PN system) and \( \hat{s} = \hat{x} / 3 \).

The sum of the number of tokens in a target marking is \( c_t \). Let us define the probability of \( c_t \) as \( f_{N}(c_t; \hat{x}, \hat{s}^2) = p \), which we use as the proportion of markings in the state space with the same sum of the number of tokens as the target marking. The values of \( p \) for our two problems under the two scenarios are shown in the table 2.

<table>
<thead>
<tr>
<th>( \text{card}(m_0) )</th>
<th>( c_t = 3 )</th>
<th>( c_t = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.1613</td>
<td>0.2516</td>
</tr>
<tr>
<td>10</td>
<td>0.0859</td>
<td>0.2097</td>
</tr>
</tbody>
</table>

The results are consistent regarding obtaining a lower number of samples as the cardinality of the initial marking increases because the mean of the normal distribution is moving to the right.

Our hypothesis test for reachability problems of PN systems will use the previous amount of samples. The way how the improved estimations affect the veracity of the outcome of the hypothesis test, the appropriateness of our analysis technique and the reduction of computational resources is not covered in this paper.

6 Conclusions

We have presented an improvement in the estimation of the size of the state space generated with PN systems for a better calculation of the number of states necessary to explore for the hypothesis test of reachability problems.

However, a possible bias in the calculation is likely to exist due to the assumed normal distribution.
distribution of the the sum of the number of tokens in every marking of the state space, so it is that a better estimation of the distribution could yield to a better calculation and possibly fewer states to explore.

Finally, despite the results belong to a rather small PN system, its usability in very larger PN systems is straightforward.

References: