Modelling of Mechanical Blocking

KERIM YUNT
P.O Box: 1070 8021 Zürich
SWITZERLAND
kerimyunt@web.de

Abstract: Blocking is a non-smooth phenomenon, which is modelled as a stick-slip transition and a totally inelastic impact, that concur temporally and spatially. The physical realization of impactive blocking can be achieved in several ways, however its modelling is realized as a Coulomb like friction force characteristics with controlled/adjustable height of the set-valued signum relation.

Key–Words: Friction, Impact Mechanics, Complementarity Analysis, Nonsmooth Analysis.

1 Introduction

Blocking is a totally inelastic impact action, because the relative velocity, in the direction in which the blocking occurs, reduces immediately to zero impulsively. This modeling of blocking is used in the development of the nonlinear programming method which is described in [5]. Recent research showed that such a nonsmooth mechanical interaction can best be described nonlinear and linear complementarities. The reference [1] provides a general overview on linear complementarity analysis.

Due to the existence of discontinuities in the mechanical dynamics, the measure-differential inclusion representation of the mechanical dynamics is used. The measure-differential inclusion for mechanical dynamics is deeply rooted in the works of Moreau such as [3], [4].

2 Manipulators with impactively blockable Links

In the sequel a class of mechanical systems are modelled, in which the only impulsive action is generated by the control action. This class of mechanical systems have blockable DOF, which are described by an impulsive set-valued control law. Set-valued force elements are used in order to derive the complementarity structure of the blocking control. The dynamics of a mechanical system with n degrees of freedom, p blockable directions of motion and s Lebesgue measurable controls is formulated as a measure-differential inclusion as follows:

\[ M(q) \frac{d\dot{q}^+}{dt} - h(q, \dot{q}^+) dt - B(q) d\Gamma = 0, \quad (1) \]
\[ -d\Gamma_{b_i} \in dN_i \text{Sgn}(\gamma_{b_i}^+), \; \forall \; i \in I_B, \quad (2) \]
\[ |\gamma_{b_i}^+| \in \text{Upr}(dN_i), \; \forall \; i \in I_B, \quad (3) \]
\[ \tau \in C_\tau. \quad (4) \]

Here \( q(t^+) \) and \( \dot{q}(t^+) \) denote the absolutely continuous (AC) generalized positions and right continuous locally bounded variation (RCLBV) generalized velocities, respectively. The matrix \( M(q) \in \mathbb{R}^{n \times n} \) and the vector \( h(q, \dot{q}) \in \mathbb{R}^{n \times 1} \) represent the mass matrix and the vector of gyroscopical and coriolis, smooth potential (gravity, spring etc.) forces, respectively. The vector \( d\Gamma \in \mathbb{R}^{s \times 1} \) is the differential measure of controls which is unbounded and impulsive. The set \( C_\tau \) denotes the bounds on the Lebesgue-measurable single-valued ordinary controls, and is assumed to be of box-constrained type. The matrix \( B(q) \in \mathbb{R}^{n \times s} \) includes the generalized control directions. The set of all blockable DOF is denoted by \( I_B \). The notations \( \text{col}\{\cdot\} \) and \( \text{row}\{\cdot\} \) denote the set of column vectors and the set of row vectors of a matrix. The relative joint velocity at any blockable direction is a linear combination of the generalized velocities:

\[ \gamma_{b_i} = w_{b_i}(q) \dot{q}, \; \forall \; i \in I_B, \quad (5) \]

where \( \gamma \in \mathbb{R}^p \) is such that \( w_{b_i}(q) \in \text{row}\{W\}, \forall i \in I_B \) and \( W \in \mathbb{R}^{n \times p} \). Without loss of generality, one can assume \( \text{col}\{W\} \subseteq \text{col}\{B\} \) if \( n \geq p \).
3 Complementarity Description of the Set-valued Impulsive Blocking Control Action

The impulsive set-valued blocking control law is described by the two set-valued relations in (2) and (3) which are depicted in figures (3) and (5), respectively. The differential measure of the normal control force is composed of a Lebesgue measurable and a Borel measurable part. In this case, relation (3) is visualized as in the figures (4) and (5). The differential measures of the normal control force $dN$, right normal control force $dN_r$ and left normal control force $dN_l$ are decomposed into Lebesgue and Borel measurable parts as given in equations (6)-(8):

$$dN = ndt + N'd\sigma = ndt + (N^+ - N^-)d\sigma$$
$$dN_r = n_r dt + N'_r d\sigma = n_r dt + (N^+_r - N^-_r)d\sigma$$
$$dN_l = n_l dt + N'_l d\sigma = n_l dt + (N^+_l - N^-_l)d\sigma$$

Here $N'$, $N'_r$ and $N'_l$ denote the Radon-Nykodym derivatives of $dN$, $dN_r$ and $dN_l$, respectively. The differential measures of the blocking control force $d\Gamma$, right normal control force $d\Gamma_r$ and left normal control force $d\Gamma_l$ are decomposed into Lebesgue and Borel measurable parts as given in equations (9)-(11):

$$d\Gamma_b = \tau_b dt + (\Gamma'_b - \Gamma^-_b) d\sigma$$
$$d\Gamma_{br} = \tau_{br} dt + (\Gamma'_{br} - \Gamma^-_{br}) d\sigma$$
$$d\Gamma_{bl} = \tau_{bl} dt + (\Gamma'_{bl} - \Gamma^-_{bl}) d\sigma$$

Here $\Gamma'$, $\Gamma'_r$ and $\Gamma'_l$ denote the Radon-Nykodym derivatives of $d\Gamma$, $d\Gamma_{br}$ and $d\Gamma_{bl}$, respectively. The evaluation of the Lebesgue-Stieltjes integral of $dN$ and $d\Gamma$ over an atomic instant of time, at which an impulsive control action is applied on the system, yields:

$$\int_{\{t_1\}} dN = N^+ - N^- = \hat{N},$$
$$\int_{\{t_1\}} d\Gamma_b = \Gamma^+_b - \Gamma^-_b = \hat{\Gamma}_b,$$

respectively. The entities $\hat{N}$ and $\hat{\Gamma}_b$ are denoted by impulsive normal blocking force and impulsive control force, respectively. The decomposition of the UPR (unilateral primitive)-type relation between $|\gamma^+_b|$ and $\hat{N}$, into two UPR relations as depicted in Fig. (3) is expressed by the following set of relations (14)-(17):

$$\hat{N} = \hat{N}_r + \hat{N}_l,$$  \hspace{1cm} (14)
$$\gamma^+_b = \gamma^+_{br} - \gamma^+_{bl},$$  \hspace{1cm} (15)
$$\gamma^+_b \hat{N}_r = 0, \hat{\Gamma}^+_b \geq 0, \hat{N}_r \geq 0,$$  \hspace{1cm} (16)
$$\gamma^+_b \hat{N}_l = 0, \hat{\Gamma}^+_b \geq 0, \hat{N}_l \geq 0.$$  \hspace{1cm} (17)

In a non-impactive phase of motion, the action of keeping blocked or applying no blocking force is formulated as an UPR between $|\gamma^+_b|$ and $n$. The decomposition of the UPR-type relation between $|\gamma^+_b|$ and $n$, into two UPR relations as depicted in Fig. (7) is expressed by the relations (18)-(21):

$$n = n_r + n_l,$$  \hspace{1cm} (18)
$$\gamma^+_b = \gamma^+_{br} - \gamma^+_{bl},$$  \hspace{1cm} (19)
$$\gamma^+_b n_r = 0, \gamma^+_b n_l \geq 0, n_r \geq 0,$$  \hspace{1cm} (20)
$$\gamma^+_b n_l = 0, \gamma^+_b n_l \geq 0, n_l \geq 0.$$  \hspace{1cm} (21)

The relations given in (14)-(17) are equivalently expressed as:

$$\gamma^+_b = \gamma^+_{br} - \gamma^+_{bl},$$  \hspace{1cm} (22)
$$\gamma^+_b \hat{N} = 0, \hat{\Gamma}^+_b \geq 0, \hat{N} \geq 0,$$  \hspace{1cm} (23)
$$\gamma^+_b \hat{N} = 0, \hat{\Gamma}^+_b \geq 0, \hat{N} \geq 0.$$  \hspace{1cm} (24)

The relations given in (18)-(21) are equivalently expressed as:

$$\gamma^+_b n = 0, \gamma^+_b n \geq 0, n \geq 0,$$  \hspace{1cm} (25)
$$\gamma^+_b n = 0, \gamma^+_b n \geq 0, n \geq 0.$$  \hspace{1cm} (26)

$$\gamma^+_b n = 0, \gamma^+_b n \geq 0, n \geq 0.$$  \hspace{1cm} (27)

The figure (3) shows the relation between the differential measures of normal control force and the blocking control force. It depicts the decomposition of the set-valued signum characteristics into two unilateral primitives graphically, which are stated in the relations (28)-(32):

$$\hat{\Gamma}_{br} = \hat{\Gamma}_{bl} + \hat{N}_i, \forall i \in \mathcal{I}_B,$$  \hspace{1cm} (28)
$$\hat{\Gamma}_{bl} = -\hat{\Gamma}_{br} + \hat{N}_i, \forall i \in \mathcal{I}_B,$$  \hspace{1cm} (29)
$$\gamma^+_{bi} = \gamma^+_{br} - \gamma^+_{bl}, \forall i \in \mathcal{I}_B,$$  \hspace{1cm} (30)
$$\gamma^+_{br} \hat{\Gamma}_{bi} = 0, \hat{\Gamma}_{br} \geq 0, \gamma^+_{br} \geq 0, \forall i \in \mathcal{I}_B,$$  \hspace{1cm} (31)
$$\gamma^+_{bl} \hat{\Gamma}_{bl} = 0, \hat{\Gamma}_{bl} \geq 0, \gamma^+_{bl} \geq 0 \forall i \in \mathcal{I}_B.$$  \hspace{1cm} (32)

In a non-impactive phase of motion, the signum characteristics is expressed by the set of relations (33)-
Figure 1: (a) If $\hat{N}$ rises to infinity, blocking occurs, (b) For values of $\hat{N}$ less than infinity and greater zero, the blocking action has friction characteristics with respect to the relative velocity, (c) if $\hat{N} = 0$ there is no blocking in the next moment.

Figure 2: (a) A perfect bilateral constraint, (b) For values of $n$ less than infinity and greater zero, the blocking action has friction characteristics, (c) if $n = 0$ there is no braking force.

Figure 4: The unilateral primitive about the relation of the blocking normal force to the relative acceleration $\dot{\gamma}^+_b$.

Figure 5: The UPR about the relation of the differential measure of blocking normal impulsive force to the relative velocity $\gamma^+_b \in RCLBV$.

Figure 6: The decomposition of the Upr relation in Fig. (5) into two unilateral primitives.

Figure 7: The decomposition of the Upr relation into two unilateral primitives.
In a similar way, the behaviour of the set-valued blocking control in the absense of impulsive action is captured by the combination of the relations given from (25) to (27) and from (33) to (37) to reformulate the set-valued blocking control in the complementarity framework on acceleration level:

\[
\begin{align*}
\tau_{br} &= \tau_b + n, \quad \forall i \in I_B, \quad (33) \\
\tau_{bl} &= -\tau_b + n, \quad \forall i \in I_B, \quad (34) \\
\dot{\gamma}_{br}^+ &= \gamma_{br}^+ - \gamma_{bl}^+, \quad \forall i \in I_B, \quad (35) \\
\dot{\gamma}_{br}^+ \tau_{br} &= 0, \quad \gamma_{br}^+ \geq 0, \quad \tau_{br} \geq 0, \quad (36) \\
\dot{\gamma}_{bl}^+ \tau_{bl} &= 0, \quad \gamma_{bl}^+ \geq 0, \quad \tau_{bl} \geq 0, \quad (37)
\end{align*}
\]

The relations given from (22) to (24) and from (28) to (32) are combined in (38)-(44) to reformulate the impulsive set-valued blocking control law given in equations (2) and (3) in the complementarity framework:

\[
\begin{align*}
\dot{\Gamma}_{br} &= \hat{\Gamma}_b + \hat{N}_i, \quad \forall i \in I_B, \quad (38) \\
\dot{\Gamma}_{bl} &= -\hat{\Gamma}_b + \hat{N}_i, \quad \forall i \in I_B, \quad (39) \\
\gamma_{br}^+ &= \gamma_{br}^+ \quad \forall i \in I_B, \quad (40) \\
\gamma_{br}^+ \hat{\Gamma}_{br} &= 0, \quad \hat{\Gamma}_{br} \geq 0, \quad \gamma_{br}^+ \geq 0, \quad \forall i \in I_B(41) \\
\gamma_{bl}^+ \hat{\Gamma}_{bl} &= 0, \quad \hat{\Gamma}_{bl} \geq 0, \quad \gamma_{bl}^+ \geq 0, \quad \forall i \in I_B(42) \\
\gamma_{br}^+ \hat{N}_i &= 0, \quad \hat{\Gamma}_{br} \geq 0, \quad \hat{N}_i \geq 0, \quad \forall i \in I_B. \quad (43) \\
\gamma_{bl}^+ \hat{N}_i &= 0, \quad \hat{\Gamma}_{bl} \geq 0, \quad \hat{N}_i \geq 0, \quad \forall i \in I_B. \quad (44)
\end{align*}
\]

In a similar way, the behaviour of the set-valued controls in the absence of impulsive action is captured by the combination of the relations given from (25) to (27) and from (33) to (37) to reformulate the set-valued blocking control in the complementarity framework on acceleration level:

\[
\begin{align*}
\tau_{br} &= \tau_b + n, \quad \forall i \in I_B, \quad (33) \\
\tau_{bl} &= -\tau_b + n, \quad \forall i \in I_B, \quad (34) \\
\dot{\gamma}_{br}^+ &= \gamma_{br}^+ - \gamma_{bl}^+, \quad \forall i \in I_B, \quad (35) \\
\dot{\gamma}_{br}^+ \tau_{br} &= 0, \quad \gamma_{br}^+ \geq 0, \quad \tau_{br} \geq 0, \quad (36) \\
\dot{\gamma}_{bl}^+ \tau_{bl} &= 0, \quad \gamma_{bl}^+ \geq 0, \quad \tau_{bl} \geq 0, \quad (37)
\end{align*}
\]

4 Impulsive Control of Manipulators with Blockable Degrees of Freedom

The investigated mechanical system is shown in Fig. (8). The numerical algorithm for the calculation of the trajectories is introduced in [5]. It has two rotational degrees of freedom denoted by \(\alpha\) and \(\beta\). The DOF \(\alpha\) is measured in the positive sense from the \(x^e\) axis of the inertial frame I, which is placed at the point \(O\). The DOF \(\beta\) is measured in the positive sense from the \(y^e\) axis of the inertial frame I as shown in Fig. (8). The link 1 rotates around point \(O\) and link 2 rotates around point \(A\) which is attached to link 1. The motor at link \(O\) drives the system by the Lebesgue measurable control force \(\tau\), which has upper and lower limits denoted by \(\tau_{\min}\) and \(\tau_{\max}\), respectively. The brake at \(A\) connected to the links 1 and 2, has two operating modes, either it allows totally free motion or it blocks such that the relative angular velocity between links 1 and 2 given by \(\dot{\alpha} - \dot{\beta}\) reduces immediately to zero. In the presented maneuver as depicted in Fig. (9), the double pendulum is maneuvering time-optimally to the position \(\alpha = \frac{\pi}{2}\) and \(\beta = \pi\) from the origin. In this time-optimal maneuver the system starts in the blocked state. The whole maneuver takes 2.77 seconds. The discretized differential measures of the normal force \(N\) and the impulsive control force \(\Gamma\) are shown in Fig. (11).

5 Discussion and Conclusion

In this work, the complementarity modelling of impactive blocking is mathematically described. An application in numerical trajectory optimization is presented. It is shown, that in order for the blocking action to take place, the height of the set-valued signum relation must rise high enough to reduce the relative velocity to zero immediately and must be zero if unblocked.
Figure 8: The parameters of the planar double pendulum.

Figure 9: The optimal evolution of the generalized velocities $\dot{\alpha}$, $\dot{\beta}$, relative velocity $\dot{\alpha} - \dot{\beta}$, generalized positions $\alpha$, $\beta$ and relative position $\alpha - \beta$. Red lines mark the transition times.

Figure 10: The optimal evolution of the generalized velocities $\dot{\alpha}$, $\dot{\beta}$, relative velocity $\dot{\alpha} - \dot{\beta}$, generalized positions $\alpha$, $\beta$ and relative position $\alpha - \beta$. Red lines mark the transition times.
Figure 11: The optimal evolution of the discretized differential measures \( n, \Gamma \), the torque \( \tau \) and the slack velocity \( \gamma_{br} \) (red lines mark the transition times).

References:


