Cubic Superior Julia Sets

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1 Introduction

Julia sets are named after the French mathematician Gaston Julia (1893-1978). He was only 25 when he published his 199-page masterpiece in 1918, which made him famous in the mathematics center of his days. Julia sets are the striking examples of computational experiments that were far ahead of its time. These mathematical objects were seen when computer graphics became available [20]. Julia set is the place where all of the chaotic behavior of a complex function occurs [11, p. 221]. Julia sets live in complex plane and are non-empty (see [28], p. 28). Perhaps no computer experiments exceed in excitement, fascination and wonderment the graphical representations of Julia and Mandelbrot sets in a complex plane. That may be why researchers are always keen to discover more into it. For a detailed analysis of Julia and Mandelbrot sets, one may refer to complex dynamics and related work in [1, 5, 7-10, 13, 14, 19, and 28].

Julia sets may be computed for any complex map. In this paper, we focus our study on Julia sets for cubic polynomials. Many researchers have already generated, analyzed and applied cubic Julia sets. The puzzles and the tableaux techniques characterize the cubic polynomials with one bounded and one unbounded critical orbit having totally disconnected Julia sets. Para-puzzles describe the successive subdivision of parameter space. Branner (1994) [2] reviewed the construction of puzzles and para-puzzles and tableaux techniques.

Liaw (1998) [17] plotted the two-dimensional projections of the parameter spaces of the cubic mappings. The projection of the parameter points that have non-totally disconnected Julia sets can be seen as a combination of Mandelbrot-like sets. Further, Cheng and Liaw (1998) [6] demonstrated asymptotic similarity between the dynamic space (the Julia sets) and parameter spaces (the Mandelbrot set) of cubic polynomials. The parameter and dynamic spaces of cubic mappings consist of many small copies of the Mandelbrot-like sets and Julia sets, respectively of the standard quadratic mapping. Liaw (2001) [16], calculated the positions, sizes, and orientations of these small copies to understand the structure of the cubic mappings.

Yan, Liu and Zhu (1999), [30] put forward the theorems on the range of the Mandelbrot and Julia sets generated from a general complex cubic iteration. These theorems are important for plotting of the sets. Zakeri [31] wrote an interesting paper to deal with the one-parameter family of cubic polynomials. Tomova (2003) [29] proved theorems for the limit of Mandelbrot and Julia sets of higher orders of the form $z^n + c$ and cubic Mandelbrots and Julia sets for $z^3 + az + b$. Fu, Jiang and Yi (2006) [12] used fast computing algorithm to construct Mandelbrot and Julia sets of higher order. For further readings to understand the structure and connectedness of cubic mappings, one may refer to Branner and Hubbard [3, 4].

In 2004, Author, jointly with Kumar, [25] introduced superior iterations in nonlinear sciences, and gave new and considerably improved escape criterions for complex polynomials. Thus, Rani et al. computed superior Julia sets [15, 21, 23, 25, 27] and superior Mandelbrot sets [21, 22, 24, 26] for complex polynomials. The purpose of this paper is to generate superior Julia sets for cubic polynomials.
2 Preliminaries

Picard iteration method is based on one-step feedback machine. Superior iteration method is an example of two-step feedback machine and iterates are constructed by the formula $z_n = \beta_n f(z_{n-1}) + (1 - \beta_n) z_{n-1}$, where $z$ is a complex number and $0 < \beta_n \leq 1$ and $\{\beta_n\}$ is convergent to a non-zero number. Superior iterates are, essentially, due to W. R. Mann [18], and have been used by few researchers in nonlinear dynamics.

**DEFINITION 2.1. Superior Orbit:** The sequence $\{z_n\}$ constructed above is called superior sequence of iterates, denoted by, $SO(f, z_0, \beta_n)$ [15, 21-27].

Notice that $SO(f, z_0, \beta_n)$ with $\beta_n = 1$ reduces to Picard orbit. Now, we give the definition of the Julia set for a function with respect to $SO$. It is called as superior Julia set, denoted by $SJ$.

**DEFINITION 2.2. Superior Julia Set:** The set of points $SK$ whose orbits are bounded under superior iteration of a function $Q(z)$ is called the filled superior Julia set. Superior Julia set of $Q$ is the boundary of filled superior Julia set $SK$ [25].

Escape criterions play a crucial role in construction of filled Julia sets of a function. Now, new and considerably improved escape criterions are required for these sets. Rani and Kumar [25] called the same as superior escape criterions. They proved following superior escape criterion for cubic polynomials $Q_{a,b}(z) = z^3 + az + b$.

**THEOREM 2.1. (Superior escape criterion)** Suppose $|z| > \max\{|b|, (|a|+2/\beta)^{1/2}\}$. Then $|z_n| > \gamma |z|$ and so $|z_n| \to \infty$ as $n \to \infty$.

3 Cubic Superior Julia Sets

We iterate cubic polynomials $Q_{a,b}(z) = z^3 + az + b$ in $SO$, and define prisoner set using cubic superior escape criterion. Thus, cubic superior Julia sets, denoted by $CSI$, are generated. Here in all the generations, prisoner set is shown by black pixels. We present few $CSI$s in Fig. 1 – 6 generated at different $(a, b)$.

Further, author makes improvements in $CSI$s that she gave in [25]. At triplet $(\beta, a, b) = (0.5, -i, -2.75i)$, $(0.5, 0, -3.55i)$, $(0.5, -0.255+1.35i, -0.122-0.75i)$ and $(0.5, 0, -0.55)$, improved $CSI$s are given in Fig. 7, 8, 9 and 10 respectively. Further, at $(\beta, a, b) = (0.5, -5.3-5.3i, -1)$, $(0.5, -3.5-3.5i, 0)$, $(0.5, -0.5-3.7i, -1-2i)$ and $(0.5, -3-3i, -1)$, $CSI$s do not exist.
Fig. 3: CSJ at $(\beta, a, b) = (0.5, 2.5+i, 1+0.5i)$

Fig. 4: CSJ at $(\beta, a, b) = (0.5, -1-i, -1.4+0.5i)$

Fig. 5: CSJ at $(\beta, a, b) = (0.5, 1, 0.5-0.1i)$

Fig. 6: CSJ at $(\beta, a, b) = (0.5, 1, 0.5)$
4 Conclusion

In 2004, Author jointly with Kumar [25] gave improved escape criterion for cubic polynomials, when they are iterated in superior orbit. In this paper, interesting Julia sets have been visualized for cubic polynomials $Q_{a,b}(z) = z^3 + az + b$ in superior orbit.

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References:


