NEW ALGORITHM OF FIFTH-ORDER HERONIAN MEAN RUNGE KUTTA METHOD

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Abstract- The aim of this paper is focused on developing a new fifth order Runge-Kutta method based on Heronian Mean to compute numerical solutions of Initial Value Problems (IVPs) in ODEs. The stability polynomial and the stability region of the new formula are obtained. A modest effort has been taken to examine the suitability, adoptability and accuracy of the method. Based on the numerical results, it is observed that the presently developed new method is superior compared to some existing methods including fifth order Runge-Kutta methods based on Arithmetic Mean, geometric mean, harmonic Mean and Contra Harmonic Mean.

Key words: Runge-Kutta, Heronian Mean, Fifth order, Stability Region, Stability Polynomial.

2000 AMS Subject Classification: 35F10; 49M30; 65L05; 65L06; 65L20

I INTRODUCTION

It is well known that most of the Initial Value Problems (IVPs) are solved by Runge-Kutta methods which in turn being applied to compute numerical solutions for variety of problems that are modeled as the differential equations and their systems (Alexander and Coyle[1]). Runge-Kutta algorithms are used to solve differential equations efficiently that are equivalent to approximate the exact solutions by matching ‘n’ terms of the Taylor series expansion. Shampine and Watts[2] have developed mathematical codes for the Runge-Kutta fourth order method. Ponalagusamy and Senthilkumar[3,4] have successfully demonstrated the applicability of Runge-Kutta methods to investigate the robot arm problem and the problem of multilayer raster CNN simulation.

In order to compute highly accurate numerical solutions, Ponalagusamy and Senthilkumar [5] have introduced a new embedded Runge-Kutta RK (4, 4) method which is actually two different RK methods but of the same order p = 4. This embedded method has been developed using Runge-Kutta methods based on arithmetic mean (RKAM) and Heronian mean (RKHeM). Ponalagusamy and Senthilkumar [6] introduced embedded fourth order Runge-Kutta root mean square (RKARMS) to investigate raster CNN simulation. Evans and Yaakub [7] introduced embedded fourth order contra harmonic mean and Yaacob and Sanugi [8] adopted embedded fourth order harmonic mean.

Butcher [9] has developed Runge-Kutta formula of fifth order. A new fifth order Runge-Kutta RK (5, 5) method with error control has been introduced by Evans and Yaakub [10]. They computed numerical results and compared with other well known methods RKF (4, 5) and RK (4, 5) Merson. Razali et al.[11] have applied the fifth order Runge-Kutta method to investigate the problem of Lorenz system. Evans and Yaakub[12] computed approximate solutions of
several types of differential equations using fifth order Runge-Kutta method based on contraharmonic mean. It is of importance to point out here that the errors involved in the numerical solutions of ordinary differential equations computed by using fifth order Runge-Kutta methods based on arithmetic mean, contraharmonic mean ([12], [13]) are found to be high in comparison with that of heronian mean. In view of this, a modest effort has been made to develop a new fifth order Runge-Kutta method based on heronian mean which out performs well as compared to the exiting fifth order Runge-Kutta methods based on various means ([12], [13]).

II A NEW FIFTH ORDER HERONIAN MEAN RUNGE-KUTTA FORMULA

The standard fifth order Heronian Mean (HeM) Runge-Kutta formula for solving initial value problems of the form

\[ \frac{dy}{dx} = f(x, y) \]

may be written as follows:

\[
y_{n+1} = y_n + \frac{h}{14} \left[ k_1 + 2(k_2 + k_3) + 2(k_3 + k_4) + k_5 \right] + \sqrt{k_1 k_2} + \sqrt{k_2 k_3} + \sqrt{k_3 k_4} + \sqrt{k_4 k_5}
\]

where,

\[
k_1 = f(x_n, y_n)
\]

\[
k_2 = f(x_n + a_1 h, y_n + a_1 h k_1)
\]

\[
k_3 = f(x_n + a_2 h, y_n + a_2 h k_1 + a_3 h k_2)
\]

\[
k_4 = f(x_n + a_4 h, y_n + a_4 h k_1 + a_5 h k_2 + a_6 h k_3)
\]

\[
k_5 = f(x_n + a_7 h, y_n + a_7 h k_1 + a_8 h k_2 + a_9 h k_3 + a_{10} h k_4)
\]

and the parameters \(a_1, a_2, ..., a_{10}\) are to be determined. It is to be noticed that for simplicity of the algebra the function \(f\) is considered as a function of \(y\) only, without loss of generality. Taylor series expansion of an exact solution \(y(x_n + h)\) up to sixth order is given by

\[
y(x_n + h) = y_n + h f + \frac{1}{2} h^2 f f_y + \frac{1}{6} h^3 (f f_{yy} + f^2 f_{yy}) + \frac{1}{24} h^4 (f^3 f_{yyyy} + 4 f^2 f_y f_{yyyy} + f f_{yy} f_{yyyy}) + \frac{1}{120} h^5 (f f_{yyyy} + 11 f^2 f_y f_{yyyy} + 4 f^3 f_{yy} f_{yyyy} + 7 f^4 f_{yyy} f_{yyyy}) + \frac{1}{720} h^6 (f^5 f_{yyyyy} + 11 f^4 f_y f_{yyyyy} + 15 f^4 f_{yy} f_{yyyyy} + 32 f^3 f_y f^2 f_{yyyyy} + 34 f^3 f_{yy} f^2 f_{yyyy} + 26 f^2 f^2 f_y f_{yyyy} + f^2 f^2 f_{yy} f_{yyyy}) + O(h^7)
\]

(2.2)

Expanding \(k_1, k_2, k_3, k_4\) and \(k_5\) in Taylor series about \(x_n\) and substituting in equation (1) and comparing the coefficients of the same in equation(2.2), we can obtain 11 equations with 10 parameters. Solving 11 equations, we get

\[
a_1 = 0.2378777235038103, a_2 = -0.2511873529868023, a_3 = 0.7098880827120877, a_4 = -0.35348824655937775, a_5 = 0.990732405242792, a_6 = 0.3595437671239466, a_7 = 1.61816408923644, a_8 = -1.3905838215846875, a_9 = -0.1613445573089054, a_{10} = 0.602096438387009.
\]

(2.3)

By substituting the values of above parameters(equation(2.3)) in equation(2.1), we get a new fifth order Runge-Kutta method based on Heronian Mean as follows:

\[
y_{n+1} = y_n + \frac{h}{14} \left[ k_1 + 2(k_2 + k_3) + 2(k_3 + k_4) + k_5 \right] + \sqrt{k_1 k_2} + \sqrt{k_2 k_3} + \sqrt{k_3 k_4} + \sqrt{k_4 k_5}
\]

where,

\[
k_1 = f(x_n, y_n)
\]

\[
k_2 = f(x_n + 0.23787772350381h, y_n + 0.23787772350381h k_1)
\]

\[
k_3 = f(x_n + (-0.25118735298680 + 0.7098880827120877), y_n - 0.2511873529868023h k_1 + 0.7098880827120878h k_2)
\]

\[
k_4 = f(x_n + (-0.3534882465593775 + 0.990732405242792 + 0.3595437671239466), y_n - 0.3534882465593775k_1 + 0.990732405242792h k_2 + 0.3595437671239466k_3)
\]

\[
k_5 = f(x_n + (1.61816408923644 - 1.3905838215846875 - 0.1613445573089054 + 0.602096438387009), y_n + 1.61816408923644k_1 - 1.3905838215846875k_2 - 0.1613445573089054k_3 + 0.602096438387009k_4)
\]

(2.4)
III ERROR ANALYSIS

The Local Truncation Error for this method is defined as

$$\text{LTE} = y(x_n + h) - y_{n+1}$$  \hspace{1cm} (3.1)

with the help of equations (2.2) and (2.4), the LTE for new fifth order Runge-Kutta method based on Heronian Mean is obtained as,

$$\begin{align*}
\text{LTE} &= h^6 [0.0004306800578897194 f^5 f_{yyyyy} \\
&+ 0.002130365775938158 f^4 f_y f_{yyyy} \\
&- 0.00020463443439519904 f^4 f_{yy} f_{yy} \\
&+ 0.0010892514265619171 f^3 f_y^2 f_{yy} \\
&- 0.003931853811405234 f^3 f_y f_{yy} f_{yy} \\
&- 0.007263843578635368 f^2 f_y^2 f_{yy} \\
&- 0.0012557830777265488 f f_y^2 f_{yy} ]
\end{align*}$$  \hspace{1cm} (3.2)

IV STABILITY POLYNOMIAL OF THE NEW FIFTH ORDER RK METHOD

We discuss the stability region for the new fifth order Runge-Kutta method. To check on the stability, the simple test equation \( y' = y_0 \) is used. In this case, equation(2.5) becomes

$$k1 = \lambda(y_n)$$

$$k_2 = \lambda(y_n + 0.23787723038103h k_1)$$

$$k_3 = \lambda(y_n - 0.251187329868h k_1 + 0.709888027121h k_2)$$

$$k_4 = \lambda(y_n + 0.35348824655934h k_1 + 0.9907328420524h k_2 + 0.35954376712395h k_3)$$

$$k_5 = \lambda(y_n + 1.618164080924h k_1 - 1.390583215847h k_2 - 0.161344455703891h k_3 + 0.602096438370h k_4)$$  \hspace{1cm} (4.1)

Using equations(2.5) and (4.1), the corresponding stability polynomial is obtained as

$$y_{n+1} = y_n + h \lambda \ y_n + 0.5 (h \lambda)^2 y_n + 0.162594 (h \lambda)^3 y_n + 0.034733 (h \lambda)^4 y_n + 0.004883 (h \lambda)^5 y_n + (h)^6$$  \hspace{1cm} (4.2)

By substituting \( h = z \) in equation (4.2), we get,

$$y_{n+1} = y_n + y_n [z + 0.5 z^2 + 0.162594 z^3 + 0.034733 z^4$$

$$+ 0.004883 z^5] + O(z)^6$$  \hspace{1cm} (4.3)

Defining

$$\frac{y_{n+1}}{y_n} = Q,$$  \hspace{1cm} (4.4)

equation (4.3) becomes

$$Q = 1 + z + 0.5 z^2 + 0.162594 z^3 + 0.034733 z^4$$

$$+ 0.004883 z^5 + O(z)^6$$  \hspace{1cm} (4.5)

Similarly, one can obtain the stability polynomial for fifth order Runge-Kutta method based on contraharmonicmean(CoM) and it is expressed as

$$Q = 1 + z + 0.5 z^2 + 0.166667 z^3 + 0.0416667 z^4$$

$$+ 0.0083333 z^5 + O(z)^6$$  \hspace{1cm} (4.6)

The stability polynomial for fifth order Runge-Kutta method based on arithmetic mean(AM) [10] is

$$Q = 1 + z + 0.5 z^2 + 0.166667 z^3 + 0.0416667 z^4$$

$$+ 0.0083333 z^5 + O(z)^6$$  \hspace{1cm} (4.7)

To determine the stability region of the fifth order RK formula in the complex plane that satisfies the condition

$$\left| \frac{y_{n+1}}{y_n} \right| = Q < 1$$

e.g.,

$$\left| 1 + z + 0.5 z^2 + 0.162594 z^3 + 0.034733 z^4$$

$$+ 0.004883 z^5 \right| < 1$$  \hspace{1cm} (4.8)

V STABILITY REGION OF THE NEW FIFTH ORDER RK METHOD

With the help of stability polynomials, the stability regions for the fifth order Runge-Kutta formula based on arithmetic mean (shown as square format), the fifth order Runge-Kutta method based on
contraharmonic mean (shown as circle format) and the proposed fifth order Runge-Kutta method based on Heronian mean (shown as triangle format) are depicted in Fig. 1. It is observed that the present fifth order RK method based on heronian mean has a wider stability region in comparison with other two fifth order methods.

It is concluded from TABLE 1 that our new fifth order method (HeM) has got the better stability region in the negative real axis and both in the positive and negative imaginary axis as compared to the existing two fifth order methods (AM and CoM).

VI NUMERICAL EXPERIMENTATION

A. Problem

Electric Circuit Problems (Time-Varying Network)

1. Case

Consider the simple network shown in Fig. 2. The input \( u(x) \) is a current source and the output \( y(x) \) is the voltage across the terminating resistor.

The input-output equation of the network is

\[
\frac{dy}{dx} = -\frac{y(x)}{R(x)} + u(x)
\]  (6.1)

where \( R(x) = 1 + x \) with \( y(0) = 0 \) and \( u(x) = 1.0 \)

The exact solution of the equation (6.1) is given by

\[
y = \frac{x(x + 2)}{2(x + 1)^2}
\]  (6.2)

2. Case

If the bias source be a time-varying current source \( y(x) = 1 + \sin(x) \), then the equation (6.1) along with the initial condition \( y(0) = 0 \) will have the exact solution which is given by

\[
y(x) = \frac{x + x^2 - \cos(x) - x \cos(x) + \sin(x) + 1}{x + 1}
\]  (6.3)

The exact solution of the electric circuit problem case-I and the numerical solution computed by three various Runge-Kutta algorithms with errors have been tabulated in TABLE (2) for step size \( h = 0.1 \) and in TABLE (3) for step size \( h = 0.2 \). In view of the error involved in the numerical results, it is observed that our proposed new fifth order Runge-Kutta algorithm based on Heronian Mean out performs well in comparison with other two well-known fifth order methods based on arithmetic mean and contraharmonic mean.

In the TABLE (4) and (5), we have presented the exact solution of the electric circuit problem case-II and the numerical solution obtained by three various Runge-Kutta algorithms with errors. It is seen that the numerical results obtained by newly proposed fifth order Runge-Kutta method based on Heronian mean are very much close to the exact solution.
VII CONCLUSION

In the present paper, we have developed a new fifth order Runge-Kutta technique based on heronian mean and obtained the stability polynomial. Several practically applicable problems have been considered to test the suitability, adoptability and accuracy of the proposed method. It is noticed from the discussion that the new fifth order Runge-Kutta method based on heronian mean is more efficient than the well know new fifth order Runge-Kutta method based arithmetic mean and new fifth order weighted Runge-Kutta method based on contra harmonic mean. A remarkable result is that the new fifth order Runge-Kutta method based on heronian mean guarantees the most efficient numerical solution in all of the problems tested.

### TABLE 1
RANGES OF THE STABILITY REGIONS FOR THE VARIOUS METHODS

<table>
<thead>
<tr>
<th>Fifth order methods</th>
<th>Real-Axis</th>
<th>Imaginary-Axis</th>
</tr>
</thead>
<tbody>
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<td>Positive</td>
</tr>
<tr>
<td>AM</td>
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</tr>
<tr>
<td>CoM</td>
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<td>0.5</td>
</tr>
<tr>
<td>HeM</td>
<td>-3.72</td>
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</tr>
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### TABLE 2
NUMERICAL SOLUTION OF THE ELECTRICAL CIRCUIT PROBLEM CASE-I WITH STEP SIZE h=0.1

<table>
<thead>
<tr>
<th>step size</th>
<th>Exact solution</th>
<th>AM</th>
<th>CoM</th>
<th>HeM</th>
<th>Error by AM</th>
<th>Error by CoM</th>
<th>Error by HeM</th>
</tr>
</thead>
<tbody>
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<td>0.00000000</td>
</tr>
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### TABLE 3
NUMERICAL SOLUTION OF THE ELECTRICAL CIRCUIT PROBLEM CASE-I WITH STEP SIZE h=0.2

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<tr>
<th>Step Size</th>
<th>Exact solution</th>
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<th>CoM</th>
<th>HeM</th>
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<th>Error by CoM</th>
<th>Error by HeM</th>
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### TABLE 4
NUMERICAL SOLUTION OF THE ELECTRICAL CIRCUIT PROBLEM CASE-II WITH STEP SIZE h=0.1

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<tr>
<th>Step Size</th>
<th>Exact solution</th>
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<th>CoM</th>
<th>HeM</th>
<th>Error by AM</th>
<th>Error by CoM</th>
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### TABLE 5
NUMERICAL SOLUTION OF THE ELECTRICAL CIRCUIT PROBLEM CASE-II WITH STEP SIZE h=0.2

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<th>Exact solution</th>
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<th>CoM</th>
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### REFERENCES


