The Minmax Equality and its Technical and Economic Applications

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Preface

The minmax problem is the most interesting part of the general problem of equilibrium. It is related mainly to the adoption of a prudent strategic behavior and the starting points are the axioms of rational behaviour of decision makers who participate in the decisional process.

The paper proposes a comprehensive approach of the problem starting with the analysis of algebraic and topological conditions which contain solutions, continuing with the presentation of the main algorithms that can be used to solve the problem and ending with solving actual minmax problems encountered in technical economic practice.

The book provides a summary of the main theoretical results found in specialized literature, while the elements of novelty and originality of the work are highlighted as follows: presenting new entropic optimality criteria and specifying the conditions of equivalence; extending the results obtained to a criterion less studied in specialized literature – the equalization criterion; the introduction of a new alignment criterion in the decision theory – the maximum probability criterion - and the analysis of its degree of generality; presenting an entropic solution to the problem of distribution of final outcomes (an issue commonly seen in the context of cooperative game theory); the analysis of the stability of coalitions formed by a special technique based on algebraic and entropic calculations.

We also hope that the paper is a useful tool for professionals working in economic and technical fields, because it provides complete solutions for several important applications, such as: determining the moments of equipment failure, the analysis of mining stability in open mining, establishing the market equilibrium interest, solving important capitalization problems in terms of variable interest.

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May 2011 Ilie Mitran

Introduction

Decision theory is an important area of applied mathematics, the development of which begins during the first decades of the last century. It is based on fundamental results from mathematical analysis, probabilistic and statistical calculations, numerical calculations, optimization theory etc.

In fact, this new mathematical theory tries to model the most complex movement, the social movement (obviously, in terms of known economic and technical restrictions). In this respect, there were introduced rational behaviour concepts, the concept of utility, optimal decisional behaviour criteria in various situations (cautious behavior, risk behavior, total and partially cooperative cases).

The paper aims to highlight and develop the main results of sequential decision theory emphasizing an important category of optimal points - the equilibrium points.

It consists of six chapters; the first three are theoretical and the last chapters are concerned with actual application. Each chapter concludes with a separate paragraph entitled "Bibliographical Notes and Comments" which is a brief summary of the main results obtained by other authors and of the main open problems encountered (if any).

Chapter 1 is an introductory chapter and presents the main concepts and results needed to develop the next chapters. Primarily, it dwells on the problem of entropy, a concept that is really fascinating for a wide range of mathematicians, engineers and economists.

The problem is not completely analyzed, but it highlights the main types of entropy encountered (weighted and unweighted) and entropic concepts used in the system theory (degree of organization, degree of concentration, forecast).

A separate paragraph of this chapter is dedicated to the analysis of some reference decisional processes; this analysis is being performed only at the level of basic concepts and fundamental results.

The second chapter is entitled “The Minmax Inequality and Equality”. It starts from the fact that for a zero-sum game and two decision makers, a saddle point represents, actually, a particular case of equilibrium point. The double inequality that characterizes a saddle point is equivalent (under some algebraic and topological conditions) to a special optimal equation also known as the minmax equation. One can define the existential conditions of the minmax equality and one may characterize analytically the properties of the solutions of this equation. These goals are achieved both in the situation when there is no information exchange between players and in the (more complicated) case when exchanging information between players is allowed. There are some numerical methods for solving the minmax problem (both in simple strategies and in mixed strategies) and the emphasis falls on two constructive methods (actually two combined methods) due to the author.

The third chapter is entitled “Minmax Decisions” and it constitutes a reference chapter of the paper. Basically, it is an extension of some results presented in the specialized literature and also of some issues approached by the author. First, the principles of optimal behavior are considered for a sequential decision problem in all possible stages: the non-cooperative case, the total cooperative and partially cooperative case. The reference optimal principles for the non-cooperative case are analyzed and extended (the maximum probability criterion, the criterion of maximum profit, the principle of stability etc.) and the results of the equalization principle are also developed (a lesser-known optimal principle). A new coalition criterion is being introduced in decision theory and its properties and its degree of generalization can be analyzed. The novelty of the chapter and, thus, of the entire paper, is the study of coalition stability (through an algebraic-probabilistic method and an entropic method), as well as giving an entropic solution of final gains, knowing that the distribution of final gains remains an open problem of cooperative games.

The last chapters of this work have an applicatory characteristic.

The fourth chapter is entitled “Applications of Minmax Equalities and of Equilibrium Points in the context of Analyzing the Operation Safety of a System”.

The problem of determining the moments of a system failure is extremely difficult, especially when taking into account the structure of the system and the possibility of renewal. Basically, it starts from the graph associated with a system made up of several components, a system subject to the operating - failure - renewal requirements. The graph is immediately associated with a finite difference equation system and a system of differential equations, respectively. Applying the Laplace transformation we are led to solving an extremely difficult algebraic system, the equations being transcendental. Using the method of successive approximations, one can approximate the solutions of these equations and hence the unavailability of the system caused by each component. System availability is determined as a solution of a special maxmin problem and the moments of failure can be found immediately as equilibrium points.

The last part of this chapter analyzes, using the minmax optimization technique, the dependence between the reliability of a system and power consumption. This dependence can be written analytically as a double integral while optimal maxmin and minmax problems can be solved by using Pontriaghin principle of maxmin and some methods specific to the game theory. The results are completed after determining the reliability increasing coefficients and the decrease in consumption of electricity in the case of interventions in moments of equilibrium, as well as the estimating the costs associated to these interventions.

Chapter five is entitled "Applications of Minmax Equality within Problems Regarding Capitalization of Compound Interest" and it provides economic insight for this specific issue.

It starts from the analytical expression of the compound interest capitalization polynomial and it presents the main results of optimal maxmin and minmax problems that occur.

A milestone of this chapter is that it analyzes the situations when the unit interest is variable while the efficiency functions adopted do not allow the use results specific to differential calculus. The results are important and provide interesting economic interpretations.

There are several options to approach the market equilibrium interest calculation; the most rigorous version is when it starts from the analytical expressions of the elasticity coefficients of credit demand and supply.

It is analyzed the situation in which the elasticity coefficients are of polynomial type and unit interest is of equilibrium type; however the market equilibrium interest can only be approximated, not calculated (as the solution of a transcendental equation).

The last chapter of this paper is entitled “Maxmin Optimal Method for Analyzing the Stability of Works in Open Pit Mining”. It dwells on a very important issue from practical point of view, because the exploitation of quarries is often more advantageous (in terms of costs and risk conditions) then the exploitation of underground deposits. On the other hand, the specialized literature does not know a rigorous method of analyzing the stability of mining pits. Basically, in the case of sliding on plane surfaces the mining work is considered to be stable if the stability coefficient is a proper fraction. Since the actual stability coefficient can only be approximated (mainly because of approximate quantification of the working conditions), one cannot say precisely what happens if this coefficient is approximately 1 or even greater. For this reason, Forster's idea seemed interesting: the breaking curve (in section) is not an arc but a section of a normal distribution for which a precise method for calculating the average and the dispersion value is given. Therefore, this chapter further develops this method and it also presents a new method of approximate calculation which determines the center and the radius of the circle according to which the slope is sliding (in fact it is a sequential method).

Acknowledgement

Games theory is the mathematical theory which deals with the substantiation of the decisional methods and strategies used within modeling conflictual situations on global markets. One of the most apparent phenomena in global economy nowadays is the emergence of multinational companies against the globalization context. Many of the multinationals operate within an oligopoly market and the game theory is an indispensable tool for studying and forecasting their behaviour. Researches conducted in Chapter III - "Minmax Decisions", are part of the research activity carried out by the author during the international project: COST 281/ 2009 Brussels, Action IS0905 - The Emergence of Southern Multinationals and their Impact on Europe. The problem of studying and forecasting the strategic behaviour of multinationals within the globalization context shall be approached in a future paper work in which the emphasis is laid on modeling competitive situations by applying methods and concepts from the game theory.

The Author

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BASIC NOTIONS AND FUNDAMENTAL RESULTS

The purpose of this introductory part is to show some reference results in the subsequent approaches of the paper.

1.1 Entropy and entropic quantities used in the analysis of systems

1.1.1 The Concept of Entropy

Let us start a certain experiment ξ, the outcomes of which are the states $E_1, E_2, ..., E_n$; we shall note $p_1, p_2, ...,p_n$ the probability of occurrence of these events. Therefore, the following conditions must be met:

$$p_i \geq 0, i = \overline{1,n},$$

$$\sum_{i=1}^{n} p_i = 1$$

The entropy associated with the experiment is denoted by $H(\xi)$, or simply H when there is no risk of mistaking $H(\xi)$ for H notations. The entropy measures:
- the amount of indetermination contained before the experiment realization;
- the information obtained after the experiment.

The first measure of entropy was given in 1928 by Hartley, starting from the study of some communication problems.

The concept of entropy was introduced rigorously by Shannon in 1948, based on the study of communication problems; practically, entropy is obtained as the solution of the following functional equation:
$H(xy) = H(x) + H(y)$,
meeting the condition: $H(1) = 0$, where H is a monotonously increasing function.

The solution of this equation is $H(x) = a \ln x$, $a \in \mathbb{R}$, while the entropy according to Shannon is defined as the average value of the discrete random variable $X\begin{pmatrix} x_1 & x_2 ... & x_n \\ p_1 & p_2 ... & p_n \end{pmatrix}$ where $x_i = - \ln p_i$, $i = \overline{1,n}$.

If p is the distribution of probabilities $(p_1, p_2, ..., p_n)$, then Shannon entropy associated with this distribution is written H and it is defined using the following equality:

$$H(p_1, p_2, ..., p_n) = -\sum_{i=1}^{n} p_i \ln p_i \qquad (1)$$

Remark 1.1 Shannon's maximum entropy is made for an equi-probable distribution, i.e. when

$$p_1 = p_2 = ... = p_n = \frac{1}{n}$$

and consequently

$$H\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right) = -\ln\frac{1}{n} = \ln n$$

It turned out that the entropy introduced by Shannon does not allow clear interpretation from technical – economic point of view and in order to avoid the subjectivity of such interpretations, other types of entropy were subsequently introduced.

Basically, we can talk about these types of entropy:
- unweighted entropy (which does not take into account the efficiency of states);
- weighted entropy (which take into account the efficiency of states).

1.1.1.1 Unweighted Entropies

The most well-known unweighted entropies (besides Shannon entropy) are the so-called entropies of order α (attributed to Renyi) and α type entropies (attributed to Daroczy) .

***The entropy of order α* [73], [84]**

Let H_α be the entropy defined as follows:

$$H_\alpha\left(p_1,p_2,...,p_n\right)=\begin{cases}\dfrac{1}{1-\alpha}ln\sum_{i=1}^{n}p_i^{\alpha},\alpha\neq 1\\ -\sum_{i=1}^{n}p_i\,ln\,p_i,\alpha=1\end{cases}\qquad(2)$$

It can be verified immediately that Shannon entropy is a borderline case of α-order entropy, that is to say the following equality holds true:

$$\lim_{\alpha\to 1}H_\alpha\left(p_1,p_2,...,p_n\right)=H\left(p_1,p_2,...,p_n\right)$$

Renyi's entropy retains the remarkable property of being additive.

$$H_\alpha(\xi_1+\xi_2)=H_\alpha(\xi_1)+H_\alpha(\xi_2)$$

This property can be interpreted as follows: the indetermination, i.e. the informational gain corresponding to the sum of events equals the sum of indeterminations, i.e. the sum of informational gains of events.

α Type entropy **[84]**

Let H^α be the entropy defined as follows:

$$H^\alpha\left(p_1,p_2,...,p_n\right)=\begin{cases}\left(\dfrac{1}{2^{1-\alpha}}-1\right)\sum_{i=1}^{n}(p_i^{\alpha}-1),\alpha\neq 1\\ -\sum_{i=1}^{n}p_i\,ln\,p_i,\alpha=1\end{cases}\qquad(3)$$

Even in this case the Shannon entropy proves to be a borderline case of Daroczy entropy:

$$\lim_{\alpha\to 1}H^\alpha\left(p_1,p_2,...,p_n\right)=H\left(p_1,p_2,...,p_n\right)$$

Unlike Renyi entropy, Daroczy entropy does not retain the property of additivity, that is to say that the following equalities occur:

$$H^\alpha(\xi_1+\xi_2)>H^\alpha(\xi_1)+H^\alpha(\xi_2),\qquad \text{if}\qquad \alpha>1;$$

$$H^\alpha(\xi_1+\xi_2)=H^\alpha(\xi_1)+H^\alpha(\xi_2),\qquad \text{if}\qquad 0<\alpha<1.$$

Remark 1.2 Although they admit different construction, entropies of order α and those of type α lead to similar results and therefore the possibility of being interpreted differently can be reduced.

1.1.1.2 Weighted Entropies

Let us consider the experiment ξ which is associated with the states $E_1, E_2, ..., E_n$ with the probabilities $p_1, p_2, ..., p_n$. The following conditions are necessary:

$$p_i\geq 0, i=\overline{1,n},\quad \sum_{i=1}^{n}p_i=1$$

If u is the efficiency function associated with the experiment ξ, we shall note $u_1, u_2, ..., u_n$ the efficiency of the states corresponding to ξ,:

$$u_1=u(E_1),u_2=u(E_2),...,u_n=u(E_n)$$

Weighted entropy (S. Guiasu) **[44]**

This is denoted by H_G and it is defined by the following equality:

$$H_G(u,p)=-\sum_{i=1}^{n}u_ip_i\,ln\,p_i\qquad(4)$$

where u and p represent the vectors $(u_1, u_2, ..., u_n)$ and $(p_1, p_2, ..., p_n)$, respectively.

Besides the fact that the analytical expression (4) shows clearly the contribution of state efficiency, Guiasu entropy has other properties as well, among which the most important are:

1. $H_G(u,p)\geq 0$;
2. if $u_1=u_2=...=u_n=1$, then it coincides with Shannon entropy, meaning $H_G(u,p)=H(\xi)$;
3. $H_G(\lambda u,p)=\lambda H_G(u,p)$ which proves that this entropy is homogenous in the first argument.

Relative entropy (Watanabe)

This concept is an adaptation of Gibbs' entropy from thermodynamics.

This is defined somewhat differently from the previous entropy but the outcome is about the same, therefore the interpretations generated do not differ. We shall note it H_w and it is defined as follows:

$$H_W(u,p)=\sum_{i=1}^{n} p_i \, ln \frac{Au_i}{p_i} \tag{5}$$

Where:

$$u_i \geq 0, i=\overline{1,n}, \sum_{i=1}^{n} u_i = 1$$

$$A = A(u_1,u_2,...u_n) \geq 0$$

In this case, Au_i plays the role of the utility function for the state $E_i, i=\overline{1,n}$. In the case of $A(u_1,u_2,...u_n)=1$, one can obtain a particular form of relative entropy approached by Tövissi, starting form an original point of view.

By computing the right component of the relation (5) we get for $A(u_1,u_2,...u_n)=1$:

$$H_W(u,p)=-\sum_{i=1}^{n} p_i \, ln \, p_i + \sum_{i=1}^{n} p_i \, ln u_i$$

It is to be noted that the first member of the expression above is exactly the Shannon entropy and the second term can be interpreted as the average value of the utility logarithms.

Remark 1.3 If utility logarithms can be interpreted as utilities, this means that Watanabe entropy can be defined as the sum of Shannon entropy and the average value of state utilities.
In addition to these properties, the entropy is nonnegative and additive.

Aggregate entropy (S. Birlea)

It is an entropic concept derived from the study of the degree of organization of systems. Suppose that a system *S* consists of *n* subsystems $S_1, S_2, ..., S_n$, and the weights (importance coefficients) of these systems are $p_1, p_2, ..., p_n$. Obviously:

$$p_i \geq 0, i=\overline{1,n}, \qquad \sum_{i=1}^{n} p_i = 1$$

We shall note $H(S)$, $H(S_1)$, $H(S_2)$, ..., $H(S_n)$ the Shannon entropies associated with the system and the subsystems, respectively.

Birlea entropy is denoted by H_B and it is defined as follows:

$$H_B(S)=\sum_{i=1}^{n} p_i H_i(S_i) - H(S) \tag{6}$$

which means the difference between the average of entropies of subsystems and the entropy of the system.

It is actually a variant of the degree of organization according to Watanabe which means that the system is more organized if the overall behavior is better known, regardless of the behavior of subsystems.

1.1.2 Entropic Concepts Used in the Analysis of Systems

Taking into consideration the concept of entropy, in the specialized literature there were several attempts to study some reference properties of systems through this concept: degree of organization, degree of concentration, prediction, uncertainty.

The first two properties are used most often, that is the degree of organization and the degree of concentration, while prediction is a probabilistic variant of the forecast and uncertainty is a relatively new concept about which there are few results.

1.1.2.1 Degree of Organization

The organization issue - one of the newest and the most useful problems related to the optimal decision framework - raises difficulties in terms of mathematical approach, which is explained by the fact that it has appeared only recently in specialized literature.

Often, the notion "degree of concentration" is used as the degree of organization, which is incorrect because the concepts of organization and concentration mean different things; therefore they should be interpreted differently.

Suppose we have a system *S* that consists of a finite number of subsystems $S_i, i=\overline{1,n}$.
The degree of organization of system *S* was introduced by Watanabe under the name of interdependence (probability) between the corresponding probability fields of *S* and those corresponding to $S_i, i=\overline{1,n}$.

$$W(S_1,S_2,...,S_n)=\sum_{i=1}^{n}H(S_i)-H(S) \tag{7}$$

where: $H(S_i)$ – the entropy of the system;
$H(S)$ - is the entropy of system *S* and it is defined using the product probability.

The entropy of the system will be denoted by *H,* the entropy of the system *i,* $i=\overline{1,n}$ will be written H_i, while the degree of organization is *W* making it easier in this cases:

$$W=\sum_{i=1}^{n}H(S_i)-H(S)$$

Properties:

1) $W\geq 0$, if $W=0$ then the subsystems S_i are probabilistically independent;

2) If subsystem S_i consists of the elements $a_i^j, i=\overline{1,n}, j=\overline{1,k_i}, k_i$, being the number of components of subsystem S_i, then:

$$W\left(S;a_1^1,a_1^2,...,a_n^{kn}\right)=W\left(S;S_1,S_2,...,S_n\right)+W\left(S_1;a_1^1,a_1^2,...,a_1^{k1}\right)+$$
$$+W\left(S_2;a_2^1,a_2^2,...,a_2^{k2}\right)+...+W\left(S_n;a_n^1,a_n^2,...,a_n^{kn}\right)$$

3) the organization degree of the system regarding its component elements, equals the sum of organization degrees of the system regarding subsystems and the organization degrees of subsystems regarding their components.

$$W\left(S;a_1^1,a_1^2,...,a_n^{kn}\right)>W\left(S_i;a_i^1,a_i^2,...,a_i^{ki}\right),\quad i=\overline{1,n}$$

This property is a consequence of property 2) and it expresses the fact that the organization degree of the subsystem on its components is higher than any of the degrees of organization of subsystems on the component elements.

4) in the case of games with three players (triad) there are some interesting results about the degree of organization of triads. We shall consider the system S made up of three players $\left(S=\{1,2,3\}\right)$ split into two coalitions.

We shall name cooperative solution of the triad the partition for which:

$$W\left(\{1,2,3\};\{i_0,j_0\};\{k_0\}\right)=\max_{\substack{i,j,k=1,3\\ i\neq j\neq k}}\left\{W\left(\{1,2,3\};\{i_0,j_0\};\{k_0\}\right)\right\} \tag{8}$$

It is noted that the usefulness of the concept of degree of organization of a system as a difference between the sum of the indeterminacy of the system components and the indeterminacy of the system remains dependent on a lack of connection between the indeterminacy of the system (subsystem) regarding the attainment of a state and its usefulness. For example, in the case of a decisional process involving multiple decision makers, it is natural for them to exercise options for certain subsets of the set of final states taking into account not only the state of indeterminacy towards reaching a certain position within the favourite subsets but also their associated utilities.

If we change the concept of degree of organization of the system by replacing $W=\sum_{i=1}^{n}H_i-H$ with $W_p=\sum_{i=1}^{n}H_i^p-H$, where H_i^p is the weighted entropy of subsystem *i*, then:

$$W_p=W+\sum_{i=1}^{n}\sum_{j=1}^{ki}h_i^j p_i^j \tag{9}$$

Where

h_i^j - the weight of component *j* of the subsystem *i*;

p_i^j - the probability of component *j* of the subsystem *i.*

Thus, a system is better organized if the overall behavior of the system is better known, despite the undetermined behavior of the components and the average weights of component subsystems. Since the weight of the entropy can be determined with the help of (normalized) utilities associated with subsystems, we conclude that a system is more organized if the difference between the sum of the degrees of indeterminacy of the component subsystems and the degree of indeterminacy of the system – on the one hand - as well as if the average value of (normalized) utilities of subsystems are higher, on the other hand.

These concepts of degree of organization combine in a certain way the concepts of entropy and utility:

$$h_i^j, p_i^j \geq 0, \quad i=\overline{1,n}, \quad j=\overline{1,k}, \quad \sum_{i=1}^{n}\sum_{j=1}^{ki} h_i^j \sum_{i=1}^{n}\sum_{j=1}^{ki} p_i^j = 1 \tag{10}$$

1.1.2.2 Degree of Concentration

Suppose we have a system that evolves over time, and has different classes of values corresponding to different periods of time.

The degree of concentration is a measure that characterizes the distribution on different classes of values, namely if the subsystem S is assigned, at time t_0, the classes $C_1, C_2, ..., C_m$ associated with the corresponding probabilities $p_1^0, p_2^0, ..., p_m^0, \ p_i \geq 0, \ i=\overline{1,m}, \ \sum_{i=1}^{m} p_1^0 = 1$, while at time $t > t_0$ the classes $C_i, \ \ i=\overline{1,m}$ are associated with the probabilities $p_1^0, p_2^0, ..., p_m^0, \ p_i \geq 0, \ i=\overline{1,m}, \ \sum_{i=1}^{m} p_1^0 = 1$, the specialized literature recognizes two different degrees of concentration:

a) degree of concentration that characterizes the distribution on different classes of values at a certain time having the general form $C = C(y), \quad y = \sum_{i=1}^{m} p_i^2$ (we considered *t* the period of time) as shown below:

1) Renyi degree of concentration obtained for $C(y) = -ln\, y$. This degree of concentration can be obtained from the entropy of order α, for $\alpha = 2$, in case all states are noticeable;

2) Onicescu degree of concentration obtained for $C(y) = y$. This degree of concentration has been introduced by scientist Onicescu under the name of information energy and it is worth noticing that this degree of concentration generalizes the Herfindahl concentration degree introduced in 1950 at a relative frequency.

3) Hirschman degree of concentration, is given by $C(y) = y$; however, it should be noted that Hirschman did not work with probabilities but with relative frequencies.

b) the degree of concentration characterizing the decrease or increase of the distribution on classes of values at different times, which can be defined using the concept of entropy in two different manners:

$$C = 1 - \frac{H_t}{H_0} \text{ or } C = \frac{H_t}{H_0}$$

where H_0 - the entropy associated with the system at time t_0;
H_t - the entropy associated with the system at time *t.*

Assuming that we have a system *S* consisting of a finite number of subsystems $S_i, i=\overline{1,n}$, the degree of organization of system *S* can be introduced (under the name of probabilistic interdependence between the fields probability associated with *S* and $S_i, i=\overline{1,n}$) as follows:

$$W(S_1, S_2, ..., S_n) = \sum_{i=1}^{n} H_i - H$$

where *H* and H_i represent the entropy of the system *S* and of the subsystem $S_i, i=\overline{1,n}$, respectively.

Properties:

1) $W \geq 0, W = 0$ then the subsystems S_i are probabilistically independent;

2) if the subsystem S_i consists of elements $a_i^j, i=\overline{1,n}, j=\overline{1,k_i}, k_i$ being the number of elements of the subsystem S_i, then:

$$W\left(S;\, a_1^1, a_1^2, \ldots, a_1^{k1};\, a_2^1, a_2^2, \ldots, a_2^{k2};\, a_n^1, a_n^2, \ldots, a_n^{kn}\right) = \\ = W\left(S;\, S_1, S_2, \ldots, S_n\right) + \sum_{i=1}^{n} W\left(S_i;\, a_i^1, a_i^2, \ldots, a_i^{ki}\right) \tag{11}$$

In other words, the degree of organization of the system *S* regarding its components equals the sum of the degree of organization of the system on the component subsystems and the degree of organization of the subsystems on the components.

3) $W\left(S;\, a_1^1, a_1^2, \ldots, a_1^{k1};\, a_2^1, a_2^2, \ldots, a_2^{k2};\, a_n^1, a_n^2, \ldots, a_n^{kn}\right) \ge W\left(S_i;\, a_i^1, a_i^2, \ldots, a_i^{ki}\right)$, $i = \overline{1,n}$. Further on, there is a change in the concept of degree of organization of the system previously introduced by replacing $W = \sum_{i=1}^{n} H_i - H$ with $W_p = \sum_{i=1}^{n} H_i^p - H$, where H_i^p is the weighted entropy of the subsystem $S_i, i = \overline{1,n}$. After an immediate calculation we get:

$$W_p = W + \sum_{i=1}^{n} \sum_{j=1}^{ki} h_i^j p_i^j \tag{12}$$

where h_i^j and p_i^j represent the weight and the normalized utility of component *j* of the subsystem $S_i, i = \overline{1,n},\ j = \overline{1,k}$, respectively.

Therefore, a system is better organized if the overall behavior of the system and the average value of the weights of the component subsystems are better known, even though these components have a random behavior. We apply the concept of degree of organization (which depends in this way on the linear utility) to a sequential decision process described by $s_i = f_i\left(s_{i-1}, d_{i-1}\right)$, $s_i \in S$, $d_i \in D\left(s_i\right)$, $D\left(s_i\right)$ represents the multitude of decisions that may be adopted in the state $s_i \in S$.

We shall use the notification *U* for the sum of the normalized utilities obtained by the *n* decision-makers taking part in the decision process and the notification $W_k = \sum_{i=1}^{n} H_i - H^k$ for the degree of organization of the system in the sequence *k*.

Theorem 1.1 [47]

The following properties are revealed:

1) If decision-makers act together, then $U = \lim_{k} W_k$;

2) If decision-makers take turns, then:

$$U = \lim_{k} \left[W_k + \sum_{i=1}^{n} \sum_{j=1}^{k} I_i\left(j-1, j\right) - \frac{1}{H^0} \prod_{j=1}^{k} C_j - \sum_{i=1}^{n} H_i^0 \right] \tag{13}$$

we have written: $I_i\left(j-1, j\right)$- the average informational gain (Renyi) reached by the decider *i* through the transition from state s_{k-1} to state s_k

$$I\left(k-1, k\right) = \sum_{i=1}^{k} p_i^k \ln \frac{p_i^k}{p_i^{k-1}} \tag{14}$$

C_j - the degree of concentration of the system *S* in phase *K*.

Remark 1.4

1. In case decision-makers decide in turns, this theorem states that the sum of normalized utilities represents the limit of a nonlinear functional with the following variables: the degree of organization of the system, the sum of the initial entropies of the *n* decision makers, the sum of information gains of *n* decision-makers, the initial entropy of the system, relative concentrations of the system.

2. In case decision-makers act simultaneously, the total normalized utilities of *n* decision-makers will be the limit of the degree of organization the system.

The concentration degree of a system characterizes the distribution. If the system *S* is composed of subsystems S_i with the weights p_i at time t_0, $p_i \ge 0$, $i = \overline{1,n}$, $\sum_{i=1}^{n} p_i = 1$,and at time ,$t > t_0$ the weights associated with the component subsystems are $p_i^t \geq 0, \quad i=\overline{1,n}, \quad \sum_{i=1}^{n}\left(p_i^t\right)=1$, we can distinguish the following types of degrees of concentration:

1) The degree of concentration characterizing the dispersal at a certain time and which has the form below:

$$C=C(y), \quad y=\sum_{i=1}^{n}\left(p_i^t\right)^2$$

The following situations are pointed out:

- if $C(y)=-\ln y$, the degree of concentration is called Renyi degree of concentration
- if $C(y)=y$, the degree of concentration is called the information energy (this measure was introduced by scientist Onicescu).

2) Degrees of concentration characterizing the variation of the dispersal by classes at different times used in two different ways:

$$C=1-\frac{H_t}{H_0}; \quad C=\frac{H_t}{H_0}$$

where H_0 and H_t denote the entropies of the system S at times t_0 and t.

Application:

Let us consider the system S a "mining enterprise" consisting of the following subsystems:
S1 = preparatory subsystem,
S2 = mine opening subsystem,
S3 = stopping subsystem.

Let us assume that the subsystem S_i is associated with repartitions $\begin{pmatrix} a_i & 0 \\ p_i & q_i \end{pmatrix}, \quad p_i+q_i=1$ regarding the success or failure in carrying out tasks at level $a_i, i=\overline{1,3}$.

We intend to calculate the degree of organization of S if one of the subsystems totally completes its tasks (assuming $p_3=1$ and $p_1=q_2$).

$$W=H_1+H_2+H_3-H=H_1+H_2-H$$

$$H_1=-p_1\ln p_1-q_1\ln q_1$$

$$H_2=q_1\ln q_1-p_1\ln p_1=-p_1^2\ln p_1^2-2p_1q_1-q_1^2\ln q_1^2=-2\left(p_1\ln p_1+q_1^2\ln q_1^2+p_1q_1\ln 2p_1q_1\right)$$

$$W=H_1+H_2-H=2p_1q_1\ln 2$$

It is noticeable that the maximum degree of organization corresponds to repair works:

$$\begin{pmatrix} a_1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ 1 & 0 \end{pmatrix} \text{ and it will be } W_{max}=\frac{1}{2}\ln 2$$

1.1.2.3 Prediction

Mathematical prediction is a core concept in applied mathematics; the specialized literature offers a rich material in this area.

We shall tackle the prediction problem closely related to entropic concepts presented so far. We should point out a prediction concept used by S. Guiaşu regarding a hypothesis in a three-person game in which players decide on several occasions.

Suppose that the number of participants in a game is k, and by $X^1, X^2, ..., X^k$ we understand the set of strategies of these players; x_j^1 is the strategy of the decider i_j in the sequence i.

Considering j the set of hypotheses $H=(h_1,h_2,...,h_t)$ about the future game, we shall note $p_0(h_i)$ the initial probability of the hypothesis $i, \; i=\overline{1,t}$ (in case we start from a maximum indeterminacy, initial probabilities are equal $p_0(h_i)=\frac{1}{t}, \; i=\overline{1,t}$).

According to hypothesis $h \in H$, the prediction probability on the development of the future of game, based on this assumption, knowing that *i* steps were taken:

$$x_{i1}^1, x_{i2}^1, ..., x_{ik}^1; x_{i1}^2, x_{i2}^2, ..., x_{ik}^2; ...; x_{i1}, x_{i2}, ..., x_{ik}, x_{ij} \in X$$

can be determined using Bayes' formula [68]:

$$P\left(h / x_{i1}^1, x_{i2}^1, ..., x_{ik}^1; x_{i1}^2, x_{i2}^2, ..., x_{ik}^2; ...; x_{i1}^l, x_{i2}^l, ..., x_{ik}^l\right) =$$

$$= \frac{P_0\left(h\right)\prod_{j=1}^{l} p\left(x_{i1}^j, x_{i2}^j, ..., x_{ik}^j; x_{i1}^l, x_{i2}^l, ..., x_{ik}^l; ...; x_{i1}^{j-1}, x_{i2}^{j-1}, ..., x_{ik}^{j-1}\right)}{\sum_{h \in H} p_0\left(h\right) p\left(x_{i1}^j, x_{i2}^j, ..., x_{ik}^j; x_{i1}^l, x_{i2}^l, ..., x_{ik}^l; ...; x_{i1}^{j-1}, x_{i2}^{j-1}, ..., x_{ik}^{j-1}\right)} = \tag{15}$$

$$= \frac{P_{l-1}\left(h / x_{i1}^1, x_{i2}^1, ..., x_{ik}^1; x_{i1}^2, x_{i2}^2, ..., x_{ik}^2; ...; x_{i1}^{l-1}, x_{i2}^{l-1}, ..., x_{ik}^{l-1}\right) A_l\left(h\right)}{\sum_{h \in H} p_{l-1}\left(h / x_{i1}^1, x_{i2}^1, ..., x_{ik}^1; x_{i1}^2, x_{i2}^2, ..., x_{ik}^2; ...; x_{i1}^{l-1}, x_{i2}^{l-1}, ..., x_{ik}^{l-1}\right) A_l\left(h\right)}$$

where

$$A_l\left(h\right) = P\left(x_{i1}, x_{i2}, ..., x_{ik} / h; x_{i1}^1, x_{i2}^1, ..., x_{ik}^1; ...; x_{i1}^{l-1}, x_{i2}^{l-1}, ..., x_{ik}^{l-1}\right), \forall h \in H$$

Let us take into consideration another point of view regarding the prediction :
If the event ζ is characterized by states $E_1, E_2, ..., E_k$ and probabilities $p_1, p_2, ..., p_k$, the average value of information supplied by ζ is $H(\zeta)$.

$$H\left(\zeta\right) = -\sum_{i=1}^{n} p_i \ln p_i$$

Furthermore, if the event ζ^0 is characterized by states $E_1^0, E_2^0, ..., E_l^0$ and probabilities $p_1^0, p_2^0, ..., p_l^0$, the average value of information supplied by ζ is:

$$H\left(\zeta^0\right) = -\sum_{j=1}^{l} p_j \ln p_j^0$$

The prediction regarding the change in the information provided by ζ, knowing that it was preceded by ζ^0 is:

$$P\left(\xi / \xi^0\right) = H\left(\xi\right) - H\left(\xi / \xi^0\right) = \sum_{i=1}^{k} P\left(E_i\right) \ln p\left(E_i\right) - \sum_{i=1}^{k}\sum_{j=1}^{l} P\left(E_i E_j\right) \ln \frac{P\left(E_i E_j^0\right)}{P\left(E_j^0\right)} =$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{l} P\left(E_i E_j\right)\left[\ln P\left(E_i\right) - \ln \frac{P\left(E_i E_j^0\right)}{P\left(E_j^0\right)}\right] \tag{16}$$

Since $P\left(E_i\right) = \sum_{j=1}^{l} P\left(E_i E_j^0\right)$

The entropic problem of prediction, which is very important in terms of practical applicability, will be discussed below in connection with Renyi in a sequential decision process $G = \left(x, F\right)$ described by the recurrence relation:

$$x_k = F\left(x_{k-1}, d_{k-1}\right), d_{k-1} \in D\left(X_{k-1}\right)$$

a) Where *X* is at most a countable set and it designates the set of positions or states x_k meaning that the decision process has reached the sequence *k*;

b) $F : X \to X$ is the transition function; due to F one can define the following sets:

$x_0 = \left\{x \in X \mid F^{-1}\left(x\right) = \varnothing\right\}$ also called the initial position set;

$\bar{x} = \left\{x \in X \mid F\left(x\right) = \varnothing\right\}$ also called the final position set.

c) $D\left(x_k\right)$ is the set of decisions d_k which can be made in the state $x_k \in X$.

Suppose the decision-making process involves *n* decision-makers whose utility is denoted by U_i, $i = \overline{1,n}$, while the set $\bar{x}$ is finite, consisting of *m* positions: $\bar{x}\left\{x_1, x_2, ..., x_m\right\}$.

Using the notation $a_1, a_2, ..., a_n$ for the lower limited gains proposed by the n decision-makers and the notation $\xi^k\left(E_1^k, E_2^k, ..., E_m^k\right)$ for the event attached to the possibilities of reaching $\bar{x}$ in the sequence k, the Renyi prediction in this sense will be:

$$P_k\left(\xi^k / \xi^1, \xi^2, ..., \xi^{k-1}\right) = H\left(\xi^k\right) - H\left(\xi^k / \xi^1, \xi^2, ..., \xi^{k-1}\right) \tag{17}$$

where $H\left(\xi^k\right), H\left(\xi^k / \xi^1, \xi^2, ..., \xi^{k-1}\right)$ denotes the entropy of the event ζ^k and the conditioned entropy associated with the event $\xi^k / \xi^1, \xi^2, ..., \xi^{k-1}$.

Considering that:

$$\begin{aligned} H\left(\xi^k / \xi^1, \xi^2, ..., \xi^{k-1}\right) &= H\left(\xi^1, \xi^2, ..., \xi^k\right) - H\left(\xi^2 / \xi^1\right) - \\ &- H\left(\xi^3 / \xi^1, \xi^2\right) - ... - H\left(\xi^{k-1} / \xi^1, \xi^2, ..., \xi^{k-2}\right) \end{aligned} \tag{18}$$

the entropic measure of the prediction is:

$$\begin{aligned} P\left(\xi^k / \xi^1, \xi^2, ..., \xi^{k-1}\right) &= H\left(\xi^k\right) + H\left(\xi^1\right) + H\left(\xi^2 / \xi^1\right) + \\ &+ ... + H\left(\xi^{k-1} / \xi^1, \xi^2, ..., \xi^{k-2}\right) - H\left(\xi^1, \xi^2, ..., \xi^k\right) \end{aligned} \tag{19}$$

On the other hand, we have:

$$H\left(\xi^2 / \xi^1\right) = H\left(\xi^2\right) - P\left(\xi^2 / \xi^1\right)$$
$$H\left(\xi^3 / \xi^1, \xi^2\right) = H\left(\xi^3\right) - P\left(\xi^3 / \xi^1, \xi^2\right)$$
$$H\left(\xi^k / \xi^1, \xi^2, ..., \xi^{k-1}\right) = H\left(\xi^{k-1}\right) - P\left(\xi^{k-1} / \xi^1, \xi^2, ..., \xi^{k-2}\right)$$

thus:

$$P\left(\xi^k / \xi^1, \xi^2, ..., \xi^{k-1}\right) = \sum_{i=1}^{k} H\left(\xi^k\right) - H\left(\xi^1, \xi^2, ..., \xi^k\right) - \sum_{i=2}^{k-2} P\left(\xi^i / \xi^1, \xi^2, ..., \xi^{i-1}\right)$$

Writing:

$$H\left(\xi^1, \xi^2, ..., \xi^k\right) = H_k$$
$$H\left(\xi^i\right) = H_i, \quad i = \overline{1,k}$$
$$P\left(\xi^i / \xi^1, \xi^2, ..., \xi^{i-1}\right) = P_i, \quad i = \overline{1,k}$$

the prediction in sequence k will be:

$$P_k = \sum_{i=1}^{k} H_i - H_k - \sum_{i=1}^{k-1} P_i \tag{20}$$

which means that prediction in an intermediate point will be the difference between the total entropies of intermediate events, the entropy of the system formed by the k decision-makers acting in k sequences and the sum of intermediate predictions.

Remark 1.5 The last equality can take the following form:

$$\sum_{i=1}^{k} P_i = \sum_{i=1}^{k} H_i - H_k$$

which means that the sum of predictions equals the difference between the total entropies of intermediate events and the entropy of the system consisting of the k decision-makers.

We shall analyze further on the problem of determining the prediction in the sequence k, knowing that within the previous sequences the predictions were insignificant (we shall assume their significance - negligible – equals $\varepsilon > 0$); the significance threshold in the sequence k is α.

We shall note W_k the degree of organization (according to Guiaşu) of the system consisting of the k decision-makers and C_i the degree of concentration of the system in the sequence i, $1 \le i \le k$, therefore:

$$W_k = \sum_{i=1}^{k} H_i - H_k$$
$$C_i = \frac{H_i}{H_1}, \quad i = \overline{1,k}$$

We shall also note $p = P\left(X_1 \geq a_1,...,X_n \geq a_n\right), \quad q = 1 - p$ where X_i is the random variable associated with the event ξ_i, $i = \overline{1,n}$.

Theorem 1.2

a) If $k \leq n$, the prediction at time k is significant (meaning $\alpha > 0$) in case $W_k \geq \alpha + (k-2)\varepsilon$;

b) If $k\left(k \gg n\right)$ is high enough, the prediction is significant at time k with the significance α if (Bereanu):

$$\sum_{i=1}^{k} C_i \geq \left[\alpha + (k-2)\varepsilon - 2\left(p^2 \lg p\right) - q^2 \lg q + pq \lg pq\right] \tag{21}$$

$$p = \frac{1}{k - 2^n} \sum_{j=1}^{k} \prod_{i=1}^{n} e^{-\frac{\left(a_i - A_{i,j}\right)}{n^{a_i}}}, \quad 0 < a_i < \frac{1}{n}, \tag{22}$$

where $A_{i,j}$ represents the value of the variable X_i in sequence j so that $P\left(X_i \geq A_{i,j}\right) > \alpha$.

Proof

Demonstration of case a) is immediate. Based on remark 1), the prediction at time k is:

$$P_k = \sum_{i=1}^{k} H_i - H_k - \sum_{i=2}^{k-1} P_i \Rightarrow P_k = W_k - \sum_{i=2}^{k-1} P_i \tag{23}$$

Since $P_k \geq \alpha$ and $P_i < \varepsilon, \quad i = \overline{1,k-1}$, we get $W_k \geq \alpha + (k-2)\varepsilon$.

The demonstration of case b) is based on the following lemma:

Lemma 1.2.1 The organization degree of a system S comprising two subsystems S_1 and S_2, characterized by probability fields *(p,q)* and *(q,p)*, $p + q = 1, \quad p \cdot q \geq 0$, respectively is $W = 2pq$.

Proof of the lemma

We shall use the notation H_1 and H_2 for the entropies of the two subsystems and H for the entropy of the system:

$$W = H_1 + H_2 - H$$

$$H_1 = -p \lg p - q \lg q; \quad H_2 = -q \lg q - p \lg p$$

(the logarithm is binary (base 2)).

The entropy of the system is defined with the help of the probability field:

$$H = -p^2 \lg p^2 - 2pq \lg 2pq - q^2 \lg q^2 = -2\left(p^2 \lg p + q^2 \lg q + 2pq \lg pq\right)$$

Then

$$\begin{aligned} W = H_1 + H_2 - H &= -p \lg p - 2q \lg q + 2p^2 \lg p + 2q^2 \lg q + 2pq \lg pq = \\ &= 2p(p-1)\lg p + 2q(q-1)\lg q + 2pq \lg 2pq = 2pq \lg 2 = 2pq \end{aligned}$$

Switching back to the demonstration of the theorem, we can notice that:

$$\sum_{i=1}^{n} H_i = \sum_{i=1}^{n} \frac{H_i}{H_l}, \quad H_l = H_l \sum_{i=1}^{n} C_i$$

According to the previous lemma, the entropy of the system composed of n decision-makers who target lower gains, is:

$$H = -2p^2 \lg p + q^2 \lg q + pq \lg pq$$

where

$$p = P\left(X_1 \geq a_1, X_2 \geq a_2,...,X_n \geq a_n\right)$$

$$q = 1 - p$$

Since we have estimated k as high enough, the decisional process tends to repeat itself, and for expressing p we shall use a known outcome according to which:

$$\frac{1}{k - 2^n} \sum_{j=1}^{k} \prod_{i=1}^{n} e^{-\frac{\left(a_i - A_{i,j}\right)}{n^{a_i}}}, \quad 0 < a_i < \frac{1}{n} \tag{24}$$

is a non-dependent asymptotic estimation, also considered asymptotically normal for p. The prediction at time k has the significance α, resulting:

$$P_k \geq \alpha \Rightarrow \sum_{i=1}^{k} C_i \geq \left[\alpha + (k-2)\varepsilon - 2\left(p^2 \lg p + q^2 \lg q + pq \lg pq\right)\right] \cdot \frac{1}{H} \tag{25}$$

Remark 1.6 In demonstrating this theorem, we have taken into consideration the case in which the system consists of two subsystems built under the two possibilities or all decision-makers obtain at least the gains estimated (possibly none) or only some of the decision makers obtain at least the lower limited gains estimated. This case allows writing the following form of prediction at time k:

$$P_k = \sum_{i=1}^{n} H_i - \left[W - \overline{H_1} + \overline{H_2}\right] - \sum_{i=2}^{k-1} P_i \tag{26}$$

where $\overline{H_1}$ and $\overline{H_2}$ are the entropies of the two subsystems mentioned above.

Remark 1.7 In the case of a maximum initial indeterminacy, $\overline{H_1}$ will be the entropy associated with the probability field.

$$\left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)$$

and thus:

$$\overline{H_i} = -\sum_{i=1}^{m} \delta_{ii} \frac{1}{m} \lg \frac{1}{m} = -\frac{m}{m} \lg \frac{1}{m} = -\lg \frac{1}{m} = \lg m \quad (\delta_n = 1)$$

The prediction will be significant at time k if:

$$\sum_{i=1}^{n} C_i \geq \left[\alpha + (k-2)\varepsilon + \left(W - \overline{H_1} - \overline{H_2}\right)\right] \cdot \frac{1}{\lg m} \tag{27}$$

1.2 Fundamental Decisional Processes

Utility. In specialized literature, the term of utility was introduced in order to clearly specify the properties that must be met by a function of efficiency.

There are two ways of introducing the concept of utility: ordinal utility (based on a axiomatic system) and cardinal utility (introduced under conditions imposed by the practice of market rules).

1.2.1 Considerations Regarding the Concept of Utility

We shall start from a set of entities E called the set of results, on which we introduce a binary relation called preference relation; if $x \geq y;\ x, y \in E$ then the outcome x is preferred to y.

The concept of experiment is also introduced by the notation $\left[p, x; 1-p, y\right]$ $x, y \in E, p \in [0,1]$, where: p represents the weight of the outcome x; $1-p$ is the weight of the outcome y.

It is obvious that if we consider the results $x_1, x_2, \ldots, x_n \in E$, the associated experiment becomes $[\, p_1, x_1;\ p_2, x_2; \ldots;\ p_n, x_n\,]$, where $p_1, p_2, \ldots, p_n$ are the weights of these outcomes.

The utility is written as a function $u : E \to R$ satisfying the following conditions (we shall consider only two results x, y, as well as the experiment generated by them):

if $x \geq y \Rightarrow u(x) > u(y)$;

$$u\left(\left[p, x; 1-p, y\right]\right) = pu(x) + (1-p)u(y)$$

This means that utility is a monotonically increasing function in relation to preference and the utility of an experiment equals the average value of the utilities that form the experiment.

Remark 1.8 If we consider the experiment $\left[p_1, x_1;\ p_2, x_2; \ldots;\ p_n, x_n\right]$ then its utility is given by the following equality:

$$u\left(\left[p_1, x_1; p_2, x_2; \ldots; p_n, x_n\right]\right) = \sum_{i=1}^{n} p_i u(x_i)$$

Axioms of rational behavior

This is an important issue for the financial practice, which is based on the concept of ordinal utility. The first system of axioms in terms of behavior was proposed by Neumann in 1935 and it was subsequently improved by Savage, Luce and Raiffa.

For the experiment $[p_1, x_1;\ p_2, x_2;\ \ldots;\ p_n, x_n]$, the axioms of rationality of Luce-Raiffa are the following:

1. Determining the order of possible results

- for any x_i and x_j we have $x_i \geq x_j$ or $x_j \geq x_i$ (any results are compatible);

- it has the property of transitivity, i.e.: $x_i \geq x_j, x_j \geq x_k \Rightarrow x_i \geq x_k$.

Based on these assumptions, we propose the concept of compound experiment.

Further on, we assume that $x_1 \geq x_2 \geq \ldots \geq x_n$ and we shall use A to note the experiment $[\,p_1, x_1; p_2, x_2; \ldots; p_n, x_n\,]$ and $A_1, A_2, \ldots, A_n$ for the experiments that have the following outcomes $x_1, x_2, \ldots, x_n$. Thus, the experiment $[\,q_1, A_1; q_2, A_2; \ldots; q_m, A_m\,]$ is called a compound experiment.

2.Reduction of compound experiments (to single experiments)

A compound experiment is equivalent to a single simple experiment

$$[\,q_1, A_1;\ q_2, A_2;\ \ldots; q_m, A_m\,] \approx [\,t_1, x_1;\ t_2, x_2;\ \ldots;\ t_n, x_n\,]$$

where: $t_j = \sum_{i=1}^{m} q_i t_j^{(i)}$, $t_j^{(i)}$ is the probability of entity x_j within the simple experiment E_i, $i = \overline{1,m}$; $j = \overline{1,n}$.

3.Continuity

For each outcome x_i a real number $s_i \in [0,1]$ is assigned so that $x_i \approx [\,s_i; x_1; 0, x_2; \ldots; 0, x_{n-1}; 1 - s_i, x_n\,]$ which can be written $x_i \approx [\,s_i, x_1; 1 - s_i, x_n\,] = X_i$. Thus, any result is equivalent to an experiment determined by the first and the last outcome.

4.Equivalence

$$[\,p_1, x_1;\ p_2, x_2;\ \ldots;\ p_n, x_n\,] \approx [\,p_1, x_1;\ p_2, x_2;\ \ldots;\ p_i, X_i;\ \ldots;\ p_n, x_n\,]$$

5.Transitivity

Preference and equivalence are transitive. Hence we have the following relations:

$$x_i \approx x_j, x_j \approx x_k \Rightarrow x_i \approx x_k$$

$$x_i \geq x_j, x_j \geq x_k \Rightarrow x_i \geq x_k$$

6.Monotony

$$[\,p_1, x;\ 1 - p_1, y\,] \geq [\,p_2, x;\ 1 - p_2, y\,] \Leftrightarrow p_1 \geq p_2$$

Based on these axioms the concept of utility function presented above can be improved by specifying the property of linearity.

Definition 1.1 The function of utility

$$u(\,p, A_1;\ 1 - p, A_2\,) = pu(\,A_1\,) + (\,1 - p\,)u(\,A_2\,) \tag{28}$$

is linear regardless of the simple experiments A_1 and A_2 and $p \in [\,0,1\,]$.

More general, if we know the experiment $A = [\,p_1, A_1; p_2, A_2; \ldots; p_n, A_n\,]$ we have $u(A) = p_1 u(A_1) + p_2 u(A_2) + \ldots + p_n u(A_n)$

This definition reveals an important characteristic of the function of utility, that is to say it is unique up to a positive linear transformation. In other words, if u_1 and u_2 are functions of utility, there will be the real and positive constants a, b so that $u_2 = au_1 + b$.

1.2.2 Non – Cooperative Games

The first preoccupations of treating mathematically the conflictual-competitional situations belong to Zermelo.

The conflictual character of a decisional situation derives from the fact that the objectives of the deciders (the players) can not be done simultaneously, and the non-cooperative character from the fact that the choices of strategies are individual acts effectuated in the lack of an information exchange and of concluding some firm agreements.

The first mathematical works consecrated to the study of the basic elements of the theory of games (problems of the analyze of the conflictual situations and the cooperative problems, too) are due to Neumann [57], [58].

The two-person games (two players) with a null sum analyzed by Neumann have been extended by Nash to $n-$ person games by Nikaide and Isoda [60], Karlin [37] to the convex games and generalized convex and Shapley [77] to the stochastic games.

Other models of non-cooperative games with the function of particular paying can be found in [14], [29], [40].

The model of non-cooperative games under a normal form which we shall deal with can be considered as a particular case of the non-cooperative game in an extension form [82].

Definition 1.2

We shall consider an $n-$ person (players) game in a normal form, the ensemble:

$$\Gamma = \{x_i, U_i, i \in M\}$$

where:

- $M = \{1, 2, ..., m\}$ represents the set of players;

- X_i is a non-void set named the set simple strategies of the decider i, $i = \overline{1,m}$.

We shall denote by $X = \prod_{i=1}^{m} X_i$.

$U_i : X \to \mathbb{R}$ is named the function of utility of the decider i, $i = \overline{1,m}$.

From strategic point of view, the decisional optimal behavior of the deciders of the set M is linked of the adoption of those strategies $x \in X$ which are equilibrium points for Γ.

Necessary and sufficient conditions of the existence of equilibrium points have been shown in [82].

Definition 1.3

Γ is named matrix game if the following conditions are met:

1) $n = 2$

2) $U_1 + U_2 = 0$ (the game is a zero-sum one)

3) X_1 and X_2 are finite. Let's assume that we have:

$$X_1 = \{x_1^1, x_1^2, ..., x_1^m\}; X_2 = \{x_2^1, x_2^2, ..., x_2^m\}$$

If we note $U = U_1 = -U_2$, to the matrix game Γ will be associated the matrix A:

$$A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}, \ a_{ij} = U(x_1^i, x_2^j)$$

Remark 1.9

The association of matrix A to the game Γ, justifies the terminology of a matrix game. If (x_1^0, x_2^0) is an equilibrium point for Γ then, from (6), it results immediately that (x_1^0, x_2^0) is a saddle point for U on $X_1 \times X_2$, and (x_1^0, x_2^0) is the solution of the problem:

$$\max_{x_1 \in X_1} \min_{x_2 \in X_2} U(x_1, x_2) = \min_{x_2 \in X_2} \max_{x_1 \in X_1} U(x_1, x_2) \tag{30}$$

Let's consider $M(\Gamma)$ the set of the equilibrium points of the game Γ and $(\overline{x}_1, \overline{x}_2) \in M(\Gamma)$, $(\overline{\overline{x}}_1, \overline{\overline{x}}_2) \in M(\Gamma)$.

Theorem 1.3 [4], [82]

The following properties take place:

1) $U(\overline{x}_1, \overline{x}_2) = U(\overline{\overline{x}}_1, \overline{\overline{x}}_2)$

2) $(\overline{x}_1, \overline{\overline{x}}_2) \in M(\Gamma)$, $(\overline{\overline{x}}_1, \overline{x}_2) \in M(\Gamma)$

Definition 1.4

$p = (p_1, p_2, ..., p_m)$, $p_i \geq 0$, $i = \overline{1,m}$, $\sum_{i=1}^{m} p_i = 1$ is called mixed strategy for the decider 1; the choice of the mixed strategy, by the decider 1, means the use of strategy x_1^i with the probability p_i, $i = \overline{1,m}$.

The notion of mixed strategy is defined in the same way for the decider 2.

We denote by $\tilde{X}_1$, $\tilde{X}_2$ the sets of the mixed strategies of both deciders.

For the case in which the requirements of the existence of equilibrium points for Γ are not met, the optimum solution is sought as a saddle point for the mean value of the game on the set $\tilde{X}_1 \times \tilde{X}_2$.

Definition 1.5

The mean value of the game associated to the pair $\left(\overline{p},\overline{q}\right)$ is the value of the function V in $\left(\overline{p},\overline{q}\right)$, where:

$$V : \tilde{X}_1 \times \tilde{X}_2 \to \mathbb{R}, V\left(p,q\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} p_i q_j,\ p = \left(p_1, p_2,..., p_n\right) \in \tilde{X}_1,\ q = \left(q_1, q_2,..., q_n\right) \in \tilde{X}_2$$

If $\left(p^0,q^0\right) \in \tilde{X}_1 \times \tilde{X}_2$ is a saddle point of the V on $\tilde{X}_1 \times \tilde{X}_2$, $V\left(p^0,q^0\right)$, then this will be called the value of the game.

In mixed strategies, every matrix game has saddle points and these can be found as solutions of a linear programming problem.

Theorem 1.4 (Ghermeier [29])

The solving of a matrix game in mixed strategies with the matrix $A = \left(a_{ij}\right)_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$ is equivalent with solving of a couple of dual linear programming problems.

$$\max\sum_{j=1}^{n} y_i \qquad\qquad \min\sum_{i=1}^{m} x_i$$

$$\begin{cases} \sum_{j=1}^{n} a_{ij} y_j \le 1,\ i = \overline{1,m} \\ y_j \ge 0 \qquad ,\ j = \overline{1,n} \end{cases} \qquad \begin{cases} \sum_{i=1}^{m} a_{ij} x_i \ge 1,\ j = \overline{1,n} \\ x_i \ge 0 \qquad ,\ i = \overline{1,m} \end{cases} \tag{31}$$

If $y^0 = \left(y_1^0, y_2^0,..., y_n^0\right)$, $x^0 = \left(x_1^0, x_2^0,..., x_m^0\right)$ are the optimal solutions of both problems, then the value of the game is:

$$V = \frac{1}{\sum_{j=1}^{n} y_j^0} = \frac{1}{\sum_{i=1}^{m} x_i^0},$$

and the optimal mixed strategies are:

$$p^0 = \left(p_1^0, p_2^0,..., p_m^0\right),\ p_i^0 = x_i V,\ i = \overline{1,m}$$
$$q^0 = \left(q_1^0, q_2^0,..., q_n^0\right),\ q_j^0 = y_j V,\ j = \overline{1,n}$$

Remark 1.10

An iterative method of solving the matrix game is shown in [63].

1.2.3 Cooperative Games under the Form of a Characteristic Function with Rewards

The cooperative games are those which allow the exchange of information and the concluding of agreements between firms.

The degree of cooperative in such a game depends on the possibility to realize or not the transfer of utility.

Based on this requirement, the cooperative games are classified as follows:

1) cooperative games without compensations;

2) cooperative games with compensations.

There isn't a unitary point of view regarding the theory of cooperative games. Further on, we shall consider ourselves in the case of cooperative games in the form of a characteristic function with compensations; the other models of cooperative games (total cooperative game, cooperative games in the form of a characteristic function without compensations) are presented in [29], [49], [63].

The basic elements which are going to be analyzed are connected with the notions of characteristic function, imputation, domination, nucleus, solution.

Let us consider $\Gamma = \left\{X_i, U_i, i \in M\right\}$, a $n-$person game in a normal form.

Definition 1.6

Any subset C of M is named a coalition; if $C = M$ the coalition is called a total coalition; if $C = \left\{i\right\}$, i being any element of M the coalition will be named a banal coalition.

Definition 1.7

The application $v : \mathcal{P}(M) \to \mathbb{R}$ which verifies the properties:

$$\begin{aligned} & v(\varnothing) = 0 \\ & v(A \cup B) \geq v(A) + v(B), \forall A, B \in \mathcal{P}(M) \end{aligned} \tag{32}$$

is called the characteristic function.

Definition 1.8

$Z = (z_1, z_2, ..., z_m) \in \mathbb{R}^m$ is considered to be a determined imputation of v if the following conditions are met:

$$\begin{aligned} & 1)\ z_i \geq v(i), \forall i \in M \\ & 2) \sum_{i=1}^{m} z_i = v(M) \end{aligned} \tag{33}$$

We denote by Z_v the set of the imputations determined by v.

Definition 1.9

We shall call a cooperative game in the form of a characteristic function the ensemble:

$$\bar{\Gamma} = \{M, v, Z_v\}$$

Let us assume $C \in \mathcal{P}(M)$, $C \neq \varnothing$ on the set Z_v; we shall introduce a relation of a partial order as follows:

$z \in Z_v$ dominates relative to C on $\bar{z} \in Z_v$ $\left(z \succ^C \bar{z}\right)$ if the conditions are verified:

$$\begin{aligned} & \sum_{i \in C} z_i \leq v(C) \\ & z_i \geq \bar{z}_i, \forall i \in C \end{aligned}$$

We denote by $dom_C\, z = \left\{\bar{z} \in Z_v \,\middle|\, z \succ^C \bar{z}\right\}$.

We shall consider $z, z' \in Z_v$.

Definition 1.10

We say that z dominates on z' $(z \succ z')$ if there exists $C \in \mathcal{P}(M)$ so that $z \succ^C z'$.

Definition 1.11

$S \subseteq Z_v$ is named the solution of the game (a stable set) if the following condition holds true:

$$\{z' \in Z_v \,|\, \exists z \in S, z \succ z'\} = Z_v / S$$

We note $\mathcal{S}$ the set of coalitions of the game $\bar{\Gamma}$.

Definition 1.12

$\mathcal{N} \subseteq Z_v$ is called a nucleus for $\bar{\Gamma}$ if for any $z \in Z_v$, $C \in \mathcal{P}(M)$ we have:

$$\sum_{i \in C} z_i \geq v(C) \tag{34}$$

Theorem 1.5 (Owen [63])

$\mathcal{N}$ is the set of undominated imputations and for any $S \in \mathcal{S}$, $S \neq \varnothing$ we have $\mathcal{N} \subset S$.

Theorem 1.6 (Ştefãnescu [82])

For any two-person game the next equalities take place:

$$\mathcal{N} = Z_v = \mathcal{S}$$

Theorem 1.7 (Ştefãnescu [82])

Any three-person game admits at least a non-void solution.

Other solutions of the cooperative game $\bar{\Gamma}$ have been given by Shapley [77], Caplow [11], Auman and Maschler [3], Mitran [47].

Returning to the game in normal form Γ, let us consider $C \in \mathcal{P}(M)$, $C \neq \varnothing$.

We shall note $U_C = \sum_{i \in C} u_i$, $X_C = \prod_{i \in C} X_i$, $\bar{X}_C = \prod_{i \in M \setminus C} X_i$.

The application $v : \mathcal{P}(M) \to \mathbb{R}$, given by:

$$v(C) = \max_{x \in X_C} \min_{y \in \overline{X}_C} U_C(x, y) \tag{35}$$

is a characteristic function [63].

Other construction of the function of utility U_C for the coalition C can be found in [44] and [47].

The gain of the coalition is the one maximum guaranteed in the zero-sum game from the coalitions C and $M \setminus C$ (the antagonist game between C and $M \setminus C$).

This point of view is debatable. First of all, the formation of the coalition C doesn't compulsory lead to the formation of the coalition $M \setminus C$. Secondly, it isn't compulsory that all the deciders of the set $M \setminus C$ adopt a strategically behavior opposite to the achievement of objectives of the deciders from C.

Thirdly, the adoption of the maximum criterion as a criterion of optimality for the deciders from coalition C (which is a providing criterion and its adoption generally leads to little gains) is not a unanimous accepted objective in real decisional situations.

1.2.4 The Sequential Decision Problem

The first formulation of a problem of the sequential decision is due to Shapley [60]. This was accomplished by the formulation of a model of the stochastic game, a decisional sequential process with n deciders of a private kind for which Shapley demonstrated the existence of the equilibrium points.

Another decisional sequential model was formulated by Wald [88] in connection with a problem of statistical decision with experiments.

The decisional sequential model which is to be presented next is the same to those from [6], [35] and [50]. Let's consider X the linear topological space (real) and $\mathcal{B}_X$, $\Gamma-$ algebra generated by the topology of space X. We shall associate to the measurable space $(X, \mathcal{B}_X)$ the set $\mu(\mathcal{B}_X)$ of all the measures of probability defined on $\mathcal{B}_X$.

The set X will be named the set of positions.

We denote by X_0, $\overline{X}$ the set of initial positions, respectively final positions; we assume that X_0, $\overline{X}$ are compact sets. To each position $x \in X$, we shall associate the measure of probability $P_x \in \mu(\mathcal{B}_X)$; for every $x \in X$ we denote by $\mathcal{L}_2^x(X, \mathcal{B}_X, P_x)$ the normed Hilbert space of all real random variables defined on X by the moment of order 2 finite [35], [47].

The set $M = \{1, 2, ..., m\}$ designates the set of the deciders that take part to the decisional process and $u_i \in \mathcal{L}_2^x(X, \mathcal{B}_X, P_x)$, $\forall x \in X$ represents the function of utility of the decider $i \in M$.

To each decider $i \in M$, we shall associate the measure $a_i \in \mathbb{R}_+$ named ceiling, which signifies the fact that the participation of the decider i to the decisional process is linked to his intention to obtain a gain which has to increase the ceiling.

The set $\overline{X}_i = \{\overline{x} \in \overline{X} \mid u_i(\overline{x}) \geq a_i\}$ is called a target set of the decider $i \in M$.

The evolution of the decisional process is described by means of the relations of recurrence:

$$x_{n+1} = f_n(x_n, d_n), x_0 \in X_0 \ (any), n \in \mathbb{N} \tag{36}$$

where: $d_n \in \mathcal{D}_{(x_n)} = \prod_{i=1}^{m} D^i(x_n)$; $D^i(x_n)$ represents the set of decision that can be adopted in position $x_n \in X$ by the decider $i \in M$ (hoping that for any $x \in X$, the set of strategies $D^i(x_n)$ is linear topological space).

Applications $f_n : X \times \mathcal{D}_{(x)} \to X$, $n \in \mathbb{N}$, are called functions of transition and we shall consider them to be continuous and bounded [35]. $\mathcal{D}_X = \bigcup_{x \in X} \mathcal{D}_{(x)}$. If $x_n \in \overline{X}$, then $f_n(x_n, d_n) = x_n$, $\forall d_n \in \mathcal{D}_{(x_n)}$.

When there exists no risk of confusion, we denote $D^i(x_n)$ by D_n^i and $\mathcal{D}_{(x_n)}$ by $\mathcal{D}_n$.

Definition 1.13

We name a decisional sequential problem the ensemble:

$$S = \{X; X_0, \overline{X}; M; U_i, i \in M; D_n^i, i \in M, n \in \mathbb{N}; f_n, n \in \mathbb{N}\}$$

where the elements of S have been specified before.

Remark 1.11

Using a result from [71], (36) we can write:

$$x_{n+1} = Ax_n + Bd_n + \lambda C_n x_n \tag{37}$$

in which the linear part was separated out of the non-linear part (we considered, in addition that from $dim\, X = K$, $dim\, \mathcal{D}_n = m$, $\forall n \in \mathbb{N}$).

In (37) A and B are matrixes with the dimensions $K \times K$, respectively $K \times m$ (with real elements), C_n is a matrix whose elements are non-linear discrete functions, $n \in \mathbb{N}$, and λ is a scalar matrix of the weights of the non-linearities [35].

Remark 1.12

If we assume that the set of the deciders is made up of a single decider and we associate with the dynamic equation (36) a functional of efficiency, we are led to a discrete system of command.

Definition 1.14

We call a trajectory, starting from $\tilde{x}_0 \in X_0$, any sequence $\left(x_n\right)_n \in X$, defined by (36) with $x_0 = \tilde{x}_0$.

Definition 1.15

We call a trajectory of duration, starting from $\tilde{x}_0 \in X_0$, any sequence $\left(\overline{x}_n\right)_n \subset X$ defined as follows:

$$\overline{x}_0 = \tilde{x}_0$$

$$\overline{x}_n = \begin{cases} x_n \text{ , } x_n \textit{ defined by (36), if } 1 < n \le N \\ 0 \qquad\qquad\qquad\qquad \textit{, if } n > N \end{cases}$$

We denote by $F\left(x_0\right)$ the set of trajectories that start from $x_0 \in X_0$ and with $F^N\left(x_0\right)$ the set of trajectories of duration N that start from $x_0 \in X_0$.

We shall construct by means of recurrence the sequence of multivocal applications $\left(B_n\right)_n$ in the following approach (fig.1):

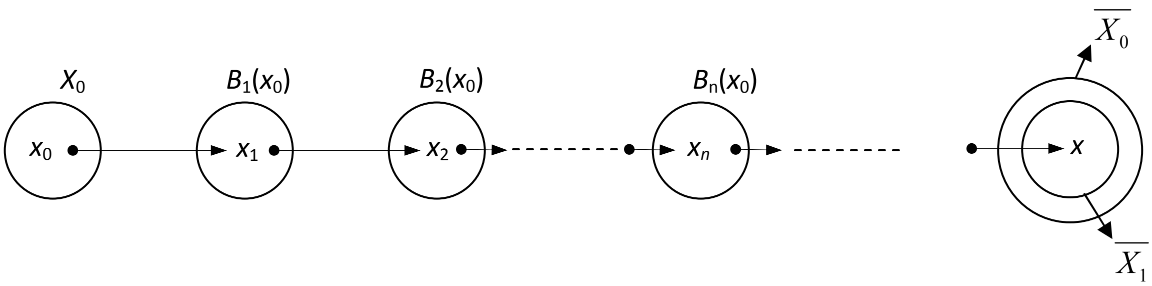

Figure 1.

With the help of fixed quantities $a_1, a_2, \ldots, a_n$, one can build the sets $\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_m$ as follows:

$$\begin{cases} \overline{X}_1 = \left\{x \in \overline{X} : P\left(x\right) \ge a_1\right\} \\ \overline{X}_2 = \left\{x \in \overline{X} : P\left(x\right) \ge a_1\right\} \\ \vdots \\ \overline{X}_2 = \left\{x \in \overline{X} : P\left(x\right) \ge a_1\right\} \end{cases}$$

In terms of decision theory, the real positive numbers $a_1, a_2, \ldots, a_n$ are limits (i.e. the limit from which the participation of decision-makers to the decisional process becomes interesting). The sets $\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_m$ are called target sets of decision makers.

Remark 1.13

If target sets form a partition of the set $\overline{X}$, then the following conditions are met:

$$\bigcup_{i=1}^{m} \overline{X}_i = \overline{X} \text{ , } \overline{X}_i \bigcap \overline{X}_j = \varnothing$$

Remark 1.14

For any $n \in \mathbb{N}$ the following equality takes place:

$$F^n(x_0) = \{x_0\} \times \prod_{i=1}^{n} B_i(x_0)$$

The set $B_n(x_0)$ represents the set of the positions in which it is possible to get in n sequences starting from the initial position $x_0 \in X_0$. It is noted:

$$\mathcal{D}_n(x_0) = \mathcal{D}_{(x_n)};\ x_n \in B_n(x_0)$$

Application regarding the money market equation

Suppose one knows the estimated incomes $V_0, V_1, V_2, \ldots, V_n$ at moments $t = 0, t = 1, \ldots, t = n$. The unit interests from the money market on the intervals $[0,1), [1,2), \ldots, [n-1, n)$ are denoted by $i_1, i_2, \ldots, i_n$.

We are interested in the maximum incomes $\overline{V}_0, \overline{V}_1, \ldots, \overline{V}_n$ that can be achieved at certain given moments. These revenues will always be a solution to the money market equation.

This problem can be immediately interpreted as a sequential decision problem in which the states represent the maximum incomes and the strategies adopted represent the interest rates on the money market. Maximum incomes are calculated gradually according to figure 2.

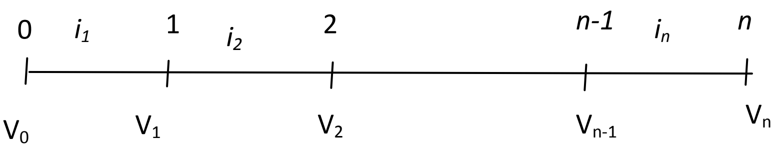

Figure 2

$$t = 0 \qquad \overline{V_0} = V_0$$

$$t = 1 \qquad \begin{cases} \overline{V_0} = V_0 + V_1(1+i_1)^{-1} \\ \overline{V_1} = V_1 + V_0(1+i_1) \end{cases} \tag{38}$$

$$t = 2 \qquad \begin{cases} \overline{V_0} = V_0 + V_1(1+i_1)^{-1} + V_2(1+i_1)^{-1}(1+i_2)^{-1} \\ \overline{V_1} = V_1 + V_0(1+i_1) + V_2(1+i_2)^{-1} \\ \overline{V_2} = V_2 + V_0(1+i_1)(1+i_2) + V_1(1+i_2) \end{cases} \tag{39}$$

$\vdots$

$$t = n \begin{cases} \overline{V_0} = V_0 + V_1(1+i_1)^{-1} + V_2(1+i_1)^{-1}(1+i_2)^{-1} + \ldots + V_n(1+i_1)^{-1}(1+i_2)^{-1}\ldots(1+i_n)^{-1} \\ \overline{V_1} = V_1 + V_0(1+i_1) + V_2(1+i_2)^{-1} + \ldots + V_n(1+i_2)^{-1}\ldots(1+i_n)^{-1} \\ \vdots \\ \overline{V_n} = V_n + V_0(1+i_1)(1+i_2)\ldots(1+i_n) + \ldots + V_{n-1}(1+i_n) \end{cases} \tag{40}$$

The money market equation is determined as follows:

Case $t = 2$:

The targeted equation is actually the equation of a straight line determined by points $\left(\overline{V_0}, 0\right)$ and $\left(0, \overline{V_1}\right)$:

$$(D) \qquad \begin{vmatrix} x & y & 1 \\ \overline{V_0} & 0 & 1 \\ 0 & \overline{V_1} & 1 \end{vmatrix} = 0 \tag{41}$$

After calculations we get:

$$(D) \qquad y = -\left(1+i_1\right)x + V_0\left(1+i_1\right) + V_1 \tag{42}$$

Case $t = 3$:

The targeted equation is actually the equation of a plane determined by points $\left(\overline{V_0},0,0\right),\left(0,\overline{V_1},0\right),\left(0,0,\overline{V_2}\right)$:

$$(P) \qquad \begin{vmatrix} x & y & z & 1 \\ \overline{V_0} & 0 & 0 & 1 \\ 0 & \overline{V_1} & 0 & 1 \\ 0 & 0 & \overline{V_2} & 1 \end{vmatrix} = 0 \tag{43}$$

that is to say:

$$(P) \qquad x\left(1+i_1\right)\left(1+i_2\right) + y\left(1+i_2\right) + z - V_0\left(1+i_1\right)\left(1+i_2\right) - V_1\left(1+i_2\right) - V_3 = 0 \tag{44}$$

The general case $t = n$:

In this case, the targeted equation is in fact the equation of a hyper-plane determined by the points $\left(\overline{V_0},0,\ldots,0\right),\left(0,\overline{V_1},0,\ldots,0\right),\ldots,\left(0,0,\ldots,0,\overline{V_n}\right)$.

As a result of calculations, the hyper-plane equation is the following:

$$(H) \qquad \sum_{i=1}^{n} x_k \prod_{j=k}^{n}\left(1+i_j\right) - \sum_{k=1}^{n} V_k \prod_{j=k}^{n}\left(1+i_j\right) = 0 \tag{45}$$

The problem of determining the maximum incomes at different time points can be approached as a sequential decision problem; the states $x_0, x_1, \ldots, x_n$ are defined by the following vectors (in accordance with (38) (39) (40)):

$$\begin{cases} x_0 = \overline{V_0} \\ x_1 = \left(x_1^1, x_2^1\right),\, x_1^1 = \overline{V_0},\, x_2^1 = \overline{V_1} \\ x_2 = \left(x_1^2, x_2^2, x_3^2\right),\, x_1^2 = \overline{V_0},\, x_2^2 = \overline{V_1}, x_3^2 = \overline{V_2} \\ \vdots \\ x_n = \left(x_1^n, x_2^n, \ldots, x_{n+1}^n\right),\, x_1^n = \overline{V_0},\, x_2^n = \overline{V_1}, \ldots, x_{n+1}^n = \overline{V_n} \end{cases} \tag{46}$$

If $Đ_k = \left\{i_1, i_2, \ldots, i_k\right\},\ k = \overline{1,n}$

1.3 Bibliographical Notes and Comments

The key issues approached in this chapter are related to the concepts of entropy and utility as well as to several important decisional processes. Being an introductory chapter, we have presented the basic notions and results and we did not insist on the details.

For a certain experiment, the entropy measures either the amount of indeterminacy within the experiment (before its occurrence) or the information obtained (after carrying out the experiment) [84]. The first concerns related to determining the measure of entropy are owing to Hartley starting from the study of communication problems. In fact, the measure of the entropy introduced by him is similar to the *H* function introduced in statistical physics by Boltzmann.

The first rigorous measure of entropy was introduced in 1948 by Shannon; he practically formulated the entropic measure as a solution to a certain functional equation. This measure is known as the first un-weighted entropy. Extensions of this entropy have been formulated by other researchers as well, the most interesting ones belonging to Renyi and Daroczy and also known as un-weighted entropy. This type of entropies has proven that it cannot be used successfully in the analysis and interpretation of some results of complex problems. It was concluded that the lack of details regarding the quantification of states is the main shortcoming.

For example, in the case of the fields of probability $\left(\frac{1}{10},\frac{9}{10}\right)$ and $\left(\frac{9}{10},\frac{1}{10}\right)$ Shannon's entropies have the same value $-ln\frac{9}{100}$ which, from practical point of view, is not correct (it is imperative to use the log base 2, and not the natural logarithm).

Therefore, there were introduced new concepts of entropy which take into account the efficiency of different states, known as weighted entropies [44] [63]. Starting from different points of view, we have obtained different results and interpretations.

This paper does not analyze the concept of entropy based on infinite fields of probability.

Taking into consideration the concept of entropy, one can introduce certain entropic measures used in systems analysis: degree of organization, degree of concentration, prediction, uncertainty [44] [84].

A term of reference within the decision theory is the utility concept. Mathematically, utility is a function defined by the set of states (ordered by preferences) and it is unique until a positive linear transformation occurs. It was introduced under an axiomatic system describing rational behavior. This behavior is usually described by a list of axioms (first formulated by Neumann and Morgenstern [58]). Subsequently, the changes made to that list of axioms were made by Savage, Suppes, Winet, Luce and Raiffa [44] [63].

Much of this work is part of the game theory (non-cooperative and cooperative). This theory was based on the work of Neumann and Morgenstern's [58] and it dwells on the following fundamental principles: *each player adopts a rational behavior* and all *players know the rules of the game.* Non-cooperative games imply the competitive nature of the decisional process. One of the principles of optimum behavior in the case of non-cooperative games with *n* players is when the optimal solution is an equilibrium point. The first result regarding the existence of an equilibrium in *n*-person games is owing to Nash [56] who showed that any finite game admits equilibrium points within mixed strategies. This result was extended by Nikaido and Isoda [60] who stated that any convex game admits equilibrium points. In the case of stochastic games "with stops" the existence of equilibrium points was demonstrated by Shapley [77] who also thought of a method to determine them. Gillette made other types of stochastic games (with updates and without stops) and the method of solving equilibrium points was given by Hoffman for two-player zero sum games. The existence of equilibrium points in general was demonstrated by Sobel while Maitre and Parthasarathi made generalizations in the infinite case [82].

Cooperative games are those games that enable the exchange of information and firm agreements.

The degree of cooperation in such a game depends on the possibility to carry out or not the utility transfer. According to this goal, cooperative games can be classified as follows: cooperative games without compensation and cooperative games with compensation. The central problem in the case of cooperative games is the distribution of individual gains. Since there are multiple ways of making coalitions, it is obvious that there are several types of relationships for final earnings Shapley [77], Caplow [11], Aumann and Mascher [3].

The sequential decision problem is a particular case of decisional process. Since the efficiency function meets certain requirements, the optimum principle adopted is owing to Bellman [6]. In stationary conditions, the solution of such a problem is determined as a solution of a functional equation; the most recommended method is that of successive approximations.

THE MINMAX INEQUALITY AND EQUALITY

2.1 Maxmin and Minmax Optimum Guaranteed Values and the Minmax Equality

2.1.1 The Minimum Function

For the beginning, we assume a game with two decision makers $J_1(D_1,D_2,F)$ where (D_1,d_{D_1}), (D_2,d_{D_2}) are metric spaces (D_1, D_2 represents the sets of strategies of the two deciders).

We construct Haussdorff's metric:

$$d:\mathcal{P}(D_2)\times\mathcal{P}(D_2)\to\mathbb{R},\, d(A,B)=max\{\rho(A,B);\rho(B,A)\},\,\forall A,B\subset D_2$$

where $\rho(A,B)=\sup_{x\in A}\inf_{y\in B} d_{D_2}(x,y)$.

Let $T:D_1\to\mathcal{P}(D_2)$ be the informational (multivocal) application of the first decider [25], [47].

Definition 2.1

The set $Td_1\subseteq D_2$ is called the informational set of decider 1 corresponding to the strategy $d_1\in D_1$ and it represents the set of the strategies that decider 2 can take if decider 1 adopted the strategy $d_1\in D_1$.

Definition 2.2

The multivocal application T is called:

1) s.c.s. in $d_1^0\in D_1$, if $\lim_{d_1\to d_1^0}\rho(Td_1,Td_1^0)=0$;

2) s.c.i. in $d_1^0\in D_1$, if from conditions:

$$\lim_n d_1^n=d_1^0,\, d_2^0\in Td_1^0$$

results that there exists $(d_2^n:d_2^n\in Td_1^n)_n$ so that $d_2^0=\lim_n d_2^0$;

3) closed in $d_1^0\in D_1$, if from conditions:

$$\lim_n d_1^n=d_1^0,\,\lim_n d_2^n=d_2^0,\, d_2^n\in Td_1^n$$

results $d_2^0\in Td_1^0$;

4) continue in $d_1^0\in D_1$, if $\lim_{d_1\to d_1^0} d(Td_1,Td_1^0)=0$.

Remark 2.1

The conditions of s.c.i. and s.c.s. according to definition 2.2 in $d_1^0\in D_1$ for the multivocal application T is generally not for assuring the continuity in d_1^0; but if D_2 is sufficient compact, the continuity is assured. We suppose that D_1 and D_2 are compact.

Consider the functional:

$$R:D_1\to\mathbb{R},\, R(d_1)=\inf_{d_2\in Td_1}F(d_1,d_2)$$

Theorem 2.1

If F is s.c.s. on $D_1\times D_2$, T is continuous in $d_1^0\in D_1$, then functional R is s.c.s. in $d_1^0\in D_1$.

Proof

Take any $\varepsilon>0$ and small enough.

As $R(d_1^0)=\inf_{d_2\in Td_1^0}F(d_1^0,d_2)$ there exists $d_2^0\in Td_1^0$ so that the inequality:

$$F(d_1^0,d_2^0)\le R(d_1^0)+\frac{\varepsilon}{2} \tag{47}$$

hold true.

From condition of s.c.s. of F in (d_1^0,d_2^0) for the chosen $\varepsilon>0$ it will exist $\delta>0$ so that:

$$F\left(d_1^0, d_2^0\right) \geq F\left(d_1, d_2\right) - \frac{\varepsilon}{2}, \forall\left(d_1, d_2\right) \in D_1 \times D_2, d_{D_1 \times D_2}\left(\left(d_1, d_2\right), \left(d_1^0, d_2^0\right)\right) \leq \delta$$

From continuity condition of T occurs that there is a $\gamma > 0$ so that $d_{D_2}\left(Td_1, Td_1^0\right) \leq \delta$ for any $d_1 \in D_1$ with the property $d_{D_1}\left(d_1, d_1^0\right) \leq \gamma$.

Take $V_{d_1^0} = \left\{d_1 \in D_1 \middle| d_{D_1}\left(d_1, d_1^0\right) \leq min(\delta, \gamma)\right\}$. Regardless $d_1 \in V_{d_1^0}$, there is a $d_2 \in Td_1$ so that $d_{D_2}\left(d_2, d_2^0\right) \leq \delta$.

So for any $d_1 \in V_{d_1^0}$ there is a $d_2 \in Td_1$ so that:

$$R\left(d_1^0\right) + \frac{\varepsilon}{2} \geq F\left(d_1^0, d_2^0\right) \geq F\left(d_1, d_2\right) - \frac{\varepsilon}{2} \tag{48}$$

as

$$R\left(d_1\right) \leq F\left(d_1, \tilde{d}_2\right), \forall \tilde{d}_2 \in Td_1 \tag{49}$$

From (48) and (49) the inequality occurs:

$$R\left(d_1^0\right) \geq R\left(d_1\right) - \varepsilon, \forall d_1 \in V_{d_1^0}$$

which means that R is s.c.s. in d_1^0.

Consequence 2.1.1

If F is s.c.s. on $D_1 \times D_2$, T is continuous on D_1, then there is a $d_1^* \in D_1$ so that:

$$R\left(d_1^*\right) = \max_{d_1 \in D_1} \inf_{d_2 \in Td_1} F\left(d_1, d_2\right)$$

Remark 2.2

We assume F continue. As the functional of gain of the second decider is $G = -F$, if it is defined the functional $\tilde{f} : D_1 \times D_2 \to \mathbb{R}$:

$$\tilde{f}\left(d_1, d_2\right) = G\left(d_1, d_2\right) - \max_{d_2 \in D_2} G\left(d_1, d_2\right)$$

then the best guaranteed result of the first decider will be [17], [24], [25]:

$$\sup_{d_1 \in D_1} \min_{d_2 \in Td_1} F\left(d_1, d_2\right)$$

if

$$Td_1 = \left\{d_2 \in D_2 \middle| \tilde{f}\left(d_1, d_2\right) = 0\right\} \neq \varnothing \tag{51}$$

The form in which the informational set of the first decider occurs in (51) leads us to consider the informational application T to be defined as follows:

$$Td_1 = \left\{d_2 \in D_2 \middle| f\left(d_1, d_2\right) \geq 0\right\} \tag{52}$$

where $f : D_1 \times D_2 \to \mathbb{R}$ is a known functional.

We denote by $T^0 d_1 = \left\{d_2 \in D_2 \middle| f\left(d_1, d_2\right) > 0\right\}$.

Theorem 2.2 [25], [42], [49]

The following results occur:

1) if F is continuous and $\overline{T^0 d_1} = Td_1$, $\forall d_1 \in D_1$, then T is continuous on D_1;

2) if in addition to 1), we assume that F is continuous on $D_1 \times D_2$ and Td_1 is compact $\forall d_1 \in D_1$, then R is continuous, so there is a $d_1^* \in D_1$ thus:

$$R\left(d_1^*\right) = \max_{d_1 \in D_1} \min_{d_2 \in Td_1} F\left(d_1, d_2\right) \tag{53}$$

Remark 2.3

The conditions of continuity of T proved at point 1) acted essentially only to assure the condition of closeness and the equalities (53).

Remark 2.4

The condition of compactity of D_1, D_2 and the continuity of F does not assure in the case of the games with an informational exchange, the contingence of the optimal guaranteed value:

$$V = \max_{d_1 \in D_1} \min_{d_2 \in Td_1} F\left(d_1, d_2\right) \tag{54}$$

(likewise in the case of the games without an informational exchange). But if F, T are continuous, $Td_1 \neq \varnothing$, $\forall d_1 \in D_1$, then the contingence of quantity V is realizable [82].

Let's notice that in comparison with conditions of theorem 2.2, these conditions are modified (in the sense that have been weaker and others are harder).

The performances obtained in the case of game J_1 can be extended to a game $J_n = \left(\prod_{i=1}^{n} D_i \times \bar{D}_i, F_1 \right)$ in which the deciders are coalized into two coalitions $C_1 = \{1,3,...,2n-1\}$, $C_2 = \{2,4,...,2n\}$ with opposite interests (we marked with F_1 – the functional of gain and D_i, $\bar{D}_i$ the set of the strategies of decider i, $i = \overline{1,n}$).

In this case, the best guaranteed result of the first decider is given by the quantity:

$$\sup_{d_1 \in D_1} \inf_{\bar{d}_1 \in \bar{D}_1} \dots \sup_{d_n \in D_n} \inf_{\bar{d}_n \in \bar{D}_n} F\left(d_1, \bar{d}_1, ..., d_n, \bar{d}_n\right) = \left[\sup_{d_i \in D_i} \inf_{\bar{d}_i \in \bar{D}_i} \right]^{(n)} F\left(d_1, \bar{d}_1, ..., d_n, \bar{d}_n\right) \tag{55}$$

Let's assume that in game J_n the informational exchange is permitted; the informational sets of those $2n$ deciders are A_i, $i = \overline{1,n}$ for the deciders from C_1 and B_i, $i = \overline{1,n}$ for the deciders from C_2, being constructed by means of the informational applications T_i (for the deciders from C_1), $\bar{T}_i$ (from the deciders from C_2), $i = \overline{1,n}$:

$$A_i = T_i\left(d_1, \bar{d}_1, ..., d_{i-1}, \bar{d}_{i-1}\right) = \left\{ d_i \in D_i \,\middle|\, f_i\left(d_1, \bar{d}_1, ..., d_{i-1}, \bar{d}_{i-1}, d_i\right) \geq 0 \right\}, i = \overline{2,n}, A_1 = D_1$$

$$B_i = \bar{T}_i\left(d_1, \bar{d}_1, ..., d_{i-1}, \bar{d}_{i-1}, d_i\right) = \left\{ \bar{d}_i \in \bar{D}_i \,\middle|\, \bar{f}_i\left(d_1, \bar{d}_1, ..., d_i, \bar{d}_i\right) \geq 0 \right\}, i = \overline{1,n}$$

$f_i : \prod_{j=1}^{i-1} \left(D_j \times \bar{D}_j\right) \times D_i \to \mathbb{R}$, $i = \overline{2,n}$, $\bar{f}_i : \prod_{j=1}^{i} D_j \times \bar{D}_j \to \mathbb{R}$, $i = \overline{1,n}$ being known.

The best guaranteed result of the decider 1 in this case is given by the quantity:

$$V_n = \sup_{d_i \in A_i} \inf_{\bar{d}_i \in B_i} F\left(d_1, \bar{d}_1, ..., d_n, \bar{d}_n\right) \tag{56}$$

Let's notice that in the case which D_i, $\bar{D}_i$, $i = \overline{1,n}$, are compact metric spaces, F, T_i, $\bar{T}_i$, f_i, $\bar{f}_i$, $i = \overline{1,n}$ are continuous, $A_i \neq \varnothing$, $i = \overline{2,n}$, $B_i \neq \varnothing$, $i = \overline{1,n}$, then the superior and inferior bounds from (55) can be reached (remark 2.4). That's why there naturally arises the problem of establishing some methods for determining of the optimal guaranteed values of the first decider in games J_n and J_1.

Other important results related to the semi-continuity of multivocal applications and to the minimum function are presented below.

Theorem 2.3 [25], [47], [48]

Let us consider that X, Y are compact metric spaces, F is defined on $X \times Y$, B is an multivocal application, $B : X \to \mathcal{P}(Y)$, $f(x) = \inf_{x} F(x, y)$.

Therefore, we shall have:

1) If F is continuous on $X \times Y$, B is s.c.s. in $x_0 \in X$, then f is s.c.i. on x_0;

2) If F is continuous on $X \times Y$, B is continuous in x_0, then f is continuous in x_0;

3) If F est s.c.s. on $X \times Y$, B is s.c.i. on x_0 and $B(x_0)$ is closed, then f is s.c.i. in x_0;

4) If F is s.c.s. on $X \times Y$, B is continuous in x_0, then f is s.c.s. in x_0.

Remark 2.5 [25]

Let us consider $(B_n)_n$, multivocal applications that are continuous on X, so that the following equality takes place:

$$\lim_n d_1\left(B(x_n),B(x)\right)=0,\ \forall x\in X,\ B_n(x)\subseteq B_{n+1}(x),\ B_1(x)\neq\varnothing,\ \forall x\in X.$$

If the multivocal application is continuous, then the sequence $(B_n)_n$ is uniformly convergent to B.

Consequently, we shall have:

$$\lim \inf_{y\in B_n(x_0)} F(x,y)=\inf_{y\in B(x)} F(x,y), \forall x\in X \tag{57}$$

Remark 2.6 [25]

Let us consider g is continuous on $X\times Y$. Therefore, we shall have

1) If $B(x)=\{y\in Y|\ g(x,y)\geq 0\}$, then B is s.c.i. on X

2) If $B(x)=\{y\in Y|\ g(x,y)\geq 0\}$, $B^0(x)=\{y\in Y|\ g(x,y)>0\}$, $\overline{B_0(x)}=B(x)$, then B is continuous.

Theorem 2.4 [25], [48]

Let us assume the sequence of functions $(f_n)_n$ which are decreasingly convergent to f_0.

1) The following equality takes place: $\lim_n \sup_{x\in X} f_n(x)=\sup_{x\in X} f_0(x)$; if the sequence $(x_n)_n$ verifies:

$$f(x_n)\geq \sup_{x\in X} f_n(x)-\varepsilon_n, \lim_n \varepsilon_n=0, \varepsilon_n>0$$

then we shall have:

$$\lim_n f_0(x_n)=\sup_{x\in X} f_0(x) \tag{58}$$

2) If X is a compact space and f is s.c.i., then

$$\lim_n \min_{x\in X} f_n(x)=\min_{x\in X} f_0(x)\ ;\ \text{if } x_n=R\min_{x\in X} f_n(x), \text{ then we have: } x^*=R\min_{x\in X} f_0(x)$$

Remark 2.7

If F is s.c.s. on $X\times Y$, B is continuous on X, then f is s.c.s. on X. As a consequence, there exists $x_0\in X$ so that

$$f(x_0)=\max_{x\in X} f(x) \tag{59}$$

which implies the following equality

$$f(x_0)=\max_{x\in X}\min_{y\in B(x)} F(x,y) \tag{60}$$

If F is continuous on $X\times Y$, B is continuous on X, then f is continuous on X.

Consequently, there exists $x_1\in X$ so that:

$$f(x_1)=\min_{x\in X}\max_{y\in B(x)} F(x,y) \tag{61}$$

and $x_2\in X$ so that:

$$f(x_2)=\max_{x\in X}\min_{y\in B(x)} F(x,y) \tag{62}$$

2.1.2 Guaranteed optimum values and their generalizations

The guaranteed optimum values were introduced by Neumann [57] starting from a non-cooperative game structure with two decision makers (players) $j_2=(F_1,F_2;D_1,D_2)$ where:

D_1,D_2 represent the decisions of both players,

$F_1,F_2: D_1\times D_2\to\mathbb{R}$ are the efficiency functions of the players.

Guaranteed optimum values are related to cautious strategic behaviour (the principle of stability); the goal of each decision maker is to reach an equilibrium point. The pair of strategies $(\overline{d}_1,\overline{d}_2)\in D_1\times D_2$ is called equilibrium point in case any deviation of a decision maker from $(\overline{d}_1,\overline{d}_2)$ would bring down their gains (63):

$$\begin{cases} F_1\left(\overline{d}_1,\overline{d}_2\right) \geq F_1\left(\overline{d}_1,d_2\right), & \forall d_2 \in D_2 \\ F_2\left(\overline{d}_1,\overline{d}_2\right) \geq F_2\left(d_1,\overline{d}_2\right), & \forall d_1 \in D_1 \end{cases} \tag{63}$$

In the case of a zero sum game $(F_1 + F_2 = 0)$, if we use the notation $F_1 = F_2 = -F_2$, then the inequalities (63) can take the following form:

$$F\left(\overline{d}_1,d_2\right) \leq F\left(\overline{d}_1,\overline{d}_2\right) \leq F\left(d_1,\overline{d}_2\right), \quad \forall (d_1,d_2) \in D_1 \times D_2 \tag{64}$$

In terms of decision theory, the guaranteed optimum values are denoted by v_1, v_2 and they are defined as follows:

$$v_1 = \sup_{d_1} \inf_{d_2} F\left(d_1,d_2\right) \tag{65}$$

$$v_2 = \inf_{d_2} \sup_{d_1} F\left(d_1,d_2\right) \tag{66}$$

Basically, v_1 represents the maximum absolute gain that can be achieved by the first player regardless of the opponent's strategic behaviour. Moreover, v_2 represents the maximum loss that may be experienced by the second decider.

When the limits v_1 and v_2 can be reached, in algebraic and typological terms, it is clear that equalities (65) and (66) can be written as follows:

$$v_1 = \max_{d_1} \min_{d_2} F\left(d_1,d_2\right) \tag{67}$$

$$v_2 = \min_{d_2} \max_{d_1} F\left(d_1,d_2\right) \tag{68}$$

Remark 2.8

The significance of the optimal and guaranteed strategies and values in the context of the theory of non-cooperative games is the following:

1) if $\left(d_1^2,d_2^1\right)$ is the optimal guaranteed strategy of the first decider, then the quantity:

$$V_1 = \max_{d_1 \in D_1} \min_{d_2 \in D_2} F\left(d_1,d_2\right) = F\left(d_1^1,d_2^1\right)$$

represents the maximum sure gain which the first decider can obtain;

2) if $\left(d_1^2,d_2^2\right)$ is the optimal guaranteed strategy of the second decider, then the quantity:

$$V_2 = \min_{d_2 \in D_2} \max_{d_1 \in D_1} F\left(d_1,d_2\right) = F\left(d_1^2,d_2^2\right)$$

represents the maximum loss the second decider could wait for.

Remark 2.9

The equilibrium point $\left(\overline{d}_1,\overline{d}_2\right) \in D_1 \times D_2$ defined by conditions (63) or, equivalently, (64) is called a saddle point. Basically, a saddle point is a particular equilibrium point; it is in fact an equilibrium point for antagonistic zero-sum games with two players.

In some algebraic and topological conditions [82], the solution $\left(\overline{d}_1,\overline{d}_2\right)$ of inequalities $F\left(\overline{d}_1,d_2\right) \leq F\left(\overline{d}_1,\overline{d}_2\right) \leq F\left(d_1,\overline{d}_2\right), \quad \forall (d_1,d_2) \in D_1 \times D_2$ is in fact the solution of the equation

$$\max_{d_1} \min_{d_2} F\left(d_1,d_2\right) = \min_{d_2} \max_{d_1} F\left(d_1,d_2\right) \tag{69}$$

also known as a minmax equation.

The existence of guaranteed optimum values v_1 and v_2 determined by (67) and (68) is assured, in a sufficiently general framework, by the next result [39], [82].

Theorem 2.5

If the conditions hold true:

1) $\left(D_1,\rho_1\right)$, $\left(D_2,\rho_2\right)$ there are compact metrical spaces;
2) F is continue in both arguments

then there exist optimal guaranteed strategies for J_1.

Proof

Let's take the functions:

$$f_1 : D_1 \to \mathbb{R},\ f_1(d_1) = \min_{d_2 \in D_2} F(d_1, d_2)$$

$$f_2 : D_2 \to \mathbb{R},\ f_2(d_2) = \max_{d_1 \in D_1} F(d_1, d_2)$$

In order to prove the existence of the optimal guaranteed strategies it is sufficient to show that f_1 and f_2 are continue (because any continue function on a compact is bounded and it reaches its bounds on the compact).

F is continue on $D_1 \times D_2$ and, consequently, it is uniformly continue.

Let us consider $\varepsilon > 0$ small enough. Then there exists $\delta(\varepsilon)$, so that the inequality takes place:

$$\left| F\left(d_1^1, d_2\right) - F\left(d_1^2, d_2\right) \right| < \varepsilon \tag{69}$$

For any $d_2 \in D_2$ and any $d_1^1, d_1^2 \in D_1$, with the property $\rho_1\left(d_1^1, d_1^2\right) < \delta(\varepsilon)$.

Let us consider $d_2^1, d_2^2 \in D_2$, so that:

$$f_1\left(d_1^1\right) = \min_{d_2 \in D_2} F\left(d_1^1, d_2\right) = F\left(d_1^1, d_2^1\right) \tag{70}$$

$$f_1\left(d_1^2\right) = \min_{d_2 \in D_2} F\left(d_1^2, d_2\right) = F\left(d_1^2, d_2^2\right) \tag{71}$$

For $\left| f_1\left(d_1^1\right) - f_1\left(d_1^2\right) \right| = \left| F\left(d_1^1, d_2^1\right) - F\left(d_1^2, d_2^2\right) \right|$, there exist the following possibilities:

1) $0 \le F\left(d_1^1, d_2^1\right) - F\left(d_1^2, d_2^2\right)$

In this case, $F\left(d_1^1, d_2^1\right) - F\left(d_1^2, d_2^2\right) \le F\left(d_1^1, d_2^2\right) - F\left(d_1^2, d_2^2\right) \le \varepsilon$, according to the inequality (69).

2) $0 \le F\left(d_1^2, d_2^2\right) - F\left(d_1^1, d_2^1\right)$

In this case, $F\left(d_1^1, d_2^1\right) - F\left(d_1^2, d_2^2\right) \le F\left(d_1^1, d_1^2\right) - F\left(d_1^2, d_2^2\right) \le \varepsilon$, according to the inequality (69).

Consequently, for any $d_1^1, d_1^2 \in D_1$, so that $\rho(d_1, d_2) < \delta(\varepsilon)$ the inequality:

$$\left| f_1\left(d_1^1\right) - f_1\left(d_1^2\right) \right| < \varepsilon \tag{72}$$

holds true and proves the fact that f_1 is continue.

Analogically, we assume that the theorem is proved.

Generalizing the concept of guaranteed optimum value can be done in two ways. First, one can speak of guaranteed optimum values without an informational exchange [25].

If we consider the game $J_{2n} = \left(F_1, \mathcal{D}_1, \mathcal{D}_2\right)$ at which take part $2n$ coalized deciders in the coalitions $C_1 = \{1, 3, ..., 2n-1\}$, $C_2 = \{2, 4, ..., 2n\}$ with opposite interest in which:

- D_i represents the set of strategies of the decider $i \in C_1$

$$\mathcal{D}_1 = \prod_{i=1}^{n} D_i$$

- $\bar{D}_j$ represents the set of strategies of decider $j \in C_2$

$$\mathcal{D}_2 = \prod_{j=1}^{n} \bar{D}_j$$

- F_1 is the functional of the gain, then we can define in the same way with the game J_1 the optimal guaranteed values for C_1 and C_2:

$$V_1 = \sup_{d_1 \in D_1} \inf_{\bar{d}_1 \in \bar{D}_1} \dots \sup_{d_n \in D_n} \inf_{\bar{d}_n \in \bar{D}_n} F\left(d_1, \bar{d}_1, ..., d_n, \bar{d}_n\right) \overset{not}{=} \left[\sup_{d_i \in D_i} \inf_{\bar{d}_i \in \bar{D}_i} \right]^{(n)} F\left(d_1, \bar{d}_1, ..., d_n, \bar{d}_n\right)$$

$$V_2 = \inf_{\bar{d}_1 \in \bar{D}_1} \sup_{d_1 \in D_1} \dots \inf_{\bar{d}_n \in \bar{D}_n} \sup_{d_n \in D_n} F\left(d_1,\bar{d}_1,\dots,d_n,\bar{d}_n\right) \overset{not}{=} \left[\inf_{\bar{d}_i \in \bar{D}_i} \sup_{d_i \in D_i}\right]^{(n)} F\left(d_1,\bar{d}_1,\dots,d_n,\bar{d}_n\right)$$

as well as the optimal guaranteed strategies.

We assume the problem of determining conditions in which there exists optimal guaranteed strategies for J_1 and J_{2n} $(n > 1)$.

Remark 2.10

The theorem 2.1 which applies to J_1 can be generalized to J_n, $n > 1$.

In case that F_1 is continue and the sets D_i, $\bar{D}_i$, $i = \overline{1,n}$, are compact metric spaces, then for J_{2n} we can prove the existence of the optimal guaranteed strategies.

An extremely important problem which appears in real decisional situations, is when from any reason, the function of efficiency F is replaced by another function of efficiency F_1 (easily to express from analytical point of view). It arises the problem of deviation from the optimal guaranteed value, knowing that two functions of efficiency verify the condition $\left|F\left(d_1,d_2\right) - F_1\left(d_1,d_2\right)\right| < \varepsilon$, $\varepsilon > 0$ small enough. In this case, we can prove the following inequality:

$$\left|\max_{d_1}\min_{d_2} F(d_1,d_2) - \max_{d_1}\min_{d_2} F_1(d_1,d_2)\right| \prec \varepsilon \tag{73}$$

The second manner of generalizing the concept of guaranteed optimum value is based on the idea of informational set; for this reason, we can talk about guaranteed optimum values with informational change.

We shall assume that we are in the case of a game with $2n$ players, $j_{2n} = \left(F, D_i, \bar{D}_i, i = \overline{1,n}\right)$, for which the meaning of the efficiency functions and the meaning of the set of strategies D_i, $\bar{D}_i$ are known from game theory. By the instrumentality of the following known functions:

$$g_i : D_i \times \bar{D}_{i-1} \to \mathbb{R}, \quad i = \overline{1,n}$$

$$h_i : D_i \times \bar{D}_i \to \mathbb{R}, \quad i = \overline{1,n}$$

we shall define as follows, the multivocal applications $A_i, B_i, i = \overline{1,n}$:

$$\left| \begin{array}{l} A_i : D_{i-1} \times \bar{D}_{i-1} \to \mathcal{P}\left(D_i\right) \\ B_i : D_i \times \bar{D}_{i-1} \to \mathcal{P}\left(\bar{D}_i\right) \end{array} \right.$$

We are led to the following problem that needs to be solved:

$$(P) \quad \max_{d_1}\min_{\bar{d}_1}\max_{d_2}\min_{\bar{d}_2}\dots\max_{d_n}\min_{\bar{d}_n} F\left(d_1,\bar{d}_1\dots d_n,\bar{d}_n\right) \tag{74}$$

Obviously, $\left(d_i,\bar{d}_i\right) \in D_i \times \bar{D}_i$, $i = \overline{1,n}$

If $\left(d_1^*,\bar{d}_1^*,\dots,d_n^*,\bar{d}_n^*\right) \in \prod_{i=1}^{n}\left(D_i \times \bar{D}_i\right)$ represents a solution for the problem (P), we shall use the following notations:

$$M = F\left(d_1^*,\bar{d}_1^*,d_2^*,\bar{d}_2^*\dots,d_n^*,\bar{d}_n^*\right)$$

and, therefore, we have the following equality:

$$M = \left[\max_{d_i}\min_{\bar{d}_i}\right]_{i=1}^{n} F\left(d_1,\bar{d}_1\dots d_n,\bar{d}_n\right) \tag{75}$$

This problem can be considered as a game which allows the informational change.

The sets $A_i, B_i, i = \overline{1,n}$ are called the informational sets of players.

Remark 2.11

If the informational change is allowed, the minmax inequality is not verified.

For example [25], let us consider $F\left(d_1,\bar{d}_1\right) = g\left(d_1\right) + h\left(\bar{d}_1\right)$ and the functions g and h, which are known. The informational sets can be defined as follows:

$$B(d_1)=\left\{\overline{d_1}:\varphi\left(d_1,\overline{d_1}\right)\ge 0\right\},\quad A\left(\overline{d_1}\right)=\left\{d_1:\varphi\left(d_1,\overline{d_1}\right)\ge 0\right\}$$

We consider that $D_1=\{d_1:B(d_1)\ne\varnothing\},\quad D_2=\left\{\overline{d_1}:A\left(\overline{d_1}\right)\ne\varnothing\right\}$

We shall give proof of the following equality:

$$\sup_{d_1}\inf_{\overline{d_1}}F\left(d_1,\overline{d_1}\right)\ge\inf_{\overline{d_1}}\sup_{d_1}F\left(d_1,\overline{d_1}\right) \tag{76}$$

In the left member of the inequality (76), we have $d_1\in D_1,\overline{d_1}\in B(d_1)$, and in the right member of the same equality, we have $\overline{d_1}\in D_2, d_1\in A\left(\overline{d_1}\right)$

Obviously, $A\left(\overline{d_1}\right)\subset D_1, B(d_1)\subset D_2, \forall d_1\in D_1, \forall d_1\in D_2$

Therefore, we shall have:

$$\sup_{d_1\in D_1}g(d_1)+\inf_{\overline{d_1}\in D_2}h\left(\overline{d_1}\right)\ge\inf_{\overline{d_1}\in D_2}\left(h\left(\overline{d_1}\right)+\sup_{d_1\in A\left(\overline{d_1}\right)}g(d_1)\right)$$

$$\inf_{\overline{d_1}\in D_2}\left(h\left(\overline{d_1}\right)+\sup_{d_1\in A\left(\overline{d_1}\right)}g(d_1)\right)=\inf_{\overline{d_1}\in D_2}\sup_{d_1\in A\left(\overline{d_1}\right)}F\left(d_1,\overline{d_1}\right)$$

which immediately yields the inequality (2.21).

Remark 2.12

If $n=2$, we shall use, further on, the following notation:

$d_1^*=R\max_{d_1}\min_{\overline{d_1}}F\left(d_1,\overline{d_1}\right)$ is a solution of the problem

$$\max_{d_1}\min_{\overline{d_1}}F\left(d_1,\overline{d_1}\right)=\min_{\overline{d_1}}\max_{d_1}F\left(d_1,\overline{d_1}\right)$$

where $d_1\in A\left(\overline{d_1}\right),\overline{d_1}\in B(d_1)$

2.2 Minmax Conditions of Optimality

2.2.1 The Case when Informational Change is not Allowed

2.2.1.1 Minmax Theorems

The minmax theorem is undoubtedly connected to the name of J. von Neumann and to the theory of games.

In a broad acceptance, the minmax theory sets the conditions in which, if D_1 and D_2 are non-void sets and $f:D_1\times D_2\to\mathbb{R}$ the equality takes place:

$$\inf_{d_2\in D_2}\sup_{d_1\in D_1}f(d_1,d_2)=\sup_{d_1\in D_1}\inf_{d_2\in D_2}f(d_1,d_2) \tag{77}$$

As a basic result in proving the minmax equality, we can consider the following one.

Theorem 2.6

If D_1, D_2 are compact and convex in linear topologic spaces and f is concave-convex on $D_1\times D_2$ and continue, separately, on D_1 and D_2, then the equality holds true:

$$\max_{d_1\in D_1}\min_{d_2\in D_2}f(d_1,d_2)=\min_{d_2\in D_2}\max_{d_1\in D_1}f(d_1,d_2) \tag{78}$$

The release of algebraic and topologic hypotheses which condition the above equality (or in the infsup variant) led to numerous minmax theorems.

In addition, we have got to notice that all the theorems of equilibrium from [82] become minmax theorems through their simple transcription for the case of zero-sum two-person games.

Theorem 2.7 [82]

If D_2 is compact in any topologic space and $f=s.c.i.$ on D_2, for every $d_1\in D_1$ fixed, then the following conditions are equivalent:

1) $\sup_{d_1\in D_1}\min_{d_2\in D_2} f\left(d_1,d_2\right)=\min_{d_2\in D_2}\sup_{d_1\in D_1} f\left(d_1,d_2\right)$;

2) For each finite pair $\left(a,D_1'\right)$, $a\in\mathbb{R}$, $D_1'\subset D$, so that:

$$a<\min_{d_2\in D_2}\max_{d_1\in D_1} f\left(d_1,d_2\right)$$

there exists a $d_1^0\in D_1$ so that

$$a\le\min_{d_2\in D_2} f\left(d_1^0,d_2\right)$$

Proof

We prove the implication $1)\Rightarrow 2)$.

Let us consider $D_1'\subset D$ finite and $a\in\mathbb{R}$, so that the inequality may take place:

$$a<\min_{d_2\in D_2}\max_{d_1\in D_1} f\left(d_1,d_2\right)$$

and, consequently, $a<\min_{d_2\in D_2}\max_{d_1\in D_1} f\left(d_1,d_2\right)=V_2$.

Writing $\varepsilon=V_2-a$, there results $a=V_2-\varepsilon$ or, in an equivalent form, $a=V_1-\varepsilon$. From the definition of V_1 results that there exists a $d_1^0\in D_1$ so that:

$$\min_{d_1\in D_1} f\left(d_1^0,d_2\right)\ge V_1-\varepsilon=a$$

Therefore, the implication $1)\Rightarrow 2)$ is proved.

Hereinfater, we shall prove the implication $2)\Rightarrow 1)$.

Let us consider $a\in\mathbb{R}$ be so that $a<\min_{d_2\in D_2}\sup_{d_1\in D_1} f\left(d_1,d_2\right)$.

For every $d_1\in D_1$, we shall consider the closed set:

$$A_a\left(d_1\right)=\left\{d_2\in D_2\,\middle|\, f\left(d_1,d_2\right)\le a\right\}$$

Obviously, $\bigcap_{d_1\in D_1} A_a\left(d_1\right)=\varnothing$. Since D_2 is compact there exists the finite set $D_1'\subset D_1$ so that $\bigcap_{d_1\in D_1'} A_a\left(d_1\right)=\varnothing$. Therefore, the following inequality takes place:

$$a<\max_{d_1\in D_1} f\left(d_1,d_2\right),\ \forall d_2\in D_2$$

which proves that $a<\min_{d_2\in D_2}\max_{d_1\in D_1'} f\left(d_1,d_2\right)$.

Let's consider $d_1^0\in D_1$ which meets the condition 2). Then:

$$a\le\min_{d_2\in D_2} f\left(d_1^0,d_2\right)\sup_{d_1\in D_1}\min_{d_2\in D_2} f\left(d_1,d_2\right)=V_1$$

Since a was arbitrary chosen, $a<V_2$, it results $V_2\le V_1$ and, thereby, the implication $2)\Rightarrow 1)$ is proved.

Theorem 2.8 [82]

Let us consider D_2 compact in a topologic space and $f:D_1\times D_2\to\mathbb{R}$ s.c.i. so that the following hypothesis is verified:

$$\forall d_1^1,d_1^2\in D_1,\exists d_1^0\in D_1: f\left(d_1^1,d_2\right)+f\left(d_1^2,d_2\right)\le 2f\left(d_1^0,d_2\right),\forall d_2\in D_2 \tag{79}$$

Moreover, we assume that for each $a\in\mathbb{R}$, $\lambda\in\left[0,1\right]$, $d_1^1,d_1^2\in D_1$ and $D_1'\subset D_1$ finite, with the property $A_a\left(D_1'\cup\left\{d_1^1\right\}\right)\cap A_a\left(D_1'\cup\left\{d_1^2\right\}\right)=\varnothing$ is true for one of the inclusions:

$$\begin{gathered}L\left(\lambda,a,d_1^1,d_1^2\right)\cap A_a\left(D_1'\right)\subseteq A_a\left(D_1'\cup\left\{d_1^1\right\}\right)\\ L\left(\lambda,a,d_1^1,d_1^2\right)\cap A_a\left(D_1'\right)\subseteq A_a\left(D_1'\cup\left\{d_1^2\right\}\right)\end{gathered} \tag{80}$$

Then the following equality takes place:

$$\min_{d_2 \in D_2} \sup_{d_1 \in D_1} f(d_1, d_2) = \sup_{d_1 \in D_1} \min_{d_2 \in D_2} f(d_1, d_2)$$

Remark 2.13

We shall note:

$$L(\lambda, a, d_1^1, d_1^2) = \left\{ d_2 \in D_2 \middle| \lambda f(d_1^1, d_1^2) + (1-\lambda) f(d_1^2, d_2) \le a \right\}$$

$$A_a(D_1') = \bigcap_{d_1 \in D_1'} A_a(d_1), D_1' \subset D_1, A_a(d_1) = \left\{ d_2 \in D_2 \middle| f(d_1, d_2) < a \right\}$$

Remark 2.14

The previous theorem is Ştefănescu's [82]. Based on this theorem, we can prove all the minmax theorems which will be presented from now on.

Theorem 2.9 [82]

If D_2 is compact on a separated topologic space, f is s.c.i. convex on D_2 and verifies the hypothesis (80) from the previous theorem, then the following inequality takes place:

$$\min_{d_2 \in D_2} \sup_{d_1 \in D_1} f(d_1, d_2) = \sup_{d_1 \in D_1} \min_{d_2 \in D_2} f(d_1, d_2) \tag{81}$$

Theorem 2.10 [24]

If D_2 is compact in a separated space, f is s.c.i. on D_2 and concave-convex on $D_1 \times D_2$, then the equality takes place:

$$\min_{d_2 \in D_2} \sup_{d_1 \in D_1} f(d_1, d_2) = \sup_{d_1 \in D_1} \min_{d_2 \in D_2} f(d_1, d_2) \tag{82}$$

Theorem 2.11 [24]

If D_1, D_2 are compact in topological spaces, f is s.c.s. on D_1 and s.c.i. on D_2. If, in addition, f is concave-convex, then we have:

$$\min_{d_2 \in D_2} \max_{d_1 \in D_1} f(d_1, d_2) = \max_{d_1 \in D_1} \min_{d_2 \in D_2} f(d_1, d_2) \tag{83}$$

We notice that beside the original proofs, these theorems admit simpler proofs, too.

Thus, theorem 2.10 allows an immediate proof, because the conditions of theorem 2.10 are met and the proof of theorem 2.11 is a consequence of the fact that the conditions of the theorem 2.10 are also met.

Indeed, the functions:

$$f_1 : D_1 \to \mathbb{R}, f_1(d_1) = \min_{d_2 \in D_2} f(d_1, d_2) \tag{84}$$

$$f_2 : D_2 \to \mathbb{R}, f_2(d_2) = \max_{d_1 \in D_1} f(d_1, d_2) \tag{85}$$

are s.c.s., respectively s.c.i. and so they will reach their bounds on D_1 sets, as well as on D_2 sets.

Remark 2.15

Let's also notice that the minmax equality formulated in the framework of the theorem 2.11 is preserved if we replace the imposed conditions for the spaces D_1, D_2 and for the function f by the following requisites:

- D_1, D_2 are convex and compact in the linear topologic spaces;
- f is concave and s.c.s. on D_1;
- f is convex and s.c.i. on D_2.

Based on this remark, we shall give a proof of the well known minmax theorem belonging to Neumann.

In order to prove this theorem, Neumann has used a mathematical apparatus sophisticated enough, based on the theorem of the fixed point.

Later on, they gave algebraic proofs much more simple, the simplest direct proof being based on the properties of duality in linear optimization.

Theorem 2.12 [82]

Let's consider the real numbers $a_{i,j}$, $i = \overline{1,m}$, $j = \overline{1,n}$. Then there exist the non-negative constants $\overline{p}_1, \overline{p}_2, ..., \overline{p}_m, \overline{q}_1, \overline{q}_2, ..., \overline{q}_n$, $\sum_{i=1}^{m} \overline{p}_i = \sum_{j=1}^{m} \overline{q}_j = 1$ so that the following equivalent conditions can be verified:

$$\sum_{j=1}^{n} a_{ij}\overline{q}_j \le \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\overline{p}_i\overline{q}_j \le \sum_{i=1}^{m} a_{ij}\overline{p}_i,\ i=\overline{1,m},\ j=\overline{1,n} \tag{86}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}p_i\overline{q}_j \le \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\overline{p}_i\overline{q}_j \le \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\overline{p}_i q_j \tag{87}$$

for any non-negative constants $p_1, p_2,..., p_m, q_1, q_2,..., q_n$, $\sum_{i=1}^{m} p_i = \sum_{j=1}^{m} q_j = 1$.

Proof

We shall consider:

$$P = \left\{ p = (p_1, p_2,..., p_m) \middle| p_i \ge 0, \sum_{i=1}^{m} p_i = 1 \right\}$$

$$Q = \left\{ q = (q_1, q_2,..., q_n) \middle| q_j \ge 0, \sum_{j=1}^{n} q_j = 1 \right\}$$

Let us define

$$f : P \times Q \to \mathbb{R},\ f(p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}p_i q_j \tag{88}$$

It results immediately that P, Q are compact and convex in $\mathbb{R}^m$, respectively $\mathbb{R}^n$.

Due to remark 2.15, we have the property 2), from where the equivalence with property 1) is evident.

Remark 2.16

The theorems presented so far imply successively Theorem 2.8 $\Rightarrow$ Theorem 2.9 $\Rightarrow$ Theorem 2.10 $\Rightarrow$ Theorem 2.11 $\Rightarrow$ Theorem 2.12, the last theorem being in the fact minmax theorem for which they gave proof (from a chronologic point of view).

Other versions of the proofs of the minmax equality have been the object of preoccupation of other mathematicians, too (in Bibliographic Notes and Commentaries we shall refer to other results obtained in this domain).

In case of aleatory extensions of the games there is nothing new from the very strict point of view of the minmax theorem, because the more general conditions of existence of the minmax equality are not pointed out.

The fact that the minmax equality is not satisfied for any game, led to the idea of defining of an extension of an initial game where the minmax equality should exist.

Practically speaking, the aleatory extensions of the zero-sum two-person games have been introduced, in an attempt of giving a universal character to the minmax theorem.

The acceptation or rejection of the modality of construction of this extension is based on the fundamental ideas of the games theory regarding the behavior of the deciders and the measurement of their utility.

Similar to the case proving the existence of the equilibrium points in the aleatory extensions of the game, we shall replace the sets D_1, D_2 with different spaces of probabilities defined on these and we shall extend conveniently the function f on their product.

We consider the aleatory extension $\tilde{J} = (P(D_1), P(D_2), F)$ where:

$$\begin{aligned} F(p,q) &= \int_{D_1\times D_2} f(d_1,d_2)\, dp \otimes q(d_1,d_2) = \int_{D_1}\int_{D_2} f(d_1,d_2)\, dq(d_2)dp(d_1) = \\ &= \int_{D_2}\int_{D_1} f(d_1,d_2)\, dp(d_1)dq(d_2) \end{aligned} \tag{89}$$

Theorem 2.13 [82]

If D_1, D_2 are compact in separated topological spaces and f is continue on $D_1 \times D_2$, then the equality takes place:

$$\min_{q\in P(D_2)} \max_{p\in P(D_1)} F(p,q) = \max_{p\in P(D_1)} \min_{q\in P(D_2)} F(p,q) \tag{90}$$

Corollary 2.13.1 (J. Ville's minmax theorem)

Let us consider $f:[0,1]\times[0,1]\to\mathbb{R}$, continue. Then, we have the equality:

$$\min_{H\in\mathcal{F}}\max_{G\in\mathcal{F}}F(G,H)=\max_{G\in\mathcal{F}}\min_{H\in\mathcal{F}}F(G,H) \quad (91)$$

where $\mathcal{F}$ is the set of functions defined on $[0,1]$ and:

$$F(G,H)=\int_0^1\int_0^1 f(d_1,d_2)\,dG(d_1)dH(d_2)=\int_0^1\int_0^1 f(d_1,d_2)\,dH(d_2)dG(d_1) \quad (92)$$

Corollary 2.13.2 (J. von Neumann minmax theorem)

If $D_1=\{d_1^1,d_1^2,...,d_1^m\}$, $D_2=\{d_2^1,d_2^2,...,d_2^n\}$ then the equality takes place:

$$\min_{q\in P(D_2)}\max_{p\in P(D_1)}F(p,q)=\max_{p\in P(D_1)}\min_{q\in P(D_2)}F(p,q) \quad (93)$$

where:

$$F(p,q)=\sum_{i=1}^{m}\sum_{j=1}^{n}f(d_1^i,d_2^j)p_iq_j,\ p=(p_1,p_2,...,p_m)\in P(D_1),\ q=(q_1,q_2,...,q_n)\in P(D_2)$$

Remark 2.17

Neumann's theorem presented as a corollary of theorem 2.13 has the following interpretation: the aleatory extension of a zero-sum finite game verifies the minmax equality.

J. Ville has given the first proof of the minmax theorem regarding the infinite games. It is he who has given the first example of an infinite game for which the minmax equality is not verified in the case of aleatory extension, either.

Other results regarding minmax equality in the case of the aleatory extensions of the game may be found in [30], [63], [82].

2.2.1.2 Optimum Minmax Conditions through Varitional Inequalities

Rockfeller [75] was the first to notice that the study of minmax problems in the differentiable gain functions could be done by means of the theory of variational inequalities with monotone operators.

Let us consider X and X^* (respectively Y and Y^*) vectorial topologic local convex and separated spaces put to duality through the bilinear form $\langle\,,\rangle_1$ (respectively $\langle\,,\rangle_2$), $\Omega_1\subset X$, $\Omega_2\subset Y$ opened, $C\subset\Omega_1$, $D\subset\Omega_2$ convex and $f_i:\Omega_1\times\Omega_2\to\mathbb{R}$ Gâteaux derivable.

We note with $\nabla_1 f(x,y)$ Gâteaux derivative of f_y in x and with $\nabla_2 f(x,y)$ Gâteaux derivative of f_x in y.

Let us consider $T:\Omega_1\times\Omega_2\to X^*\times Y^*$ defined as it follows:

$$T(z)=(\nabla_1 f(x,y)-\nabla_2 f(x,y)),\ z=(x,y)\in\Omega_1\times\Omega_2 \quad (94)$$

Theorem 2.14 (Auslender [4])

f is convex-concave (strong convex-concave) on $C\times D$ if and only if T is monotone (strong monotone) on $C\times D$.

Theorem 2.15 (Auslender [4])

$z^*=(x^*,y^*)$ is a saddle point of f on $C\times D$ if and only if the inequality takes place:

$$\langle T(z^*),z-z^*\rangle\ge 0,\ \forall z=(x,y)\in C\times D \quad (95)$$

Remark 2.18

The inequality (95) is a variational one associated to the operator T is thus generalized: take $A:C\to X^*$, monotone, $l\in X^*$, $\varphi:C\to(-\infty,\infty]$ convex and non identical $+\infty$. We have to solve to following problem:

(P) Let us determine $u:\ \in C\therefore\langle l-Au,v-u\rangle\le\varphi(v)-\varphi(u),\ \forall v\in C$ (96)

In this case, $C\times D=X\times Y$, the inequality (96) becomes:

$$T(z^*)=0 \quad (97)$$

and the problem (P) leads to the next one:

$$\left(P_l\right) \text{ in which we have to find } u \in X \therefore Au = l$$

We assume $\Gamma_0\left(X\right) = \left\{\varphi : X \to \left(-\infty,\infty\right], \textit{convex and unequal} +\infty\right\}$.

Theorem 2.16 (Auslender [4])

If $A : C \to X^*$ is monotone and semicontinue, $\varphi \in \Gamma_0\left(X\right)$, then for any $l \in X^*$, the set of the solutions of the problem $\left(\tilde{P}\right)$ is convex and closed in X.

Definition 2.3

We shall say that $A : C \to X^*$ is of type M for any sequent $\left(u_n\right)_n \subset C$, so that the following conditions have to be met:

i) $u_n \xrightarrow{s} u$

ii) $A\left(u_n\right) \xrightarrow{s} B$

iii) $\overline{\lim_n} \left\langle Au_n, u_n \right\rangle \le \left\langle B, u \right\rangle$

then $A_u = B$.

Definition 2.4

The operator $A : C \to X^*$ is called pseudomonotone if for every sequence $\left(u_n\right)_n \subset C$, so that the conditions must be verified:

i1) $u_n \xrightarrow{s} u$

i2) $\overline{\lim_n} \left\langle Au_n, u_n - u \right\rangle \le 0$

then $\left\langle Au, u - v \right\rangle \le \underline{\lim_n} \left\langle Au_n, u_n - v \right\rangle$, $\forall v \in C$.

Definition 2.5

The operator $B : X \to X^*$ is called an operator of penalization in respect with the convex, closed and non-void $C \subset X$, if is monotone, semicontinue, bounded and $Ker\, B = C$. The link between the properties of the operator $A : C \to X^*$ monotone and pseudomonotone and the type M is given by the following results.

Theorem 2.17 (Brezis [7])

If the operator A is pseudomonotone then A is of type M.

Theorem 2.18 (Brezis [7])

If operator A is monotone and semicontinue on C, then A is pseudomonotone on C.

Further on, we shall present some methods of solving the minmax problem by using the variational inequalities formulated in problem $\left(P\right)$, respectively the variational equality formulated in problem $\left(P_l\right)$.

1) The minmax variational equality with an operator of type M by using Galerin's method

Let us consider X a Banach reflexive separable space put to duality with X^* (its conjugate) through the bilinear form $\left\langle \cdot,\cdot \right\rangle$.

If $w_1, w_2, ..., w_m, ...$ is a basis of X, we note:

$$W_m = S_p\left(w_1, w_2, ..., w_m\right), m \in \mathbb{N}$$

Let us suppose $A : X \to X^*$ which verifies the conditions:

a) A it is bounded;

b) A it is coercive (i.e. $\lim_{\|v\| \to \infty} \dfrac{A\left(v\right)v}{\|v\|} = \infty$);

c) A it is of type M.

We assume the problem of solving the variational equality:

$$\left(P_l\right) \quad A\left(u\right) = l \text{, where } l \in X^* \text{ is fixed}$$

In order to solve the problem, we form the sequence $\left(u_m\right)_m$ that verifies the conditions:

(i) $u_m \in W_m$, $\forall m \in \mathbb{N}$;

(ii) $\left\langle A\left(u_m\right), w_j \right\rangle = \left\langle l, w_j \right\rangle$, $j = \overline{1,m}$.

Theorem 2.19 (Auslender [4])

1) There is a sequence u_m that verifies (i) and (ii).

2) Any weak limiting point of a subsequence $(u_m)_m$ (and there exists at least one) is the solution of problem (P_l).

Proof

1) Let's consider W_m with a Hilbertian structure given by the scalar product:

$$(x,y)=\sum_{i=1}^{m}\alpha_i w_i \text{ , where } x=\sum_{i=1}^{m}\alpha_i w_i \text{ , } u=\sum_{i=1}^{m}\beta_i w_i$$

Due to the Rietz's theorem:

$$\forall u\in X,\exists P(u)\in W_m \therefore (P(u),v)=\langle A(u),v\rangle,\forall v\in W_m$$

Consequently $(P(u),u)=\langle A(u),u\rangle$, $\forall u\in W_m$.

On the other hand, based on Rietz's theorem, $\exists y\in W$ so that:

$$(P(u),u)-\langle l,u\rangle=(P(u),u)-(y,u)=(P(u)-y,u)$$

According to Schwartz's inequality, we have:

$$\langle A(u),u\rangle-\langle l,u\rangle\geq\langle A(u),u\rangle-\|l\|\|u\|,\forall u\in V_m$$

A being coercive, $\exists\rho>0$ so that:

$$\forall u\in W_m,\|u\|=\rho \text{ we have } \langle A(u),u\rangle-\|l\|\|u\|\geq 0\Rightarrow(P(u)-y,u)\geq 0$$

As A is bounded and of type M, it can be immediately verified that it is weakly continue and, consequently, P is weakly continue, too.

Lemma 2.19.1 Let us consider F the Euclidean finite dimensional space with the scalar product $(\cdot,\cdot)$, $P:E\to E$ continue so that $\exists\rho>0\therefore(P(a),a)\geq 0$, $\forall a\in E$ with the property $\|a\|=\rho$. Thus, there is an $\overline{a}\in E\therefore\|\overline{a}\|\leq\rho$, $P(\overline{a})=0$.

Based on this lemma:

$$\exists u_m\in W_m,\|u_m\|\leq\rho\therefore(P(u_m)-y,u_m)=0\Rightarrow(P(u_m),u_m)=(y,u_m)\Rightarrow\langle A(u_m),u_m\rangle=\langle l,u_m\rangle\Rightarrow$$
$$\Rightarrow\langle A(u_m),w_j\rangle=\langle l,w_j\rangle,\forall j=\overline{1,m}$$

2)

$$\langle A(u_m),u_m\rangle=\langle l,u_m\rangle\leq\|l\|\|u\|\Leftrightarrow\frac{\langle A(u_m),u_m\rangle}{\|u\|}\leq\|l\| \tag{98}$$

A being coercive, from (98) results that the sequence $(u_m)_{m\in\mathbb{N}}$ is bounded and therefore:

$$\exists C\in\mathbb{R}^+:\|u_m\|\leq C,\forall m\in\mathbb{N} \tag{99}$$

The relationship (99) shows that there is at least a limiting point of accumulation u^0 of $(u_m)_{m\in\mathbb{N}}$ and from the reflexivity condition of space X, it occurs that u^0 is a weak limit of a subsequence $(u_{m_n})_n$ of the sequence $(u_m)_m$. On the other hand, A is bounded and therefore, there is a limiting point p of a subsequence $(A(u_k))_k$ of the $(A(u_{m_n}))_n$.

For each j which meets the condition (ii), we shall pass to the limit and we shall obtain:

$$\langle p,w_j\rangle=\lim_k\langle A(u_k),w_j\rangle=\langle l,w_j\rangle\Rightarrow p=l$$
$$\lim_k\langle A(u_k),u_k\rangle=\lim_k\langle l,u_k\rangle=\langle l,u^0\rangle$$

because $l\in X$ and $u_k\xrightarrow{s}u^0$.

Corollary 2.19.1

If A is bounded, monotone, semicontinue and coercive and the following conditions are verified too:

1) the application $v \to \|v\|$ is strict convex

2) $A(u) = A(v) \Leftrightarrow \|u\| = \|v\| = \rho$

then the problem (P_1) admits a unique solution.

2) Minmax variational inequality with pseudomonotone operators through the method of penalization

Let us assume X^* a Banach reflexive separable space, $C \subset X$ convex, closed and non-void, $A : X \to X^*$ bounded, pseudomonotone and coercive, $l \in X$.

Consider $P : X \to X^*$ an operator of penalization related to C (i.e. monotone, bounded, semicontinue and with the nucleus C). We assume the next problem:

$$(P_2)\, u \in C, \langle A(u) - l, v - n \rangle \geq 0, \forall v \in C, l \text{ is fixed}$$

The solving of the variational inequality given by (P_2) by means of the method of penalization consists in the solving of a sequence of variational equalities in X:

$$A(u_n) + K_n P(u_n) = l, K_n > 0, \forall n \in \mathbb{N}, \lim_n K_n = \infty \tag{100}$$

Theorem 2.20 [4]

1) There is a sequence (u_n) which verifies (100).

2) Any weak limiting point of a subsequence of $(u_n)_n$ (and there is at least one) is the solution of the problem (P_2).

Proof

1) The operator $A + K_n P$ is bounded as a sum of two operators and pseudomonotone as a sum between a pseudomonotone and semicontinue operator. He is also coercive relatively to $v_0 \in C$:

$$\langle A(v), v - v_0 \rangle + K_n \langle P(v), v - v_0 \rangle = \langle A(v), v - v_0 \rangle + K_n \langle P(v) - P(v_0), v - v_0 \rangle \geq \langle A(v), v - v_0 \rangle \Rightarrow$$

$$\Rightarrow \frac{\langle A(v), v - v_0 \rangle + K_n \langle P(v), v - v_0 \rangle}{v} \xrightarrow[\|v\| \to \infty]{} \infty$$

because A is coercive.

According to the theorem 2.19, for any $n \in \mathbb{N}$ there is at least a solution of the variational equality (100).

2) From conditions:

$$A(u_n) + K_n P(u_n) = l$$

$$\frac{\langle A(v), u_n - v_0 \rangle + K_n \langle P(v), u_n - v_0 \rangle}{\|u_n\|} \xrightarrow[\|u_n\| \to \infty]{} \infty$$

it is immediately obtained the boundery of the sequence $(u_n)_n$ (by reducing ad absurdum).

As the set C is closed and $(u_n) \subset C$, there is at least a limiting point u^* (weaked) of $(u_n)_n$ and a subsequence $(u_m)_m$ of $(u_n)_n$ so that $u_m \xrightarrow{s} u^*$.

Considering that A is bounded we have:

$$P(u_m) = \frac{1}{K_m}\left[l - A(u_m) \right] \xrightarrow{s} 0$$

because $l \in X^*$ is bounded in X^*. In addition, $\exists h \in X^*$ and $(u_k)_k \subset (u_n)_n$ so that $u_k \xrightarrow{s} u^*$, $A(u_k) \xrightarrow{s} h$.

P is also an operator of penalization, of type M, monotone and semicontinue and verifies the conditions:

$$\liminf_n u_m = u^* \,(weak)$$

$$\lim_n P(u_m) = 0 \,(weak)$$

$$\limsup_m \langle P(u_m), u_m \rangle \leq \langle 0, u^* \rangle$$

we obtain:

$$P\left(u^*\right)=0 \Rightarrow u^* \in C \tag{101}$$

From (101) we get for any $v \in C$:

$$\left\langle A\left(u_k\right)-l, v-u_k\right\rangle = K_n\left\langle P(v)-P\left(u_k\right), v-u_k\right\rangle \geq 0$$

because P is monotone and $K_n > 0$, $\forall n \in \mathbb{N}$, from which, making $v=u^*$ we have:

$$\liminf_k \left\langle A\left(u_k\right), u^*-u_k\right\rangle \geq \liminf_k \left(l, u^*-u_k\right)=0 \Rightarrow \limsup_k \left\langle A\left(u_k\right), u_k-u^*\right\rangle \leq 0$$

Because A is pseudomonotone, we have:

$$\left\langle A(u), u-v\right\rangle \leq \liminf_k \left\langle A\left(u_k\right), u_k-v\right\rangle \leq \liminf_k \left\langle l, u_k-v\right\rangle = \left\langle l, u-v\right\rangle$$

3) Minmax variational inequality using monotonous operators

We note with $\mathcal{P}$ the class of the convex, bounded and closed sets from a Banach reflexive space X and with I a finite set of indexes.

Let us consider $g_i : X \to (-\infty, \infty]$ convex, semicontinue inferior and non-identical ∞, $\forall i \in I$.

If $D \in \mathcal{P}$, we define by means of the family $\left(g_i\right) i \in I$, the following subset of D:

$$C=\left\{x \in D \middle| g_i(x) \leq 0, i \in I\right\}$$

We define the multivocal application $\Gamma : \mathcal{P} \to \mathcal{P}$ as it follows:

$$\Gamma A \in A, \forall A \in \mathcal{P}; \text{ if } A \neq \varnothing \Rightarrow \Gamma A \neq \varnothing$$

Suppose that we have the following problem:

$\left(P_4\right)$ Let us determine an element of ΓC

In order to solve the problem $\left(P_4\right)$ we shall proceed as it follows: let us consider J_i a family of indexes, $i \in I$ and $\left(g_{ij}\right)_{j \in J_i}$ a family of convex weakly uniform equicontinue functionals on D as that:

$$g_i(x)=\sup\left\{g_{ij}(x), j \in J_i\right\}, i \in I \tag{102}$$

Considering $Q_0 = D$, we build a sequence of sets $\left(Q_n\right)_n$, a family of sequence of indexes $\left(j_{ni}\right)_{i \in I}$, $j_{ni} \in J_i$ and a sequence of elements $\left(x_n\right)_n \in D$ as it follows:

a) $Q_n \to x_n$ through relation $x_n \in \Gamma Q_n$ if $x_n \in C$ then x_n is the solution of problem $\left(P_4\right)$;

b) $x_n \to j_{ni}$ through relation:

$$g_{ij_{ni}}\left(x_n\right) \geq \theta \sup\left\{g_{ij}\left(x_n\right), j \in J_i\right\}-\lambda_n = \theta g_i\left(x_n\right), i \in I$$

where $\theta \in (0,1)$ is fixed and $\left(\lambda_n\right) \subset \mathbb{R}$, $\lambda_n \to 0$.

c) $Q_n, j_{ni} \to Q_{n+1}$ so that:

$$Q_{n+1}=Q_n \cap \left\{x \middle| g_{ij_{ni}}(x) \leq 0, i \in I\right\}$$

Theorem 2.21 [4]

Any limiting point of the sequence $\left(x_n\right)_n$ (and there is at least one) belongs to C.

2.2.2 The Case when Informational Change is Allowed

We have to solve the following problem:

$$(P) \qquad \sup_{x \in A} \inf_{y \in B} F(x,y) \tag{103}$$

where

$$\begin{aligned} A&=\left\{x \in X \middle| g_i(x) \geq 0, i=\overline{1,m}\right\} \neq \varnothing \\ B&=\left\{y \in Y \middle| h_j(y) \geq 0, j=\overline{1,n}\right\} \neq \varnothing \end{aligned} \tag{104}$$

the functions $g_i, i=\overline{1,m}$ $h_j, j=\overline{1,n}$ are known.

Definition 2.4

The function $L(\,\cdot,\cdot,a,b): X\times Y\to\mathbb{R}$ defined by:

$$L(x,y,a,b)=F(x,y)+\sum_{i=1}^{n}a_i g_i(x)-\sum_{j=1}^{n}b_j h_j(y),\ a_i\geq 0, i=\overline{1,m}, b_j\leq 0,\ j=\overline{1,n} \tag{105}$$

is called the Lagrange function for the problem (103).

Theorem 2.22 [25], [48]

If F, g_i, h_j, $i=\overline{1,m}$, $j=\overline{1,n}$ are bounded, then we shall have:

1) $\sup_{x\in A}\inf_{y\in B}F(x,y)=\sup_{x\in A}\inf_{y\in B}\inf_{a\geq 0}\sup_{b\leq 0}L(x,y,a,b)=\sup_{x\in A}\inf_{y\in B}\sup_{b\leq 0}\inf_{a\geq 0}L(x,y,a,b)$

2) If $x^*=R\max_{x\in X}\inf_{x\in Y}\inf_{a\geq 0}\sup_{b\leq 0}L(x,y,a,b)$

then $x^*=R\max_{x\in X}\inf_{x\in Y}F(x,y)$.

Theorem 2.23 [25]

If the following conditions hold true:

1) X and Y are compact metric spaces;

2) F is the Lipschitzien on $X\times Y$;

3) Functions g_i, h_j, $i=\overline{1,m}$, $j=\overline{1,n}$ are continuous and

$$\min_{1\leq i\leq m}g_i(x)\leq -\beta d_X(x,A),\ \forall x\notin A$$
$$\min_{1\leq j\leq n}h_j(y)\leq -\beta d_y(y,B),\ \forall y\notin B$$

where $\beta>0$ is a certain constant.

Then, there exist $a,\ a^*\geq 0$ and $b,\ b^*\leq 0$ so that

$$\max_{x\in X}\min_{y\in Y}\min_{0\leq a\leq a^*}\max_{b^*\leq b\leq 0}L(x,y,a,b)=\max_{x\in X}\min_{y\in Y}F(x,y) \tag{106}$$

Theorem 2.24 [25]

1) If we have the equality

$$\sup_{\substack{x\in X\\ b\leq 0}}\inf_{\substack{y\in Y\\ a\geq 0}}L(x,y,a,b)=\inf_{\substack{y\in Y\\ a\geq 0}}\sup_{\substack{x\in X\\ b\leq 0}}L(x,y,a,b) \tag{107}$$

than we shall have:

$$\sup_{x\in X}\inf_{y\in Y}F(x,y)=\inf_{y\in Y}\sup_{x\in X}F(x,y) \tag{108}$$

2) If g_i, $i=\overline{1,m}$, h_j, $j=\overline{1,n}$ are concave functions on the compact convex space convexes X and Y, F is a concave function in respect with x and a convex function in respect with y, then the equality (108) implies the equality (107).

Assume $x^*\in X$ with the following property:

$$\max_{x\in X}\min_{y\in Y}F(x,y)=\min_{y\in Y}F(x^*,y) \tag{109}$$

Theorem 2.25 [25]

Let us consider that the following equalities hold true:

1) $X\subset E_n$ is an convex compact space, Y is an compact metric space;

2) F, $\dfrac{\partial F}{\partial x}$ are continuous in $X\times Y$.

Thus, if $x^*=R\max_{x\in X}\min_{y\in Y}F(x,y)$ is the solution of the problem

$$\max_{x\in X}\min_{y\in Y}F(x,y)=\min_{y\in Y}F(x^*,y) \tag{110}$$

there exist $p_i\geq 0$, $y_i\in Y$, $1\leq i\leq r\leq n+1$ so that

$$\sum_{i=1}^{r}p_i=1,\ -\sum_{i=1}^{r}p_i\frac{\partial}{\partial x}F(x^*,y_i)=K_X^*(x^*)\ \text{(where } y_i=R\min_{y\in Y}F(x^*,y)\text{)}$$

which is equivalent to

$$-Q\left(x^*\right) \cap K_X^*\left(x^*\right) \neq \varnothing \tag{111}$$

where $Q(x) = coH(x)$ is the convex hull of the set:

$$H(x) = \left\{ z = \frac{\partial}{\partial x} F(x,y) \middle| \, y \in R \min_{y \in Y} F(x,y) \right\}$$

Theorem 2.26 [25], [48]

The condition (111), can be written

$$\sup_{x \in X} \min_{y \in Y\left(x^*\right)} \left(\frac{\partial}{\partial x} F\left(x^*, y\right), x - x^* \right) = 0 \tag{112}$$

where $Y\left(x^*\right) = R \min_{y \in Y} F\left(x^*, y\right)$.

Application: The Excess Problem

Let us assume the cooperative game $J = (M, v)$, for which $M = \{1, 2, ..., m\}$ represents the set of players and v is the characteristic function.

We can prove that:

$$v(C_1) + v(C_2) \le v(C_1 \cup C_2), \forall C_1, C_2 \subseteq M, C_1 \cap C_2 = \varnothing \tag{113}$$

Let us assume Z the set of imputations which have the following form:

$$Z = \left\{ z = (z_1, z_2, ..., z_m) \middle| \, z_i \ge v(\{i\}), \sum_{i=1}^{m} z_i = v(M) \right\}$$

We shall consider C the set of coalitions and the excess function $e : C \times Z \to \mathbb{R}$, defined by

$$e(C, z) = v(C) - \sum_{i \in C} z_i \tag{114}$$

According to Kukuskin et Moruzov, [39] the most equitable imputation z^* is a solution of the problem:

$$\min_{z \in C} \max_{C \subseteq M} e(C, z) = \max_{C \subseteq M} e\left(C, z^*\right) \tag{115}$$

By taking into consideration theorem 2.26, we can show that the following condition is met:

$$\min_{z \in C} \max_{C \in C\left(z^*\right)} \left(\frac{\partial}{\partial z} e\left(C, z^*\right), z - z^* \right) = 0 \tag{116}$$

where $C\left(z^*\right)$ represents the set of coalitions which verify the equality :

$$\max_{C \subseteq M} e\left(C, z^*\right) = e(C_i, z), i = \overline{1, k}$$

Obviously,

$$\frac{\partial}{\partial z} e(C, z) = (0, ..., -1, ..., -1, ..., 0)$$

(here, in the position j, we shall have the value -1 if the player j takes part in the coalition C).

The condition of optimality for z^* can be written:

$$\min_{z \in Z} \max \left\{ \sum_{i \in C_1} (z_i - z_i), ..., \sum_{i \in C_k} (z_i - z_i) \right\} = 0 \tag{117}$$

If the player $j \in C_i$, then the condition 117 is satisfied for $z_j^* = v(\{j\})$.

Obviously, the coalition $(C_1 \cup \{j\}) \notin C\left(z^*\right)$.

Hence, we have the following equality:

$$e\left(C_1 \cup \{j\}, z^*\right) < e\left(C_1, z^*\right)$$

from which we are led to:

$$v(C_1 \cup \{j\}) - \sum_{i \in C_1} z_i^* - z_j^* < v(C_1) - \sum_{i \in C_1} z_i^*$$

Thus, we can show that

$$v(C_1 \cup \{j\}) < v(C_1) + v(\{j\}) \tag{118}$$

We can immediately take notice of the contradiction between the relationships (113) and (118). Accordingly, each player takes part to at least a coalition from C.

On the other hand,

$$e\left(C_1, z^*\right) = e\left(C_2, z^*\right) = \ldots = e\left(C_k, z^*\right)$$

2.3 Solving The Minmax Problem

2.3.1 The Case of Simple Strategies

2.3.1.1 The Case when Informational Change is not Allowed

2.3.1.1.1 The Penalty Method and the Method Convergence

Let us consider X a compact metric space and $\left(f_i\right)_{i=\overline{1,n}}$ a finite family of functionals which are defined on X.

If $F : X \to \mathbb{R}$, we shall solve the following problem (P):

(P) We must find $x_0 \in A$ which verifies the equality $F\left(x_0\right) = \max\limits_{x \in A} F\left(x\right)$ where $A = \left\{x \in X \,\middle|\, f_i\left(x\right) \geq 0, i = \overline{1,n}\right\} \neq \varnothing$

Let us assume the function $J\left(\cdot, C\right)$ defined by X (C is a parameter) which verifies the following properties:

1) $J\left(x, C\right) \geq 0$, $\forall x \in X$;

2) $J\left(x, C\right) \searrow 0$, if $C \to \infty$ on the dense set A:

3) $\lim\limits_{C \to \infty} J\left(x, C\right) = \infty$ uniformly, $\forall x \in X \setminus V_\delta\left(A\right)$, $\forall \delta > 0$, where $V_\delta\left(A\right)$ represents an δ-neighbourhood of A.

The function $J\left(x, C\right)$ is called the penalty function.

Let us assume $L\left(\cdot, C\right) : X \to \mathbb{R}$ defined by

$$L\left(x, C\right) = F\left(x\right) - J\left(x, C\right) \tag{119}$$

Theorem 2.27

If the functions F, f_i, $i = \overline{1,n}$ are continuous, then:

1) $\max\limits_{x \in X} F\left(x\right) = \lim\limits_{C \to \infty} \sup\limits_{x \in X} L\left(x, C\right)$

2) if the sequence $\left(x_n\right)_n$ verifies the equality:

$$L\left(x_n, C_n\right) \geq \sup_{x \in X} L\left(x, C_n\right) - \varepsilon_n$$

where $\lim\limits_n \varepsilon_n = \lim\limits_n \frac{1}{C_n} = 0$, $\varepsilon_n > 0$, $\forall n \in \mathbb{N}$; then, if $\overline{x}$ is a limit point of the sequence $\left(x_n\right)_n$, $\overline{x}$ is the solution of the problem (P).

Corollary 2.27.1

Let us assume the penalty function $J\left(x, C\right) = Cf\left(x\right)$.

Then, $F\left(x^*\left(C\right)\right)$, $f\left(x^*\left(C\right)\right)$ are monotonously decreasing for $C > 0$, where

$$x^*\left(C\right) = \left\{\overline{x} \in X \,\middle|\, L\left(\overline{x}, C\right) = \max_{x \in X} L\left(x, C\right)\right\}$$

Proof

For $0 < C_1 < C_2$ we shall have

$$L\left(x^*\left(C_1\right), C_1\right) \geq L\left(x^*\left(C_2\right), C_1\right) \tag{120}$$

$$L\left(x^*\left(C_2\right), C_2\right) \geq L\left(x^*\left(C_2\right), C_1\right) \tag{121}$$

From relations (120) and (121) we are led to the following inequality:

$$(C_1 - C_2)\left[f\left(x^*(C_2)\right) - f\left(x^*(C_1)\right)\right] \geq 0$$

which shows that:

$$f\left(x^*(C_1)\right) \leq f\left(x^*(C_2)\right) \tag{122}$$

Obviously,

$$F\left(x(C_1)\right) \geq F\left(x(C_2)\right)$$

from where we get the following inequality:

$$F\left(x^*(C_1)\right) \geq F\left(x^*(C_2)\right)$$

Remark 2.19

In particular, we can consider the following penalty functions:

$$J_1(x,C) = C\sum_{i=1}^{n}\left|min\left(0,f_i(x)\right)\right|^q,\ q > 0 \tag{123}$$

$$J_2(x,C) = \frac{C}{\left[min\left(0,f_i(x)\right)\right]^q},\ q > 0 \tag{124}$$

Let us consider the compact metric spaces X_i, Y_i, $i = \overline{1,n}$ and the functional $F : \prod_{i=1}^{n}(X_i \times Y_i) \to \mathbb{R}$; then, by taking into account notations from §2.1, we shall have:

$$A_i = A_i\left(\overline{x}^{i-1}, \overline{y}^{i-1}\right) = \left\{x_i \in X_i \,\middle|\, q_i\left(\overline{x}^{i}, \overline{y}^{i-1}\right) \geq 0\right\},\ i = \overline{1,n}$$

$$B_i = B_i\left(\overline{x}^{i}, \overline{y}^{i-1}\right) = \left\{y_i \in Y_i \,\middle|\, h_i\left(\overline{x}^{i}, \overline{y}^{i}\right) \geq 0\right\},\ i = \overline{1,n}$$

If $C > 0$, we can define the penalty functions I_i, J_i, $i = \overline{1,n}$:

$$J_i\left(\overline{x}^{i}, \overline{y}^{i-1}, C\right) = \begin{cases} 0 \quad ,\ si\ x_i \in A_i\left(\overline{x}^{i-1}, \overline{y}^{i-1}\right) \\ a(C),\ si\ x_i \notin A_i\left(\overline{x}^{i-1}, \overline{y}^{i-1}\right),\ a(C) < 0 \end{cases} \tag{125}$$

$$I_i\left(\overline{x}^{i}, \overline{y}^{i}, C\right) = \begin{cases} 0 \quad ,\ si\ y_i \in B_i\left(\overline{x}^{i}, \overline{y}^{i-1}\right) \\ b(C),\ si\ y_i \notin B_i\left(\overline{x}^{i}, \overline{y}^{i-1}\right),\ b(C) < 0 \end{cases} \tag{126}$$

$$\lim_{C\to\infty} J_i\left(\overline{x}^{i}, \overline{y}^{i-1}, C\right) = -\infty,\ si\ x_i \notin A_i\left(\overline{x}^{i-1}, \overline{y}^{i-1}\right)$$

$$\lim_{C\to\infty} I_i\left(\overline{x}^{i}, \overline{y}^{i-1}, C\right) = \infty,\ si\ y_i \in B_i\left(\overline{x}^{i}, \overline{y}^{i-1}\right)$$

Remark 2.20

Assume $J_i(\cdot,\cdot,C) : \prod_{i=1}^{n}(X_i \times Y_i) \to \mathbb{R}$ defined by

$$J_i\left(\overline{x}^{i}, \overline{y}^{i-1}, C\right) = C\,min\left(0, g_i\left(\overline{x}^{i}, \overline{y}^{i-1}\right)\right) \tag{127}$$

This application is a penalty functional.

Remark 2.21

Let us assume a game problem $\Gamma = (F,G;X,Y)$ which allows the informational change.

The set of strategies X and Y are compact metric spaces and the decisional functions F and G are continuous on $X \times Y$.

We have the following equality:

$$\sup_{x\in X}\inf_{y\in Y} F(x,y) = \sup_{x\in X}\inf_{y\in B(x)} F(x,y)$$

where $B(x) = R\max_{y\in Y} G(x,y)$.

Method Convergence

If $\overline{C}=\left(C_1,C_2,...,C_{2n}\right)$, $C_i>0$, $i=\overline{1,2n}$, we shall consider $L\left(\cdot,\cdot,\overline{C}\right):\prod_{i=1}^{n}\left(X_i\times Y_i\right)\to\mathbb{R}$ defined by

$$L\left(\overline{x}^n,\overline{y}^n,C\right)=F\left(\overline{x}^n,\overline{y}^n\right)+\sum_{i=1}^{n}J_i\left(\overline{x}^i,\overline{y}^{i-1},C_i\right)+I_i\left(\overline{x}^i,\overline{y}^i,C_{n+i}\right) \tag{128}$$

Theorem 2.28 [25], [39]

If the following conditions hold true:

1) X_i, Y_i are compact metric spaces, $i=\overline{1,n}$;

2) the functions F, g, h_i are continuous, $i=\overline{1,n}$;

3) the sets $A_i\left(\overline{x}^{i-1},\overline{y}^{i-1}\right)$, $B_i\left(\overline{x}^i,\overline{y}^{i-1}\right)$ are non-void;

4) the multivocal applications are continuous in Hausdorff's metric.

then:

1) $\left[\sup_{x_i\in A_i}\inf_{y_i\in B_i}\right]_{i=1}^{n}F\left(x_1,y_1,...,x_n,y_n\right)=\lim_{\overline{C}\to\infty}\left[\max_{x_i\in X_i}\min_{y_i\in Y_i}\right]_{i=1}^{n}L\left(\overline{x}^n,\overline{y}^n,C\right)$

2) Every adherence point x^* of the set

$$\left\{x^*\left(\overline{C}_k\right)\middle|\,\overline{C}_k\to\infty\right\}$$

where $x\left(\overline{C}_k\right)=R\left[\max_{x_i\in X_i}\min_{y_i\in Y_i}\right]_{i=1}^{n}L\left(\overline{x}^n,\overline{y}^n,\overline{C}_k\right)$, verifies the equality:

$$x^*=R\left[\max_{x_i\in A_i}\min_{y_i\in B_i}\right]_{i=1}^{n}F\left(\overline{x}^n,\overline{y}^n\right)$$

Corollary 2.28.1

All the limit points of the set $\left\{y^*\left(\overline{C}_k\right)\middle|\,\overline{C}_k\to\infty\right\}$.

$$y^*\left(\overline{C}_k\right)=R\min_{y_1\in Y_1}\left[\max_{x_i\in X_i}\min_{y_i\in Y_i}\right]_{i=2}^{n}L\left(x^*\left(\overline{C}_k\right),y_1,...,x_n,y_n,\overline{C}_k\right) \tag{129}$$

$\lim_k x^*\left(\overline{C}_k\right)=x^*$, verifies the equality:

$$y^*=R\min_{y_1\in Y_1}\left[\max_{x_i\in A_i}\min_{y_i\in B_i}\right]_{i=2}^{n}F\left(x^*,y_1,x_2,y_2,...,x_n,y_n\right)$$

Theoreme 2.29 [25], [39]

If the following conditions are met:

1) X_i, Y_i are compact metric spaces, $i=\overline{1,n}$;

2) the functions F, g_i, h_i are continuous, $i=\overline{1,n}$;

3) the sets $A_i\left(\overline{x}^{i-1},\overline{y}^{i-1}\right)$, $B_i\left(\overline{x}^i,\overline{y}^{i-1}\right)$, $i=\overline{1,n}$ are non-void;

4) the applications A_i, $i=\overline{1,n}$, are continuous (in Hausdorff's metric).

then:

1) $M=\left[\sup_{x_i\in A_i}\min_{y_i\in B_i}\right]_{i=1}^{n}F\left(\overline{x}^n,\overline{y}^n\right)=\lim_{C\to\infty}\left[\sup_{x_i\in A_i}\min_{y_i\in B_i}\right]_{i=1}^{n-1}\max_{x_n\in A_n}\min_{y_n\in B_n}\left\{F\left(\overline{x}^n,\overline{y}^n\right)+I_n\left(\overline{x}^n,\overline{y}^n,C\right)\right\}$

2) $M=\lim_{C\to\infty}\left[\min_{y_i\in B_i}\sup_{x_{i+1}\in A_{i+1}}\right]_{i=1}^{n-1}\min F\left(x^*\left(C\right),y_1,...,x_n,y_n\right)$

where the sequence $\left\{x^*\left(C_k\right)\middle|\,C_k\to\infty\right\}$ verifies the inequality:

$$\left[\min_{y_i\in B_i}\sup_{x_{i+1}\in A_{i+1}}\right]_{i=1}^{n-1}\min_{y_n\in B_n}\left\{F\left(x^*\left(C_k\right),y_1,...,x_n,y_n\right)+I_n\left(x^*\left(C_k\right),y_1,...,x_n,y_n,C_k\right)\right\}\geq$$

$$\geq\left[\sup_{x_i\in A_i}\min_{y_i\in B_i}\right]_{i=1}^{n}\max_{x_n\in A_n}\min_{y_n\in B_n}\left\{F\left(\overline{x}^n,\overline{y}^n\right)+I_n\left(\overline{x}^n,\overline{y}^n,C_k\right)-\varepsilon_k\right\}$$

and $\varepsilon_k\geq 0$, $k\in\mathbb{N}$, $\lim_k\varepsilon_k=0$.

Remark 2.22

$\left(x^*\left(C_k\right)\right)_k$ is an ε-optimum sequence of strategies.

Proof

1) For $\left(\overline{x}^n,\overline{y}^{n-1}\right)$ fixed, we have the inequality:

$$\min_{y_n\in B_n\left(\overline{x}^n,\overline{y}^{n-1}\right)}F\left(\overline{x}^n,\overline{y}^n\right)=\lim_{C_k\to\infty}\min_{y_n\in Y_n}\left\{F\left(\overline{x}^n,\overline{y}^n\right)+I_n\left(\overline{x}^n,\overline{y}^n,C_k\right)\right\}\tag{130}$$

The applications B_i are s.c.s., $i=\overline{1,n}$, in Hausdorff's metric.

Because the applications A_i are continuous, B_i are s.c.s., then, according to theorems 2.27 and 2.28 we immediately get the property 1).

2) This property is a consequence of theorem 2.28.

Remark 2.23

For the two-players game with informational change, we shall consider the application B, defined by

$$B(x)=\left\{y\in Y\,\middle|\,f(x,y)\geq 0\right\}$$

where $f:X\times Y\to\mathbb{R}$ is a continuous function.

According to the theorem 2.29, we have the equality

$$\min_{y\in B(x)}F(x,y)=\lim_{C\to\infty}\min_{y\in Y}\left\{F(x,y)+I(x,y,C)\right\}$$

where the penalty function $I:X\times Y\times\mathbb{R}_+\to\mathbb{R}$ is defined by

$$I(x,y,C)=C\left[\min\left(0,f(x,y)\right)\right]^2$$

According to the theorem 2.29, we can approximate:

$$\sup_{x\in X}\min_{y\in B(x)}F(x,y)\text{ by }\max_{x\in X}\min_{y\in Y}\left\{F(x,y)+I(x,y,C)\right\}$$

2.3.1.1.2 Combined Variational Methods of Solving Minmax Problem

In this paragraph there will be presented two combined methods of determining a solution to the minmax problem by generalizing two methods belonging to Auslender [4].

Method 1

We assume that $D_1=D_2$, we shall write D (D is the real Hilbert space endowed with the scalar product $(\cdot,\cdot)$ which generalize the norm $\|\ \|$) and we consider $f:D\times D\to\mathbb{R}$ the Gâteaux derivative. Let us take C_1, C_2, G_1, G_2 convex, bounded and closed subsets in D.

Let us suppose $h_1:D\to\mathbb{R}$, $h_2:D\to\mathbb{R}$ which verify the properties:

i1) $h_1(x)=0$, $\forall x\in C_1$, $h_2(y)=0$, $\forall y\in G_1$

i2) $h_1(x)>0$, $\forall x\in H\setminus C_1$, $h_2(y)>0$, $\forall y\in H\setminus G_1$

i3) h_1, h_2 are convex Gâteaux derivative.

If $C_0=C_1\cap C_2$, $G_0=G_1\cap G_2$, a saddle point of f on $C_0\times G_0$ is to be determined (obviously $C_0\times G_0$ is convex, bounded and weakly closed in $D\times D$).

Let us take $(a_n)_n\subset\mathbb{R}_+$, $\lim_n a_n=\infty$.

For each $n\in\mathbb{N}$, we build the following functional:

$$B^n:D\times D\to\mathbb{R},B^n(x,y)=f(x,y)+a_n\left[h_1(x)-h_2(y)\right]\tag{131}$$

Starting from its definiton, the functional B^n, $n \in \mathbb{N}$ verifies the properties:

a) $B^n(x,y) = f(x,y), \forall (x,y) \in C_0 \times G_0$

b) B^n is a Gâteaux derivative and convex-concave.

For each $n \in \mathbb{N}$, the following operator is built:

$$A_n : D \times D \to D^* \times D^*,\ A_n(x,y) = \left(\nabla_1 B^n(x,y), -\nabla_2 B^n(x,y)\right) \tag{132}$$

where:

D^* represents the dual of D;

$\nabla_1 B^n(x,y)$, $\nabla_2 B^n(x,y)$ represents the Gâteaux derivatives of B_y^n in x and B_x^n in y respectively.

We assume that operators A_n are strongly monotone and have the same constant of coerciveness $C > 0$, $\forall n \in \mathbb{N}$:

$$\left\langle A_n(z_1) - A_n(z_2), z_1 - z_2 \right\rangle \geq C\|z_1 - z_2\|, \forall z_1, z_2 \in D \times D \tag{133}$$

To each operator A_n, we shall associate the $\left(u_m^n\right)_m$ sequence defined through the recurrence relation:

$$u_{m+1}^n = P_{C_2 \times G_2}\left(u_m^n - pA_n\left(u_m^n\right)\right),\ p > 0\ (a\ fixed\ p),\ any\ u_0^n \in D \tag{134}$$

Theorem 2.30

If for any $n \in \mathbb{N}$, the operator A_n is Lipschitzian of constant M and $p \in \left(0, \frac{2c}{M^2}\right)$, then the following results take place:

1) the sequence $\left(u_m^n\right)_m$ is strongly convergent and the limit $\left(u_*^n\right)_n$ is the saddle point of B^n on $C_2 \times G_2$;

2) the sequence $\left(u_*^n\right)_n$ is made up of at least one strongly convergent subsequence and the limit of such a subsequence is the saddle point of f on $C_0 \times G_0$ (fig.3).

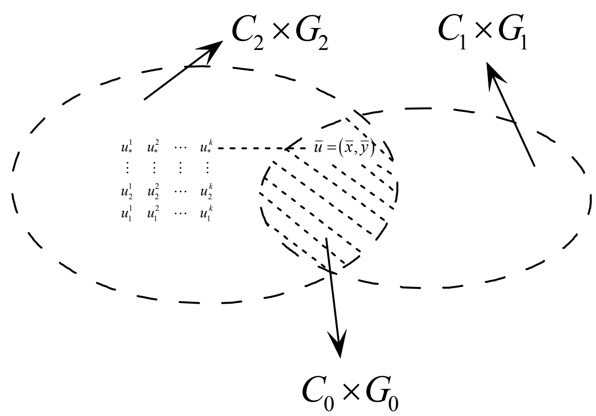

Figure 3

Proof

1) For each $n \in \mathbb{N}$ the application $V \to F_n(V) = P_{C_2 \times G_2}\left(V - pA_n(V)\right)$ is taken into consideration. As the projection operator is nonexpansive, the following inequalities takes place:

$$\begin{gathered}\left\|F_n(V_1) - F_n(V_2)\right\|^2 = \left\|V_1 - V_2 - P\left(A_n(V_1) - A_n(V_2)\right)\right\|^2 \leq \\ \leq \left\|V_1 - V_2\right\|^2 + p^2\left\|A_n(V_1) - A_n(V_2)\right\|^2 = 2p\left(A_n(V_1) - A_n(V_2), V_1 - V_2\right)\end{gathered} \tag{135}$$

Besides of the properties of the operator A_n of being strongly monotone and Lipschitzian, the following inequality results directly from (135):

$$\left\| F_n\left(V_1\right) - F_n\left(V_2\right) \right\|^2 \le K \left\| V_1 - V_2 \right\|^2, K = 1 - 2pc + p^2 M^2 \quad (136)$$

As for $p \in \left(0, \dfrac{2c}{M^2}\right)$, K is subunitary the $V \to F_n\left(V\right)$ application is a contraction and, consequently, the sequence $\left(u_m^n\right)_m$ defined by (134) is strongly convergent and the strong limit u_*^n of this sequence is a fixed point for F_n. Consequently, $u_*^n = F_n\left(u_*^n\right) = P_{C_2 \times G_2}\left(u_*^n - A_n\left(u_*^n\right)\right)$ and by applying the inequality of the projection, we get the inequality:

$$\left(A_n\left(u_*^n\right), V - u_*^n\right) \ge 0, \forall V \in C_2 \times G_2 \quad (137)$$

Considering the way in which the operator A_n has been drawn, it follows directly that u_*^n is a saddle point of B^n on $C_2 \times G_2$.

2) The fact that $\left(u_*^n\right)_n$ is made up of at least one strongly convergent subsequence can be accounted as it follows: $C_2 \times G_2$ being bounded and closed, the $\left(u_*^n\right)_n$ sequence will comprise a weakly convergent subsequence $\left(u_*^{n_{k_1}}\right)_{k_1} \subset C_2 \times G_2$ being also convex, the $\left(u_*^n\right)_n$ sequence will comprise a $\left(u_*^{n_{k_2}}\right)_{k_2}$ strongly convergent subsequence, each $u_*^{n_{k_2}}$ term of this subsequence being a convex combination built with the help of the terms of the $\left(u_*^{n_{k_1}}\right)_{k_1}$ subsequence.

Let us consider $\left(\overline{x}, \overline{y}\right) \in C_2 \times G_2$ the strong limit of the subsequence $\left(u_*^{n_{k_1}}\right)_{k_1} = \left(x_{n_{k_1}}, y_{n_{k_1}}\right)_{k_1}$.

We shall show at first that $\left(\overline{x}, \overline{y}\right) \in C_0 \times G_0$ and then that $\left(\overline{x}, \overline{y}\right)$ is saddle point of f on $C_0 \times G_0$ (for the sake of calculation easiness, we shall replace the $\left(x_{n_{k_1}}, y_{n_{k_1}}\right)_{k_1}$ subsequence by the $\left(x_n, y_n\right)_n$ sequence).

Let us take a fixed point $\left(x_0, y_0\right) \in C_0 \times G_0$. As $\left(x_n, y_n\right)$ is a saddle point of B^n on $\left(x_n, y_n\right)$ we shall have:

$$B^n\left(x_n, y\right) \le B^n\left(x_n, y_n\right) \le B^n\left(x, y_n\right), \forall \left(x, y\right) \in C_2 \times G_2 \quad (138)$$

For $\left(x_0, y_0\right) \in C_0 \times G_0$ we shall have $h_1\left(x_0\right) = h_2\left(y_0\right) = 0$ and, consequently, the inequality:

$$B^n\left(x_n, y_n\right) \le B^n\left(x_0, y_n\right)$$

can be written as:

$$f\left(x_n, y_0\right) + a_n h_1\left(x_n\right) \le f\left(x_0, y_n\right) - a_n h_2\left(y_n\right) \quad (139)$$

From (139) it follows that:

$$a_n\left(h_1\left(x_n\right) + h_2\left(y_n\right)\right) \le f\left(x_0, y_n\right) - f\left(x_n, y_0\right)$$

We shall consider the applications:

$$g_1 : H_1 \times H_2 \to \mathbb{R}, g_1\left(x, y\right) = h_1\left(x\right) + h_2\left(y\right), \forall \left(x, y\right) \in D_1 \times D_2$$

$$g_2 : D_1 \times D_2 \to \mathbb{R}, g_2\left(x, y\right) = f\left(x_0, y\right) - f\left(x, y_0\right), \forall \left(x, y\right) \in D_1 \times D_2$$

Considering the way in which these applications were built, it follows directly that g_1, g_2 are Gâteaux derivatives.

From the inequality (139), we can write:

$$g_1\left(x_n, y_n\right) = h_1\left(x_n\right) + h_2\left(y_n\right) \le \frac{f\left(x_0, y_n\right) - f\left(x_n, y_0\right)}{a_n} = \frac{g_2\left(x_n, y_n\right)}{a_n} \quad (140)$$

and passing to the limit in (140) we get:

$$g_1\left(\overline{x}, \overline{y}\right) = h_1\left(\overline{x}\right) + h_2\left(\overline{y}\right) \le 0 \quad (141)$$

Taking into account the properties of the functionals h_1 and h_2, it results from (141) that $h_1\left(\overline{x}\right) = h_2\left(\overline{y}\right) = 0$, i.e. $\left(\overline{x}, \overline{y}\right) \in C_0 \times G_0$.

We show that $\left(\overline{x},\overline{y}\right)$ is a saddle point of f on $C_0 \times G_0$. From (139) we have the inequality:

$$f\left(x_n, y_0\right) \leq f\left(x_0, y_n\right) \tag{142}$$

hence, by passing to the limit we obtain:

$$f\left(\overline{x}, y_0\right) \leq f\left(x_0, \overline{y}\right) \tag{143}$$

Because $\left(x_0, y_0\right) \in C_0 \times G_0$ has been chosen as any $\left(x_0, y_0\right) \in C_0 \times G_0$, from (143) it results that:

$$f\left(\overline{x}, y_0\right) \leq f\left(\overline{x}, \overline{y}\right) \tag{144}$$

$$f\left(\overline{x}, \overline{y}\right) \leq f\left(x_0, \overline{y}\right) \tag{145}$$

which proves that $\left(\overline{x},\overline{y}\right)$ is a saddle point of f on $C_0 \times G_0$.

Remark 2.24

The $G-$ derivability condition of the f, h_1, h_2 functionals is important only to prove point 1) of theorem 1.

If a saddle point $\left(x_n, y_n\right)$ of B^n on $C_2 \times G_2$ is known, the fact that f has a saddle point on $C_0 \times G_0$ can be demonstrated under weaker condition imposed upon the functionals f, h_1, h_2 [1].

Method 2

We shall assume that the real Hilbert spaces D_1 and D_2 are endowed with the scalar products $\left(\cdot,\cdot\right)_1$, $\left(\cdot,\cdot\right)_2$, $C \subset D_1$, $D \subset D_2$, which are convex and closed and f is subdifferentiable and has saddle points on $C \times D$.

We want to find a saddle point of f on $C \times D$. We write:

$$\nu f_x\left(y\right) = \left\{ y^* \in H_2 \,\middle|\, f_x\left(y'\right) - f_x\left(y\right) \leq \left(y, y' - y\right), \forall y' \in D_2 \right\}$$

$$\nu f_y\left(x\right) = \left\{ x^* \in H_1 \,\middle|\, f_y\left(x'\right) - f_y\left(x\right) \leq \left(x, x' - x\right), \forall x' \in D_2 \right\}$$

(νf_x, νf_y represents the set of the subgradients of the functional f_x and f_y, respectively).

Let us take $A : D_1 \to D_1^*$, $B : D_2 \to D_2^*$ and assume that operators A, B verify the conditions:

$$\left(Ax, x-u\right)_1 \leq 0, \forall x \in D_1, \forall u \in C \tag{146}$$

$$\left(By, y-v\right)_2 \geq 0, \forall y \in D_2, \forall v \in D \tag{147}$$

Let us consider $\left(a_n\right)_n \subset \mathbb{R}_+$, $\lim\limits_n a_n = 0$.

We assume that the following inequality is verified:

$$\left(H\right)\left(\left\|Ax_n - c_n\right\|_1^2 + \left\|By_n + d_n\right\|_2^2\right) \leq a_n, \forall c_n \in \nu f_{y_n}\left(x_n\right), \forall d_n \in \nu f_{x_n}\left(y_n\right)$$

In order to determine a saddle point of f on $C \times D$, we proceed as it follows.

Algorithm 2.1

Let us consider $p > 0$. Starting from $\left(x_0, y_0\right) \in C \times D$, the sequence $\left(x_n, y_n\right)_n \in C \times D$ is built:

$$x_{n+1} = P_C\left(x_n - p\left(c_n - Ax_n\right)\right), c_n \in \nu f_{y_n}\left(x_n\right) \tag{148}$$

$$y_{n+1} = P_D\left(y_n + p\left(d_n + By_n\right)\right), d_n \in \nu f_{x_n}\left(y_n\right) \tag{149}$$

The algorithm stops if $\left(x_{n+1}, y_{n+1}\right) = \left(x_n, y_n\right)$.

Theorem 2.31

The following properties take place:

1) If $\left(x_{n+1}, y_{n+1}\right) = \left(x_n, y_n\right)$ then $\left(x_n, y_n\right)$ is a saddle point of f on $C \times D$.

2) If f is strongly convex-concave on $C \times D$ and the $\left(H\right)$ hypothesis is verified, then the set of the saddle points of f on $C \times D$ is a singleton marked $\left(x^*, y^*\right)$ and for $p \in \left(0, \frac{1}{2}k\right)$, the sequence $\left(x_n, y_n\right)_n$

converge strongly at $\left(x^*,y^*\right)$, $k=min\{k_1,k_2\}$, k_1, k_2 being the coefficients if coercion of the strong convex-concavity condition of f.

Proof

1) If $\left(x_n,y_n\right)=\left(x_{n+1},y_{n+1}\right)$, then the following equalities take place:

$$x_{n+1}=P_C\left(x_n-p\left(c_n-Ax_n\right)\right) \tag{150}$$

$$y_{n+1}=P_D\left(y_n+p\left(d_n+By_n\right)\right) \tag{151}$$

and by applying the property of the projection operator to a convex set ($z=P_Cx\Rightarrow\left(x-z,y-z\right)\le 0,\forall y\in C$) we have:

$$\left(x_n-x_n+p\left(c_n-Ax_n\right),y-x_n\right)_1\ge 0\Rightarrow\left(c_n,y-y_n\right)_1\ge 0,\forall y\in C \tag{152}$$

$$\left(y_n-y_n-p\left(d_n+By_n\right),y'-y_n\right)_2\le 0\Rightarrow\left(d_n,y'-y_n\right)_2\le 0,\forall y'\in D \tag{153}$$

Taking into account the properties of subgradients, conditions (152) and (153) lead to the inequalities:

$$f\left(x_n,y_n\right)\le f\left(x,y_n\right),\forall x\in C \tag{154}$$

$$f\left(x_n,y_n\right)\ge f\left(x_n,y\right),\forall y\in D \tag{155}$$

which means that $\left(x_n,y_n\right)$ is the saddle point of f on $C\times D$.

2) This property is proved through reduction ad absurdum.

Indeed, if there were $\left(\overline{x}_1,\overline{y}_1\right)$, $\left(\overline{x}_2,\overline{y}_2\right)$ saddle points of f on $C\times D$, then there would be $\overline{c}_1\in\nu f_{\overline{y}_1}\left(\overline{x}_1\right)$, $\overline{d}_1\in\nu f_{\overline{x}_1}\left(\overline{y}_1\right)$, $\overline{c}_2\in\nu f_{\overline{y}_2}\left(\overline{x}_2\right)$, $\overline{d}_2\in\nu f_{\overline{x}_2}\left(\overline{y}_2\right)$, so that the following inequalities take place:

$$\left(\overline{c}_1,\overline{x}_2-\overline{x}_1\right)_1\ge 0,-\left(\overline{d}_1,\overline{y}_2-\overline{y}_1\right)_2\ge 0 \tag{156}$$

$$\left(\overline{c}_2,\overline{x}_1-\overline{x}_2\right)_1\le 0,-\left(\overline{d}_2,\overline{y}_1-\overline{y}_2\right)_2\le 0 \tag{157}$$

From (156) and (157) we obtain:

$$\left(\overline{c}_1-\overline{c}_2,\overline{x}_1-\overline{x}_2\right)-\left(\overline{d}_1-\overline{d}_2,\overline{y}_1-\overline{y}_2\right)\le 0 \tag{158}$$

But the equality (158) contradicts the fact that f is strongly convex-concave on $C\times D$ because in this case it is known that for any $x',x''\in C$, $y',y''\in D$, $c'\in\nu f_{y'}\left(x'\right)$, $c''\in\nu f_{y''}\left(x''\right)$, $d'\in\nu f_{x'}\left(y'\right)$, $d''\in\nu f_{x''}\left(y''\right)$, there exist $k_1,k_2>0$ so as to verify the inequality:

$$\left(c'-c'',x'-x''\right)_1-\left(d'-d'',y'-y''\right)_2\ge 2k_1\left\|x'-x''\right\|_1+2k_2\left\|y'-y''\right\|_2 \tag{159}$$

From (158) and (159), we draw the conclusion that the assumption (there are two saddle points $\left(x_1,y_1\right)$, $\left(x_2,y_2\right)$ of f on $C\times D$) is false and consequently the set of saddle points of f on $C\times D$ is a singleton (marked with$\left(x^*,y^*\right)$).

We shall prove that $\left(x^*,y^*\right)$ is the strong limit of the sequence $\left(x_n,y_n\right)$. Let us consider $s_n=\left\|x_2-x^*\right\|_1+\left\|y_n-y^*\right\|_2$.

Writing $\overline{c}_n=p\left(c_n-Ax_n\right)$, $\overline{d}_n=p\left(d_n+By_n\right)$, $u=x_n-\overline{c}_n$, $v=y_n+\overline{d}_n$ we have:

$$\begin{aligned}\left\|x_{n+1}-x^*\right\|_1&=\left\|P_C\left(x_n-\overline{c}_n\right)-x^*\right\|_1^2=\left\|x^*-P_C\left(x_n-\overline{c}_n\right)\right\|_1^2=\left(x^*-P_C\left(x_n-\overline{c}_n\right),x^*-P_C\left(x_n-\overline{c}_n\right)\right)_1=\\&=\left(x^*-P_Cu,x^*-P_Cu\right)_1=\left(u-P_Cu+x^*-u,u-P_Cu+x^*-u\right)_1=\\&=\left(u-P_Cu+x^*-u,u-P_Cu\right)_1+\left(u-P_Cu+x^*-u,x^*-u\right)_1=\\&=\left\|u-P_Cu\right\|_1^2+2\left(x^*-u,u-P_Cu\right)_1+\left\|x^*-u\right\|_1\end{aligned} \tag{160}$$

The last two terms of the right member of the inequalities (160) can be rewritten in the following way:

$$\left\|x^* - u\right\| = \left(x^* - x_n + \overline{c}_n, x^* - x_n + \overline{c}_n\right)_1 = \left\|x^* - x_n\right\|_1^2 + \left\|\overline{c}_n\right\|_1^2 - 2\left(x^* - x_n, \overline{c}_n\right)_1 = \\ = \left\|x^* - x_n\right\|_1^2 + p^2\left\|\overline{c}_n - Ax_n\right\| - 2p\left(c_n - Ax_n, x_n - x^*\right)_1 \tag{161}$$

$$\left(x^* - u, u - P_C u\right)_1 = \left(u - P_C u, x^* - u\right)_1 = \left(u - P_C u, P_C u - u\right)_1 + \left(u - P_C u, x^* - P_C u\right)_1 = \\ = \left\|u - P_C u\right\|_1^2 + \left(u - P_C u, x^* - P_C u\right)_1 \tag{162}$$

Taking into account (159), (160) and (161), (162) we shall get:

$$\left\|x_{n+1} - x^*\right\|_1 \le \left\|x^* - x_n\right\|_1^2 + p^2\left\|c_n - Ax_n\right\|_1^2 - 2p\left(c_n, x_n - x^*\right)_1 \tag{163}$$

By means of a similar reasoning, we can prove that:

$$\left\|y_{n+1} - y^*\right\|_2 \le \left\|y^* - y_n\right\|_2^2 + p^2\left\|d_n + By_n\right\|_2^2 + 2p\left(d_n, y_n - y^*\right)_2 \tag{164}$$

From (163) and (164) the following inequalities are obtained:

$$s_{n+1} \le s_n + p^2\left(\left\|c_n - Ax_n\right\|_1^2 + \left\|d_n + By_n\right\|_2^2\right) + 2p\left(c_n, x_n - x^*\right)_1 - \left(d_n, y_n - y^*\right)_2$$

On the other hand, taking into account the fact that there is $c^* \in \nu f_{y^*}\left(x^*\right)$, $d^* \in \nu f_{x^*}\left(y^*\right)$ so that the following inequalities takes place:

$$\left(c^*, x_n - x^*\right)_1 \ge 0, \left(d^*, y_n - y^*\right)_2 \le 0 \tag{165}$$

we are led to the inequalities:

$$s_{n+1} \le s_n + p^2 a_n - 2p\left[\left(c_n - c^*, x_n - x^*\right)_1 - \left(d_n - d^*, y_n - y^*\right)_2\right] \le \\ \le s_n + p^2 a_n - 2pks_n - s_n\left(1 - 2pk\right) + p^2 a_n \le s_0\left(1 - 2pk\right)^{n+1} + \frac{pa_n}{2k} \tag{166}$$

where $k = min\left(k_1, k_2\right) > 0$.

It results from (166) that for any $\varepsilon > 0$, $\exists n_0\left(\varepsilon\right) \in N$ so that the following inequalities take place for $n \ge n_0\left(\varepsilon\right)$:

$$\left\|x_n - x^*\right\| \le \frac{\varepsilon}{2}; \left\|y_n - y^*\right\| \le \frac{\varepsilon}{2} \ i.e.\ \left(x_n, y_n\right)_n \to \left(x, y\right) \tag{167}$$

for $p \in \left(0, \frac{1}{2k}\right)$ which proves the theorem.

Remark 2.25

Method 1 represents in fact a combination of a projection method and a penalization method and it generalizes a method belonging to Auslender [4].

Method 2 represents a combination of a projection method and a subgradient method and it generalizes a subgradient method also belonging to Auslender [4]

The advantage of using these methods lies in the fact that these constructive methods being iterative, they allow the use of certain known numerical methods, in order to determin the approximate solution of the minmax problem (with a fixed error).

2.3.1.2 The Case when Informational Change is Allowed

Let us consider a game problem $\Gamma = \left(F, G; X, Y\right)$ for which informational change is allowed.

The sets of strategies X and Y are compact metric spaces if the decision functions F and G are continuous on $X \times Y$.

The best guaranteed result for the first decision maker is :

$$\sup_{x \in X} \inf_{y \in B(x)} F\left(x, y\right), \text{ where } B\left(x\right) = R\max_{y \in Y} G\left(x, y\right)$$

Let us assume the penalty function

$$I\left(x, y, C\right) = C\left[\max_{z \in Y} G\left(x, z\right) - G\left(x, y\right)\right] \tag{168}$$

According to theorem 2.28, we have the following equality:

$$\sup_{x\in X}\min_{y\in B(x)} = \lim_{C\to\infty}\max_{x\in X}\min_{y\in Y}\left\{F(x,y)+I(x,y,C)\right\}$$

$$\max_{x\in X}\min_{y\in Y}\left\{F(x,y)+I(x,y,C)\right\} = \max_{\substack{x\in X\\ z\in Y}}\min_{y\in Y}\left\{F(x,y)+CG(x,z)-G(x,y)\right\}$$

For the penalty function

$$I(x,y,C)=C\int_Y\left[\min\left(0,G(x,y)-G(x,z)\right)\right]^2 d\mu(z) \tag{169}$$

we have the following theorem:

Theorem 2.32 [25]

If F and G are lipschitziens in respect with y, then

$$1)\ \sup_{x\in X}\min_{y\in B(x)} F(x,y) = \lim_{C\to\infty}\max_{(x,u)\in X\times U}\left\{u = z(C)\int_Y\left|\min\left(0,F(x,y)+I(x,y,C)-u\right)\right|^q dy\right\} \tag{170}$$

where $z(C)$ verifies the condition $\lim_{C\to\infty}\frac{C^n}{z(C)}=0$.

2) For every $\varepsilon>0$, there exists C_0, so that for $C\ge C_0$, reaching the maximum ($x_q(C)$, $u_q(C)$) in (170) is an ε - optimum strategy and ε -approximation for $\sup_{x\in X}\inf_{y\in B(x)} F(x,y)$).

We can assume, without restricting the generality of the problem, that the application B has a particular form.

If $h: X\times Y\to\mathbb{R}$ is continuous, let us consider

$$B(x)=\left\{y\in Y\,\middle|\,h(x,y)=0\right\}$$

Theorem 2.33

If

$$L(x,y,C,D)=F(x,y)+Ch(x,y)-D\min_{y\in Y}h(x,y) \tag{171}$$

then we shall have the following equality:

$$\lim_{C\to\infty}\max_{x\in X}\min_{y\in Y}L(x,y,C,z(C)) = \max_{x\in A}\min_{y\in B(x)}F(x,y)$$

where the fonction z verifies

$$\lim_{C\to\infty}\frac{z(C)}{C}>1 \tag{172}$$

Proof

We shall use the notations:

$$W^* = \max_{x\in A}\min_{y\in B(x)}F(x,y)$$

$$W(C)=\max_{x\in X}\min_{y\in Y}L(x,y,C,z(C))$$

Let us consider

$$\bar{x} = R\max_{x\in A}\min_{y\in B(x)}F(x,y)$$

and, consequently,

$$W(C)\ge\min_{y\in Y}\left[F(\bar{x},y)+Ch(x,y)\right]$$

we shall have (see theorem 2.8):

$$\lim_{C\to\infty}\min_{y\in Y}\left[F(\bar{x},y)+Ch(\bar{x},y)\right]=W^*$$

which implies

$$\underline{\lim}_{C\to\infty}W(C)\ge W^* \tag{173}$$

On the other hand, $\exists C_0>0$, $\varepsilon_0>0$ so that $z(C)\ge(1+\varepsilon_0)C$, $\forall C\ge C_0$ and, consequently,

$$W(C)\le\max_{x\in X}\min_{y\in Y(x)}\left[F(x,y)-\varepsilon_0 Ch(x,y)\right]$$

where $Y(x)=\left\{(x,\overline{y})\in X\times Y \,\middle|\, \overline{y}=R\min_{y\in Y} h(x,y)\right\}$.

Because $Y(x)=B(x)$, $\forall x\in A$, we shall have

$$\lim_{C\to\infty}\max_{x\in X}\min_{y\in Y(x)}\left[F(x,y)-\varepsilon_0 Ch(x,y)\right]=W^*$$

and, accordingly,

$$\overline{\lim_{C\to\infty}} W(C)\le W^*$$

This inequality and (173) implies that

$$W^*=\lim_{C\to\infty} W(C) \tag{174}$$

and, therefore, we have the equality:

$$\max_{x\in X}\min_{y\in B(x)} F(x,y)=\lim_{C\to\infty}\max_{x\in X}\min_{y\in Y} L(x,y,C,z(C)) \tag{175}$$

Corollary 2.33.1

If x^* is a limit point for the following set

$$\left\{x_k^*=R\max_{x\in X}\min_{y\in Y} L(x,y,C_k,z(C_k)) \mid C_k\to\infty\right\}$$

then $x^*\in A$ and $x^*=R\max_{x\in X}\min_{y\in B(x)} F(x,y)$.

2.3.2 The Case of Mixed Strategies (for Matrix Games)

The central result used in order to solve this problem consists in the theorem of approximation of the continuous games through finite games [16], [25].

Based on this result, the solving of some types of particular games (games with a separable function of paying, games with a convex and generalized – convex function of paying, temporal games) can be found (at least from a theoretical point of view) in [25], [29], [85].

Due to this reason, beside the presentation of the theorem that allows the approximation of the continuous games through the finite games there will be issued some methods of solving the matrix games. One of the methods refers to the abridgement of the solving of a matrix game to the solving of a couple of problems of linear dual programming. The other method to be presented within this paragraph is the iterative method.

The iterative method is constructive and represents, in fact, a version of Brown's method and it refers to repeated games in which the accumulated experience in the previous games can be used as a restrictive number of simple strategies. For the beginning, we assume that in the game $J_1=(f,D_1,D_2)$, the sets of the strategies are bounded and closed in the Euclidean space E^n, respectively E^m. For the aleatory extension $\tilde{J}_1$ of J_1 we note:

$$V_1=\sup_{\nu}\inf_{\mu}\int_{D_1}\int_{D_2} f(d_1,d_2)\,d\mu d\nu$$

$$V_2=\inf_{\mu}\sup_{\nu}\int_{D_1}\int_{D_2} f(d_1,d_2)\,d\mu d\nu$$

where ν and μ are probabilistic measures defined on compact X, respectively Y and f is a continue function.

We can easily prove the inequality $V_1\le V_2$.

There exists the equality $V_1=V_2$; sup and inf will be changed with max, respectively min when $\tilde{J}_1$ has got a saddle point (ν_0,μ_0). We can prove that the strategy ν_0 of the first decider is a probabilistic quantity for which the equality takes place:

$$V_1=\min_{\mu}\int_{D_1}\int_{D_2} f(d_1,d_2)\,d\mu d\nu_0 \tag{176}$$

In the strategy μ_0 of the second decider there is a probabilistic quantity for which the following equality occurs:

$$V_2=\min_{\nu}\int_{D_1}\int_{D_2} f(d_1,d_2)\,d\mu_0 d\nu \tag{177}$$

The existence of the pair $\left(\mu_0,\nu_0\right)$ which represents a saddle point for $\tilde{J}_1$ games is a consequence of the results presented in [16], [25].

Remark 2.26

We can easily prove the equalities:

$$\max_{\nu}\min_{\mu}\int_{D_1}\int_{D_2} f\left(d_1,d_2\right)d\mu d\nu=\max_{\nu}\min_{\mu}\int_{D_1} f\left(d_1,d_2\right)d\nu$$

$$\min_{\mu}\int_{D_1}\int_{D_2} f\left(d_1,d_2\right)d\mu d\nu=\min_{\mu}\int_{D_1} f\left(d_1,d_2\right)d\nu$$

$$\min_{\mu}\max_{\nu}\int_{D_1}\int_{D_2} f\left(d_1,d_2\right)d\mu d\nu=\min_{\mu}\max_{\nu}\int_{D_2} f\left(d_1,d_2\right)d\mu$$

$$\max_{\nu}\int_{D_1}\int_{D_2} f\left(d_1,d_2\right)d\mu d\nu=\max_{\nu}\int_{D_2} f\left(d_1,d_2\right)d\mu$$

As we have mentioned above, the simplest direct proof of Neumann's theorem is based on the properties of duality in the linear optimization.

It is important to mention that the phenomenon of duality in the problems of optimization was firstly pointed out in the context of the minmax problem for the finite games.

Therefore, let's consider the matrix game $\tilde{J}=\left(f,D_1,D_2\right)$, $D_1=\left\{d_1^1,...,d_1^n\right\}$, $D_2=\left\{d_2^1,...,d_2^m\right\}$, where we note $f\left(d_1^i,d_2^j\right)=a_{ij}$, $i=\overline{1,n}$, $j=\overline{1,m}$.

We assume the problem of determining the mixed strategies:

$$P_0=\left\{p_1^0,...,p_n^0\right\},Q_0=\left\{q_1^0,...,q_1^m\right\}$$

which determine a saddle point of f for $\tilde{J}$.

We are led to solving the following two problems:

a) the determination of a strategy P_0 so that:

$$\min_{1\le j\le m}\sum_{i=1}^{n}a_{ij}p_i^0=\max_{P}\min_{1\le j\le m}\sum_{i=1}^{n}a_{ij}p_i=V$$

where $P=\left\{p_1,p_2,...,p_n\right\}$;

b) the detremination of a strategy Q_0 so that:

$$\max_{1\le i\le n}\sum_{j=1}^{m}a_{ij}q_j^0=\min_{Q}\max_{1\le i\le n}\sum_{j=1}^{m}a_{ij}q_j=V$$

where $Q=\left\{q_1,q_2,...,q_m\right\}$.

Let us consider $a_{ij}>0$.

If we write:

$$V_1\left(P\right)=\min_{1\le j\le m}\sum_{i=1}^{n}a_{ij}p_i \tag{178}$$

$$V_2\left(Q\right)=\max_{1\le i\le n}\sum_{j=1}^{m}a_{ij}q_j \tag{179}$$

the first problem may be approached as a problem of maximization of $V_1\left(P\right)$ with the constraints:

$$V_1\left(P\right)\le\sum_{i=1}^{n}a_{ij}p_i,\ j=\overline{1,m},\ \sum_{i=1}^{n}p_i=1,\ p_i\ge 0$$

We introduce the variables:

$$x_i=\frac{p_i}{V_1\left(P\right)}\Rightarrow 1\le\sum_{i=1}^{n}a_{ij}x_i,\ x_i\ge 0$$

Consequently, the problem of maximization of $V_1\left(P\right)$ leads to that of minimization of $\sum_{i=1}^{n}x_i$.

Analog, by writing $y_i = \frac{q_i}{V_2(Q)}$ the problem b) is reduced to the minimization of $\sum_{i=1}^{m} y_i = \frac{1}{V_2(Q)}$, having got the relationships:

$$1 \ge \sum_{j=1}^{m} a_{ij} y_j \,,\, y_j \ge 0$$

Therefore, any solution (P_0, Q_0) of a game of value V gives, at the same time, a solution of the two problems of linear optimization in respect with variables x_i, y_j and $\sum_{i=1}^{n} x_i^0 = \sum_{j=1}^{m} y_j^0 = \frac{1}{V}$.

Contrarily, if $\{x_i^0\}$, $\{y_j^0\}$ are optimal solutions of the two problems of linear optimization and $\sum_{i=1}^{n} x_i^0 = \sum_{j=1}^{m} y_j^0 = \frac{1}{V}$, then for $p_i^0 = Vx_i^0$, $q_j^0 = Vy_j^0$ it results that $\sum_{i=1}^{n} p_i^0 = \sum_{j=1}^{m} q_j^0 = 1$, $\sum_j a_{ij} q_j^0 \le V \le \sum_i a_{ij} p_i^0$, $i \le n$, $j \le m$. From here, we can write $\min_j \sum_i a_{ij} p_i^0 \ge V \ge \max_i \sum_j a_{ij} q_j^0$ and thus the inequalities form below take place:

$$\max_P \min_j \sum_{i=1}^{n} a_{ij} p_i \ge V \ge \min_Q \max_i \sum_{j=1}^{m} a_{ij} q_j \tag{180}$$

Therefore, $P_0 = \{p_i^0\}$, $Q_0 = \{q_j^0\}$ are optimal mixed strategies.

Consequence

The solving of a matrix game with the matrix of paying $(a_{ij})_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}}$ is equivalent to the solving of a couple of dual linear optimization problems:

1) $\min \sum_i x_i$

$$x_i \ge 0, \sum_i a_{ij} x_i \ge 1, \, j = \overline{1,m}$$

2) $\max \sum_j y_j$

$$y_j \ge 0, \sum_j a_{ij} y_j \le 1, i = \overline{1,n}$$

The value V of the game is $V = \frac{1}{\sum_i x_i^0} = \frac{1}{\sum_j y_j^0}$ and the optimal strategies p_i^0, q_j^0 are given by $p_i^0 = Vx_i^0$, $q_j^0 = Vy_j^0$.

Remark 2.27

If the player 2 carries only simple strategies and the player 1 adopts only mixed strategies, we shall consider

$$V = V(P, q) = \sum_{i=1}^{n} a_i p_i$$

the mean value of the game (here $P = (p_1, p_2, ..., p_n)$) is a mixed strategy for the player 1.

We are led to the following problem:

$$\sum_{i=1}^{n} a_i p_i \ge V \to \max$$

$$\sum_{i=1}^{n} p_i = 1, \, p_i \ge 0, i = \overline{1,n}$$

We denote by $x_i = \frac{p_i}{V}$; we have the problem:

$$\sum_{i=1}^{n} a_i x_i \geq 1$$

$$\sum_{i=1}^{n} x_i = \frac{1}{M},\ x_i \geq 0,\ i = \overline{1,n}$$

Because $M \to max$, we are led to a linear optimization problem:

$$\begin{cases} min \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{m} a_i x_i \geq 1 \\ x_i \geq 0,\ i = \overline{1,m} \end{cases}$$

Remark 2.27 (The Equivalence Matrix Game – Linear Optimization)

Let us consider the matrix $A = \left(a_{ij}\right)_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$ and the following linear optimization problem:

$$\begin{cases} max \sum_{i=1}^{m} c_i x_i \\ \sum_{i=1}^{n} a_{ij} x_i \leq b_j,\ j = \overline{1,n} \\ x_i \geq 0,\ i = \overline{1,m} \end{cases} \qquad \begin{cases} min \sum_{j=1}^{n} b_j y_j \\ \sum_{j=1}^{n} a_{ij} y_j \geq c_i,\ i = \overline{1,m} \\ y_j \geq 0,\ j = \overline{1,n} \end{cases}$$

Let us assume the matrix:

$$B = \begin{array}{c} \\ m\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ n\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ 1\ \{ \end{array} \left\| \begin{array}{ccccccc} \overbrace{\begin{matrix} 0 & \cdots & 0 \end{matrix}}^{m} & & \overbrace{\begin{matrix} a_{11} & \cdots & a_{1n} \end{matrix}}^{n} & & \overbrace{c_1}^{1} \\ \vdots\ \vdots\ \vdots & & \vdots\ \vdots\ \vdots & & \vdots \\ 0\ \cdots\ 0 & & a_{1m}\ \cdots\ a_{nm} & & c_m \\ a_{11}\ \cdots\ a_{1m} & & 0\ \cdots\ 0 & & b_1 \\ \vdots\ \vdots\ \vdots & & \vdots\ \vdots\ \vdots & & \vdots \\ a_{n1}\ \cdots\ a_{nm} & & 0\ \cdots\ 0 & & b_n \\ c_1\ \cdots\ c_m & & b_1\ \cdots\ b_n & & 0 \end{array} \right\|$$

If x_i, $i = \overline{1,m}$, y_j, $j = \overline{1,n}$ are the solutions corresponding to problems a) and b), then $x_i^0 = V_1 x_i$, $i = \overline{1,m}$, $v_j^0 = V_1 x_j$, $j = \overline{1,n}$, where

$$V_1 = \frac{1}{1 + \sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j}$$

are optimum strategies for the game with the matrix B.

Remark 2.28

If the matrix A verifies the equality $a_{ij} = -a_{ji}$, $i = \overline{1,n}$, $j = \overline{1,m}$, then:

1) The value of the game is zero.

2) If $P_0 = \left(p_1^0, p_2^0, ..., p_n^0\right)$ is an optimum strategy for the player 1, then this is also an optimum strategy for the player 2.

2.4 Bibliographical Notes and Comments

Semi-continuity of multivocal applications was first introduced by Kuratovski on metric spaces and it was extended for topological spaces by Kakutani, Choqet etc.

Minmax theorems were originally demonstrated using fixed point theorems (Neumann, Fan, Brouwer). Moreover, a great part of the equilibrium theorems for general games appeal to the fixed point theorems (Kakutani, Fan, Ghicksberg).

The first formulation of minmax theorem is attributed to Borel starting from an antagonist game problem. Placing the issue in an inappropriate framework, Borel had concluded that the minmax equality cannot occur in a sufficiently general case.

A correct expression of the minmax problem is owned to Neumann who showed that its solution is a condition that is equivalent to the existence of a saddle point for the efficiency function adopted. The existence of the optimum solution was demonstrated using a fixed point theorem; later on, more simple algebraic demonstrations have been made. The most convenient direct demonstration of Neumann theorem is based on the dual properties of linear optimization.

The first generalization of Kakutani's minmax theorem was made under conditions of continuity for the efficiency function and according to the hypothesis that the sets of strategies are infinite. The solution was demonstrated by a fixed point for higher semi-continuous multi-voce applications.

An extremely important analysis of the conditions necessary for the minmax equality is presented in the excellent monograph of Stefanescu [82].

The first approach of minmax equality in a random extension of a finite game is owed to Neumann [57] who has shown that in such cases there is always a solution to the minmax equality. Ville [87] has made the first theorem for infinite games and Wold has shown that if a set of strategies is finite and the other set is countable then the random extension of the game satisfies the minmax equality if the efficiency function is bounded.

Nash enlarged the saddle point notion to that of an equilibrium point at the n person games [56] permitting the introduction of a new optimality concept, the stability principle, for the playing game situations with n deciders.

On the other hand, Rockfaller [75] remarked that the study of saddle point problems with differentiable utility functions is a particular case of the variation inequality theory with monotone operators. This theory was developed by Brower, Brezis, Sibony. The existence of equilibrium points was generally demonstrated in the games theory.

Thus, Nikaido [59] demonstrated the existence of equilibrium points in the convex two-person zero sum games.

In the random extensions of n player games$(n \geq 2)$, Young [92], Glicksberg [31] demonstrated the existence of equilibrium points, Shapley [77] showed that any stochastic game with stops has equilibrium points and Gillete [30] demonstrated the same thing in the case of stochastic games without stops and with updates.

An important result related to the probabilistic characterization of minmax equality is owed to Goldman [32]. In the case of finite games, this result can be characterized as follows: if the number of strategies of the first player is *m* and the number of strategies of the second player is *n,* then the probability of a saddle point existing in a simple strategy is:

$$\frac{m!n!}{(m+n-1)!}$$

Basically, in the case of a two-person zero sum game, this result proves that the existence of a saddle point is more rare than the existence of an equilibrium point which has the following probability to occur: $1-e^{-1}$.

MINMAX DECISIONS

The sequential decision problem in its general form is defined as follows:

$S = \left\{ X; X_0, \overline{X}; M; u_i, i \in M; D^i(x), i \in \mathbb{N}, x \in X; f_n, n \in \mathbb{N} \right\}$, where:

* X represents the set of states;

* $X_0, \overline{X}$ represents the set of initial states, respectively the set of final states (obviously $X_0 \subset X, \overline{X} \subset X$);

* The set $M = \{1,2,\dots,m\}$ represents the set of decision makers participating to the decision making process;

* u_i represents the utility function of the decision makers $i \in M$;

* $D^i(x)$ represents the set of strategies of the decision maker $i \in M$ in state $x \in X$ (denoted by $D(x) = \prod_{i=1}^{m} D^i(x)$;

* $f_n, n \in \mathbb{N}$ represents the transition functions. The decision making process is described by the following recurrent relation: $x_{n+1} = f_n(x_n, d_n), x_0 \in X_0, d_n \in D(x_n)$

* a_i represents the minimal gain proposed by the decision maker $i \in M$

* $\overline{X}_i = \left\{ x \in \overline{X}; u_i(x) \geq a_i \right\}$ represents the target set of the decision maker;

Let us consider the following algebraic and topological requirements:

X is a linear and topological space; ß$_X$ is σ_algebra generated by topology of space X. We shall associate to measurable space (X, ß$_X$) the set μ(ß$_X$) of all the measures of probability defined on ß$_X$. To each position $x \in X$ is associated the measure of probability $P_x \in \mu$(ß$_X$).

$\overline{X}, X_0$ are compact sets, $D^i(x)$ are linear topological spaces, $i \in \mathbb{N}, x \in X$;

Applications $f_n, n \in \mathbb{N}$ and $u_i : \overline{X} \to \mathbb{R}, i \in M$ are continuous.

We shall also use the following notations:

$T(x_0)$ represents the set of trajectories starting from x_0 and $T^k(x_0)$ represents the set of duration trajectories k starting from $x_0 \in X_0$;

$B_n : X_0 \to P(x)$ are multivocal applications due to which it is possible to establish the set of positions to be reached after n stages, starting from a certain initial position:

$$B_1(x_0) = \left\{ x_1 \in X : x_1 = f_0(x_0, d_0), x_0 \in X_0, d_0 \in D(x_0) \right\}$$

$$B_n(x_0) = \left\{ x_n \in X : x_n = f_{n-1}(x_{n-1}, d_{n-1}), x_{n-1} \in B_{n-1}(x_0), d_{n-1} \in D(x_{n-1}) \right\},\ n > 1;$$

3.1 Optimum Principles for the Uncooperative Case

The conditions that should be met by an optimality criterion were presented for the first time as a list of axioms in a finite decision problem considered as a game against nature. Based on these axioms, Milnor [46] characterizes the minmax principle and the principle of insufficient rationality. Changes to this list of axioms were formulated by Atkinson and Church [2], Harris and more recently by Preda [69] (who characterized the Bayesian principles strictly and broadly for a sequential decision problem).

The optimum criteria agreed by the decision theory, as well as their classification, will be presented hereinafter, by using a synthetic approach.

In order to simplify our presentation, we shall consider a decisional process $y = (f_1, f_2; D_1, D_2)$ which involves only two decision makers. We shall denote by f_1, f_2 the efficiency functions corresponding to the above mentioned decision makers and by D_1, D_2 the sets of their strategies. The results carried forth bellow will take into consideration the perspective of the first decision maker (player).

Under nondeterministic conditions, the maxmin criterion and the minimum regret criterion can be accepted.

According to the **maxmin** criterion, the first decision maker will choose that specific strategy which verifies the following equality:

$$a^* = \max_{d_1} \min_{d_2} f_1(d_1, d_2) \tag{181}$$

namely

$$a^* = \max_{d_1} \{a : f(d_1, d_2) \geq a, d_1 \in D_1, \forall d_2 \in D_2\} \tag{182}$$

In other words, the decision maker will select the strategy which allows him to achieve the lower ceiling of the gain a^*, regardless of the strategic behavior of the second decision maker. It is obvious that this optimum criterion implies the **minimum regret criterion (Savage)**; thus the first decision maker will choose the strategy which verifies the following equality:

$$a = \min_{d_1} \max_{d_2} \left\{ \max_{d_1} f\left(\overline{d}_1, d_2\right) - f\left(d_1 d_2\right) \right\} \tag{183}$$

This criterion is an optimum criterion as well, and it implies a cautious strategic behavior. The criterion is based on the following ratiocination: if the decision maker 1 knew that the decision maker 2 will choose the strategy d_2, then the maximum value of the gain which could be achieved by the first decision maker would be:

$$\max_{d_1} f_1\left(\overline{d}_1, d_2\right)$$

But the first decision maker doesn't really know that the second decision maker will choose the strategy d_2, therefore he will try to „minimize" in the minmax sense of the deviation.

$$\max_{\overline{d}_1} \left\{ f\left(\overline{d}_1, d_2\right) - f\left(d_1, d_2\right) \right\} \tag{184}$$

Under **uncertainty** conditions, we shall have to use the average value maximization criterion, the maximum probability criterion and the maximum level criterion.

The **average value maximization** criterion is extremely frequent, but is quite questionable to use it when the decisional process is non-repeatable.

We assume that the strategy of the decision maker 2 - d_2 is a continuous random variable for which we know the probability density f_{d_2}. Obviously, the corresponding probability distribution can be defined as follows:

$$F_{d_2}(d_1) = \int_0^{d_1} f_{d_2}(t) dt \tag{185}$$

According to the decision maker 2, the optimum solution d_1^* for the first decision maker is the solution that verifies the following equality:

$$\max_{d_1} M\left(f_{d_2}(d_1)\right) = M\left(f_{d_2}\left(d_1^*\right)\right) \tag{186}$$

The adoption of the above optimum criterion has few weak points.

Firstly, the average value is a number which cannot approximate, with an actual error, the result that is going to be obtained. Secondly, we cannot say that the result obtained with a high probability equals exactly the average value.

A version of the law of large numbers comes out in favor of adopting this optimum criterion. According to this law, if the decisional process is repeated times and often, then we shall get the result $n \cdot m$ (m is the average value) with a very high probability.

The **maximum level** principle assumes the choosing of the specific strategy which verifies the following equality:

$$\max_{d_1 V} \left\{ V : P\left(f_{d_2}(d_1) < V\right) \geq \alpha, \alpha \; fixed \right\} \tag{187}$$

Practically, this criterion implies the choosing of the optimum strategy d_1^* which allows us to get the level V, which can be obtained with a fixed probability α.

The **maximum probability** criterion will be then analyzed, by taking into consideration some conditions that are more general. This criterion is also known as the minimum risk principle and it consist in choosing that strategy d_1^* which maximizes the probability that the gain achieves a fixed ceiling *a*:

$$\max_{d_1} P\left(f_{d_2}\left(d_1\right) \geq a\right) = P\left(f_{d_2}\left(d_1^*\right)\right),\ a \text{ fixed} \tag{188}$$

Obviously, for *a* sufficiently large, this criterion can be considered as a risk criterion.

3.1.1 The Principle of Stability

According to this principle, the optimum strategic behavior consists in the adoption of these strategies which are equilibrium points.

According to this principle, the optimal strategy to be adopted is one that has the property that any deviation from it of any decider will lead to lower earnings.

For example, in the case of a decisional process with two decision makers the strategy $\left(\overline{d}_1,\overline{d}_2\right) \in D_1 \times D_2$ is an equilibrium point if the following conditions are met:

$$f_1\left(\overline{d}_1,\overline{d}_2\right) \geq f_1\left(d_1,\overline{d}_2\right), \forall d_1 \in D_1 \tag{189}$$

$$f_2\left(\overline{d}_1,\overline{d}_2\right) \geq f_2\left(\overline{d}_1,d_2\right), \forall d_2 \in D_2 \tag{190}$$

If $f_1\left(d_1,d_2\right)+f_2\left(d_1,d_2\right)=0, \forall\left(d_1,d_2\right) \in D_1 \times D_2$ is a zero-sum game, then due to the notation $f=f_1=-f_2$, the previous inequalities become:

$$f\left(d_1,\overline{d}_2\right) \leq f\left(\overline{d}_1,\overline{d}_2\right) \leq f\left(\overline{d}_1,d_2\right), \forall\left(d_1,d_2\right) \in D_1 \times D_2 \tag{191}$$

The equilibrium point in this case is called saddle point while the previous double inequality represents the inequality that characterizes a saddle point in a zero-sum two-person game.

In addition, under certain algebraic and topological conditions for f, D_1, D_2, one can demonstrate [31] [82] that this double inequality is equivalent to the following equality (minmax equality).

$$\max_{d_1} \min_{d_2} f\left(d_1,d_2\right) = \max_{d_2} \min_{d_1} f\left(d_1,d_2\right) \tag{192}$$

In the case of a game with *m* decision makers where $f_j, D_j,\ j=\overline{1,m}$ represents the efficiency function, namely the set of strategies for the decision maker *j*, then $\left(\overline{d}_1,\overline{d}_2,\ldots,\overline{d}_n\right)$ is an equilibrium point if the following inequalities occur:

$$\begin{cases} f_1\left(\overline{d}_1,\overline{d}_2,\ldots,\overline{d}_n\right) \geq f_1\left(d_1,\overline{d}_2,\ldots,\overline{d}_n\right), \forall d_1 \in D_1 \\ f_2\left(\overline{d}_1,\overline{d}_2,\ldots,\overline{d}_n\right) \geq f_2\left(\overline{d}_1,d_2,\ldots,\overline{d}_n\right), \forall d_1 \in D_1 \\ \vdots \\ f_m\left(\overline{d}_1,\overline{d}_2,\ldots,\overline{d}_n\right) \geq f_m\left(\overline{d}_1,\overline{d}_2,\ldots,d_n\right), \forall d_1 \in D_1 \end{cases} \tag{193}$$

In the case of the sequential decision problem in which the transition equations are (36), the strategy $\overline{d}_n=\left(\overline{d}_n^1,\overline{d}_n^2,\ldots,\overline{d}_n^m\right) \in \mathcal{D}_n$ (corresponding to the state x_n) is an equilibrium point provided the following inequality is verified:

$$P_{f_n\left(x_n,\overline{d}_n\right)}\left\{\overline{x} \in \overline{X} \middle| u_j\left(\overline{x}\right) \geq a_j\right\} \geq P_{f_n\left(x_n,d_n\right)}\left\{\overline{x} \in \overline{X} \middle| u_j\left(\overline{x}\right) \geq a_j\right\} \tag{194}$$

for any $j \in M$, $d_n=\left(\overline{d}_n^1,\ldots,\overline{d}_n^{j-1},d_n^j,\overline{d}_n^{j+1},\ldots,\overline{d}_n^m\right) \in \mathcal{D}_n$.

Necessary and sufficient conditions for the existence of the equilibrium points have been shown in chapter 2.

We shall be in the state $x_n \in X \setminus \overline{X}$ and in the position of decider $i \in M$. We are thus led to a situation of non-cooperative game for which we shall specify the main result.

The evolution of the decisional process through the recurrence relation:

$x_{n+1} = f_n\left(x_n, d_n\right), x_n \in X, d_n \in D_n, n \in \mathbb{N}, x_0 \in \overline{X}_0, x_0$ ensures the fulfillment of set $\overline{X}$ at one point but it does not complete the target set $\overline{X}_i$ for a certain decider $i \in M$. This is why every $B_n\left(x_0\right)$ set built through the previous recurrence relation is assigned a set of efficiency functions $G^n\left(G_1^n, G_2^n, \ldots, G_m^n\right)$ for determining precisely the state $\overline{x}_n$ from $B_n\left(x_0\right)$. Basically, according to the optimal criteria which are adopted by each decider, one can determine the actual position from $B_n\left(x_0\right)$ (fig.4)

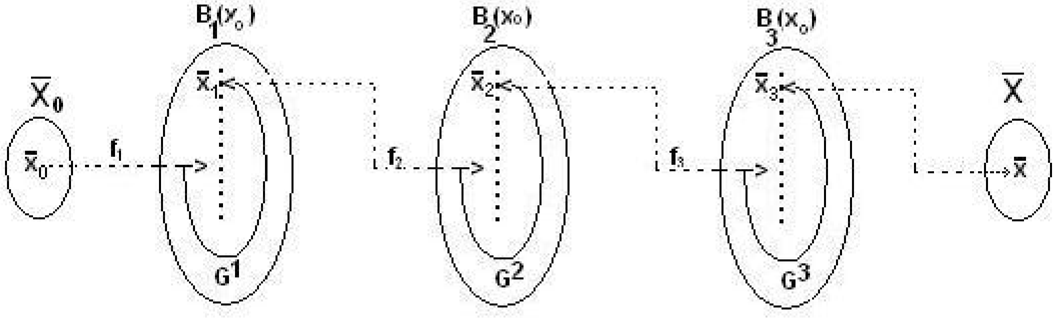

Figure 4

If $M = \{1,2\}$, then d_n^1, d_n^2 shall denote the strategies of the two deciders in the state x_n and $G^1\left(G_1^1, G_2^1\right), G^2\left(G_1^2, G_2^2\right), \ldots, G^n\left(G_1^n, G_2^n\right) \ldots$ shall denote the set of efficiency functions taken into consideration in this state $n \in \mathbb{N}$.

In order to make a convenient presentation, we shall use a case that is stationary $G_1^1 = G_1^2 = \cdots = G_1^n$ and $G_2^1 = G_2^2 = \cdots = G_2^n$. As a consequence, the criteria adopted for each stage $x_1, x_2, \ldots, x_n$ will be described by pairs of functions $\left(G_1, G_2\right)$.

For a certain state $x_k \in B_k\left(x_0\right)$, the pair of strategies $\left(\overline{d}_k^1, \overline{d}_k^2\right)$ represents the equilibrium (saddle point) if the following inequalities can be met:

$$G\left(d_k^1, \overline{d}_k^2\right) \leq G\left(\overline{d}_k^1, \overline{d}_k^2\right) \leq G\left(\overline{d}_k^1, d_k^2\right), \forall\left(d_k^1, d_k^2\right) \in D_k^1 \times D_k^2 \qquad (195)$$

It can be demonstrated that in case $\left(\overline{d}_1^1, \overline{d}_1^2\right), \left(\overline{d}_2^1, \overline{d}_2^2\right), \ldots, \left(\overline{d}_n^1, \overline{d}_n^2\right)$ are equilibrium points $x_1, x_2, \ldots, x_n$, then under certain conditions [47], there will be an equilibrium strategy $\left(\overline{d}_{n+1}^1, \overline{d}_{n+1}^2\right)$ corresponding to the state x_{n+1}. In addition, if $x^* = \lim_n x_n$, then the following properties are described:

$x^* \in \overline{X}, x^*$ is an equilibrium state for the two decision makers.

Looked upon as a special case of sequential decision processes, stochastic games were introduced in 1953 by Shapley [77]. In [77], the existence of equilibrium points is demonstrated for stochastic games with pauses and Gillette [30] demonstrated the existence of equilibrium points for stochastic games with actualizations and without pauses.

In the case of the zero sum two-player games, the finding of the equilibrium points is equivalent to the finding of the saddle points, i.e. to the solving of a minmax problem.

Application to a market equilibrium problem

Assuming that the sequential decisional process described by (36) involves two decision makers, it results that $d_n = \left(d_n^1, d_n^2\right)$.

Consequently, we have

$$x_{n+1} = f_n\left(x_n, d_n^1, d_n^2\right), \quad n \in N, x_0$$

The state x_n is associated with the efficiency function F_n; in our case we shall consider the stationary case where $F_1 = F_2 = \cdots = F_n = \cdots$ and hence their common value will be denoted by F.

If in the case of states $x_0, x_1, \ldots x_n \ldots$ one adopts strategies $\left(d_0^1, d_0^2\right), \left(d_1^1, d_2^2\right), \ldots, \left(d_n^1, d_n^2\right) \ldots$ then the efficiency function F is defined as follows (fig.3.2)

$$F\left(d_i^1(p), d_i^2(p)\right) = \left|O_i(p) - C_i(p)\right|, i \in N \tag{196}$$

Where:

p is the price

O_i, C_i represent the functions of supply and demand corresponding to the state $x_i, i \in \mathbb{N}$

$d_i^1, d_i^2 : \mathbb{R}^+ \to \mathbb{R},\ i \in \mathbb{N}$ (fig.5)

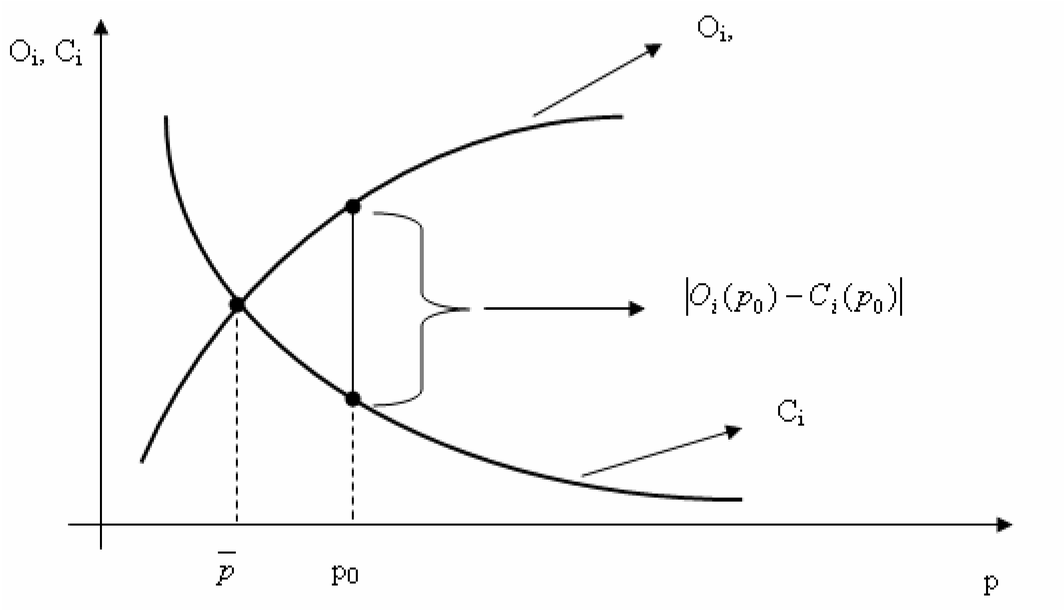

Figure 5

It is obvious that the price $\overline{p}$ represents the equilibrium point in case it is the solution of equation:

$$\max_{d_1} \min_{d_2} F\left(d_1, d_2\right) = \max_{d_2} \min_{d_1} F\left(d_1, d_2\right)$$

or, equivalently, if it verifies the following inequalities:

$$F\left(d_i^1, \overline{d}_i^2\right) \le F\left(\overline{d}_i^1, \overline{d}_i^2\right) \le F\left(\overline{d}_i^1, d_i^2\right), \forall \left(d_i^1, d_i^2\right) \in D_i^1 \times D_i^2 \tag{197}$$

Obviously $\left(\overline{d}_i^1, \overline{d}_i^2\right) = \left(d_i^1\left(\overline{p}\right), d_i^2\left(\overline{p}\right)\right)$

Performing calculations we are led to the following inequalities:

$$\begin{cases} O_i\left(\overline{p}\right) \ge C_i\left(\overline{p}\right) \\ O_i\left(\overline{p}\right) \le C_i\left(\overline{p}\right) \end{cases} \tag{198}$$

As a consequence $\overline{p}$ is the equilibrium price in the state X_i in case it is the solution of equation:

$$O_i\left(\overline{p}\right) = C_i\left(\overline{p}\right) \tag{199}$$

3.1.2 Entropic Criteria

3.1.2.1 The Maximum Probability Criterion

If the decision-making process has evolved to the state $x_n \in X \setminus \overline{X}$, the adoption of the criterion of maximum probability by decider 1 implies the adoption of the problem [47]:

$$\left(P_i^n\right) \sup_{d_n} \left\{ P \in \mathbb{R} \,\middle|\, P_{f_n\left(x_n, d_n^1, \tilde{d}_n^1\right)} \left\{ \overline{x} \in \overline{X} \,\middle|\, u_i\left(\overline{x}\right) \geq a_1 \right\} \geq P \right\}, d_n = \left(d_n^1, \tilde{d}_n^1\right) \in D_n^1 \times \prod_{j=2}^{m} D_n^j \qquad (200)$$

As decider 1 will decide first and deciders $j \in M \setminus \{1\}$ adopt the decision simultaneously, the following notations will be used:

$$D_n^1 = D_1, \prod_{j=2}^{m} D_n^j = D_2$$

The following functionals are introduced:

$$F_n : D_1 \times D_2 \to \mathbb{R}, F_n\left(d_1, d_2\right) = P_{f_n\left(x_n, d_1, d_2\right)} \left\{ \overline{x} \in \overline{X} \,\middle|\, u_1\left(\overline{x}\right) \geq a_1 \right\}, \left(d_1, d_2\right) \in D_1 \times D_2 \qquad (201)$$

$$\begin{aligned} g_n : D_1 \times D_2 \to \mathbb{R}, g_n\left(d_1, d_2\right) = P_{f_n\left(x_n, d_1, d_2\right)} &\left\{ \overline{x} \in \overline{X} \,\middle|\, \sum_{i \in M} u_i\left(\overline{x}\right) \geq \sum_{i \in M} a_i \right\} \\ - P_{x_n} &\left\{ \overline{x} \in \overline{X} \,\middle|\, \sum_{i \in M} u_i\left(\overline{x}\right) \geq \sum_{i \in M} a_i \right\}, \left(d_1, d_2\right) \in D_1 \times D_2 \end{aligned} \qquad (202)$$

and the multivocal application:

$$B_n : D_1 \to \mathcal{P}\left(D_2\right), B_n\left(d_1\right) = \left\{ d_2 \in D_2 \,\middle|\, g_n\left(d_1, d_2\right) \geq 0 \right\} \qquad (203)$$

For greater convenience, we shall write F, g, B instead of F_n, g_n, B_n. $\left(D_1, d_{D_1}\right)$, $\left(D_2, d_{D_2}\right)$ are assumed to be compact metric spaces.

The following hypotheses are issued:

1) the forming of a coalition in the sense of maximum probability is allowed (this coalization criterion will be described in detail in chapter 3, but the principal matter regarding this criterion is the fact that it implies the possibility realizing the transfer of the utility);

2) if the first decider has adopted the strategy $d_1 \in D_1$, the other decider will adopt only strategies from $B\left(d_1\right)$.

Remark 3.1 [49]

Hypothesis 2) is based on the following argument: if the choice of the pair of strategies $\left(d_1, d_2\right) \in D_1 \times B\left(d_1\right)$ increases in the state x_{n+1}:

$$P_{f_n\left(x_n, d_1, d_2\right)} \left\{ \overline{x} \in \overline{X} \,\middle|\, \sum_{j \in M \setminus \{1\}} u_j\left(\overline{x}\right) \geq \sum_{j \in M \setminus \{1\}} a_j \right\}$$

then this choice will suit deciders $j \in M \setminus \{1\}$.

If in the state x_{n+1}, the value:

$$P_{f_n\left(x_n, d_1, d_2\right)} \left\{ \overline{x} \in \overline{X} \,\middle|\, \sum_{j \in M \setminus \{1\}} u_j\left(\overline{x}\right) \geq \sum_{j \in M \setminus \{1\}} a_j \right\}$$

doesn't increase (as opposed to $P_{x_n} \left\{ \overline{x} \in \overline{X} \,\middle|\, \sum_{j \in M \setminus \{1\}} u_j\left(\overline{x}\right) \geq \sum_{j \in M \setminus \{1\}} a_j \right\}$, but $g\left(d_1, d_2\right) \geq 0$), then a coalition in the sense of maximum probability is formed and deciders $j \in M \setminus \{1\}$ will be favored again.

In his turn, decider 1 will be favored as he has the possibility to improve his control over deciders $j \in M \setminus \{1\}$.

Having introduced these notions, we can formulate the problem $\left(P_i^n\right)$ in the following way:

$\left(P_1^n\right)$ yields $d^* = \left(d_1^*, d_2^*\right) \in D_1 \times D_2$ which verifies the equality

$$F\left(d_1^*, d_2^*\right) = \sup_{\left(d_1, d_2\right) \in D_1 \times D_2} F\left(d_1, d_2\right) \qquad (204)$$

The solving of the problem $\left(P_i^n\right)$ represents however, the ideal case for decider 1 as in concrete situation it hardly ever happens for all the deciders of the set M to have the same target set (in other words, all the deciders of the set M have the same target).

Let us consider $\varepsilon \in (0,1)$. We assume that in the state x_n, the following equality is realized:

$$P_n = P_{x_n}\left\{\overline{x} \in \overline{X} \,\middle|\, u_1(\overline{x}) \geq a_1\right\} = \varepsilon \tag{205}$$

We shall write as $I_{n,n+1}^{d_n}$ the mean informational gain (in the Renyi sense) obtained through passing on form the state x_n to the state x_{n+1} as a result of adopting strategy $d_n \in D_n$:

$$I_{n,n+1}^{d_n} = \sum_{j=1}^{m} P_{n+1,j}^{d_n} \ln \frac{P_{n+1,j}^{d_n}}{P_n^j} \tag{206}$$

We have written:

$$P_{n+1,j}^{d_n} = P_{f_n(x_n,d_n)}\left\{\overline{x} \in \overline{X} \,\middle|\, u_j(\overline{x}) \geq a_j\right\}, j = \overline{1,m} \tag{207}$$

Theorem 3.1 [49]

If the following conditions are satisfied:

1) the target sets $\overline{X}_j$, $j \in M$ realize an unfolding of $\overline{X}$;

2) the strategy sets D_n^l are compact sets in $\mathbb{R}^k$, $\forall n \in \mathbb{N}$;

3) application F is continuous in both arguments and convex in the second.

then, there is a finite subset $\overline{D}_n^l \overset{not}{=} \overline{D}_l \subset D_l$ $\left(\overline{D}_l = \left\{d_1^l,...,d_l^l\right\}\right)$ so that a necessary condition for solving problem $\left(P_l^n\right)$ is the solving of the following problem:

$$\left(\tilde{P}_l^n\right) \max_{d_n \in \overline{\mathcal{D}}_n}\left(H_{n+1}^{d_n} + I_{n,n+1}^{d_n}\right) \tag{208}$$

in which $H_{n+1}^{d_n} = -\sum_{j=1}^{m} P_{n,j}^{d_n} \ln P_{n,j}^{d_n}$ represents the indeterminacy (Shannon entropy) given by the choice of the strategy:

$$d_n \in \overline{\mathcal{D}}_n = \overline{D}_n^l \times \prod_{j=2}^{m} D_n^j, P_{f_n(x_n,d_n)}\left\{\overline{x} \in \overline{X} \,\middle|\, u_j(\overline{x}) \geq a_j\right\} = P_{n,j}^{d_n}, j = \overline{1,m}$$

Remark 3.2

This theorem specifies that there is a finite set $\overline{D}_l$ of simple strategies of decider 1 so that a necessary condition for solving problem $\left(P_l^n\right)$ is the adoption of that strategy's indeterminacy $d_n^* = \left\{d_{1n}^*, d_{2n}^*\right\} \in \overline{D}_l \times \prod_{j=2}^{m} D_2^j$ which maximizes the sum between the opponents of decider 1 by the choice of the strategy d_n^* and the mean informational gain obtained by decider 1 by adopting that strategy.

Remark 3.3

If ε is sufficiently great, we say that decider 1 has a favoring situation in the sense of probability in the state $x_n \in X \setminus \overline{X}$. If the strategy $d_n^* \in \overline{D}_l \times \prod_{j=2}^{m} D_n^j$ allows the realization of the inequality:

$$H_{n+1}^{d_n^*} + I_{n,n+1}^{d_n^*} \leq -\varepsilon \ln \varepsilon - (1-\varepsilon)\ln(1-\varepsilon) \tag{209}$$

then the adoption of the strategy d_n^* in the state x_n allows the favoring situation in the sense of probability to be preserved in the state x_{n+1}. The adoption of the strategy d_n^0, the solution to problem $\left(P_l^n\right)$, allows the favoring situation in the sense of probability to be preserved even more in the state x_{n+1} and $\lim_n x_n \in \overline{X_l}$.

Remark 3.4

The following cases have been analyzed so far:

1) the decider i has got a favorable situation in state x_n. In the state x_{n+1} decider i will keep his the favorable situation in one of the situations:

a) $p_n^i = \varepsilon$, ε being sufficiently close to 1 and in the state x_{n+1} the inequality (209) is verified;

b) the decider i has a favorable situation even if in the next position there hasn't been adopted the pair of the optimum strategies $\left(d_1^*, d_2^*\right)$; he has the possibility to correct this "error" which deviates him in the following position from x_{n+1}^*;

2) the decider i decides the first $(i=1)$. In this situation, adopting the criterion of the maximum profit, he has the possibility, using mixed strategies, to estimate his interval of maximum gain which he can obtain in position x_{n+1}; in the case $V_1 \geq a_1$, the trajectory obtained using this criterion it is convergent in $\overline{X}_1$;

3) the decider i decides the last $(i=m)$. In these conditions, the criterion of the maximum probability is equivalent (cases I and II) with other two entropic criteria. If the inequality (209) is again verified, then the obtained trajectory using this criterion is convergent in $\overline{X}_m$.

3.1.2.2 The Maximum Profit Criterion

The following hypotheses are supposed to be tested:

1) the sets of simple strategies are finite: for each $j \in M$ there is $K_j < \infty$ so that $card\left(D_n^j\right) = k_j$ (we shall note the set of strategies of decider $j \in M$, $D_n^j = \left\{d_{j,n}^1, d_{j,n}^2, \ldots, d_{j,n}^{k_j}\right\}$);

2) deciders will decide successively.

We shall note $\tilde{D}_n^j$ the set of mixed strategies of the decider $j \in M$. We assume $i=1$ (decider i will decide first) and we shall associate with decider 1 the functional $R_1 : \prod_{j=1}^{m} \tilde{D}_n^j \to \mathbb{R}$ defined as follows:

$$R_1\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_m\right) = \sum_{j_1=1}^{k_1} \sum_{j_2=2}^{k_2} \ldots \sum_{j_m=1}^{k_m} P_{f_n\left(x_n, d_1, d_2, \ldots, d_m\right)} \left\{\overline{x} \in \overline{X} \middle| u_1\left(\overline{x}\right) \geq a_1\right\} d_{1,n}^{j_1} d_{2,n}^{j_2} \ldots d_{m,n}^{j_m} - \sum_{j_1=1}^{k_1} d_{1,n}^{j_1} \ln d_{1,n}^{j_1} \tag{210}$$

where $\left(d_{1,n}^{j_1}, d_{2,n}^{j_2}, \ldots, d_{m,n}^{j_m}\right) \in \prod_{j=1}^{m} \tilde{D}_n^j$,

We shall adopt as optimal criterion for decider 1 the maximization of the functional R_1 i.e. the maximization of the sum between the mean utility and the indeterminacy contained in the choice of the respective strategies.

We shall adopt the following notation:

$$u_1\left(d_n^1, d_n^2, \ldots, d_n^m\right) = P_{f_n\left(x_n, d_n^1, d_n^2, \ldots, d_n^m\right)} \left\{\overline{x} \in \overline{X} \middle| u_1\left(\overline{x}\right) \geq a_1\right\} \tag{211}$$

$$d_{i,n}^{j_i} = d_i^{j_i},\ i = \overline{1,m},\ d_n^k = d_k,\ k = \overline{1,m}$$

Lemma 3.1 [44]

The distribution of probabilities $\left(p_1, p_2, \ldots, p_m\right)$ which maximizes:

$$R\left(p_1, p_2, \ldots, p_m\right) = \sum_{i=1}^{m} a_i p_i - \sum_{i=1}^{m} p_i \ln p_i,\ a_i > 0 \text{, fixed, } i = \overline{1,m} \tag{212}$$

is given by:

$$p_i = \frac{e^{a_i}}{\sum_{j=1}^{m} e^{a_j}},\ i = \overline{1,m} \tag{213}$$

the maximum value of R being:

$$R_{max} = \ln \sum_{j=1}^{m} e^{a_j} \tag{214}$$

Theorem 3.2 [47], [49]

The following results occur:

1) If there is k', $1 < k' < m$ so that the deciders $2,3,\ldots,k'$ adopt the mixed strategies $d_2^0, d_3^0, \ldots, d_{k'}^0$, and the deciders $k'+1, k'+2, \ldots, m$ adopt a strategic behavior opposed to the maximum profit of decider 1 verifies the inequality:

$$\max_{\tilde d_1}\min_{\tilde d_{k'+1}}\ldots\min_{d_m} R_1\left(\tilde d_1^0,\tilde d_2^0,\ldots,\tilde d_{k'}^0,\tilde d_{k'+1}^0,\ldots,\tilde d_m^0\right)\geq$$

$$\geq\sum_{j_1=1}^{k_1} e^{\sum\limits_{j_2=1}^{k_2}\ldots\sum\limits_{j_{k'}=1}^{k_{k'}}\left(\min\limits_{\tilde d_{k'+1}}\ldots\min\limits_{\tilde d_m}\sum\limits_{j_{k'+1},\ldots,j_m} u_1\left(d_1,d_2^0,\ldots,d_{k'}^0,d_{k'+1},\ldots,d_m\right)\prod\limits_{i\in\{2,\ldots,k'\}} d_i^{j_i}\right)\prod\limits_{i\in\{k'+1,\ldots,m\}} d_i^{j_i}} = m\left(\tilde d_2^0,\tilde d_3^0,\ldots,\tilde d_{k'}^0\right) \tag{215}$$

the margin $m\left(\tilde d_2^0,\tilde d_3^0,\ldots,\tilde d_{k'}^0\right)$ corresponding to the mixed strategy:

$$\tau^j=\frac{e^{\sum\limits_{j_2=1}^{k_2}\ldots\sum\limits_{j_{k'}=1}^{k_{k'}}\left(\min\limits_{\tilde d_{k'+1}}\ldots\min\limits_{\tilde d_m}\sum\limits_{j_{k'+1}=1}^{k_{k'+1}}\ldots\sum\limits_{j=1}^{k_m}u_1\left(d_1^j,d_2^0,\ldots,d_{k'}^0,d_{k'+1},\ldots,d_m\right)\prod\limits_{i=2}^{k'}d_i^{j_i}\right)\prod\limits_{i=k'+1}^{m}d_i^{j_i}}}{\sum\limits_{j=1}^{k_1}e^{\sum\limits_{j_2=1}^{k_2}\ldots\sum\limits_{j_{k'}=1}^{k_{k'}}\left(\min\limits_{\tilde d_{k'+1}}\ldots\min\limits_{\tilde d_m}\sum\limits_{j_{k'+1}=1}^{k_{k'+1}}\ldots\sum\limits_{j_m=1}^{k_m}u_1\left(d_1^j,d_2^0,\ldots,d_{k'}^0,d_{k'+1},\ldots,d_m\right)\prod\limits_{i=2}^{k'}d_i^{j_i}\right)\prod\limits_{i=k'+1}^{m}d_i^{j_i}}},\ j=\overline{1,k_1} \tag{216}$$

2) If decider 1 has no information concerning the strategic behavior of the other deciders, then the maximum profit of this decider verifies the inequality:

$$V_1\leq\max_{\tilde d_1}R_1\left(\tilde d_1,\tilde d_2,\ldots,\tilde d_m\right)\leq V_2,\ \forall\left(\tilde d_2,\ldots,\tilde d_m\right)\in\prod_{j=2}^{m}\tilde D_j \tag{217}$$

$$V_1=\ln\sum_{j_1=1}^{k_1}e^{\min\limits_{\tilde d_2}\ldots\min\limits_{\tilde d_m}\sum\limits_{j_2=1}^{k_2}\ldots\sum\limits_{j_m=1}^{k_m}u_1\left(d_1^{j_1},d_2,\ldots,d_m\right)\prod\limits_{k\in\{2,\ldots,m\}}d_k^{j_k}} \tag{218}$$

$$V_2=\ln\sum_{j_1=1}^{k_1}e^{\max\limits_{\tilde d_2}\ldots\max\limits_{\tilde d_m}\sum\limits_{j_2=1}^{k_2}\ldots\sum\limits_{j_m=1}^{k_m}u_1\left(d_1,d_2,\ldots,d_m\right)\prod\limits_{k\in\{2,\ldots,m\}}d_k^{j_k}} \tag{219}$$

the margins V_1 and V_2 corresponding to the mixed strategies:

$$\tau_2^j=\frac{e^{\min\limits_{\tilde d_2}\ldots\min\limits_{\tilde d_m}\sum\limits_{j_2=1}^{k_2}\ldots\sum\limits_{j_m=1}^{k_m}u_1\left(d_1^j,d_2,\ldots,d_m\right)\prod\limits_{k\in\{2,\ldots,m\}}d_k^{j_k}}}{\sum\limits_{j_1=1}^{k_1}e^{\min\limits_{\tilde d_2}\ldots\min\limits_{\tilde d_m}\sum\limits_{j_2=1}^{k_2}\ldots\sum\limits_{j_m=1}^{k_m}u_1\left(d_1^{j_1},d_2,\ldots,d_m\right)\prod\limits_{k\in\{2,\ldots,m\}}d_k^{j_k}}},\ j=\overline{1,k_1} \tag{220}$$

$$\tau_2^j=\frac{e^{\max\limits_{\tilde d_2}\ldots\max\limits_{\tilde d_m}\sum\limits_{j_2=1}^{k_2}\ldots\sum\limits_{j_m=1}^{k_m}u_1\left(d_1^j,d_2,\ldots,d_m\right)\prod\limits_{k\in\{2,\ldots,m\}}d_k^{j_k}}}{\sum\limits_{j_1=1}^{k_1}e^{\max\limits_{\tilde d_2}\ldots\max\limits_{\tilde d_m}\sum\limits_{j_2=1}^{k_2}\ldots\sum\limits_{j_m=1}^{k_m}u_1\left(d_1^{j_1},d_2,\ldots,d_m\right)\prod\limits_{k\in\{2,\ldots,m\}}d_k^{j_k}}},\ j=\overline{1,k_1} \tag{221}$$

We have got the following scheme (fig.6):

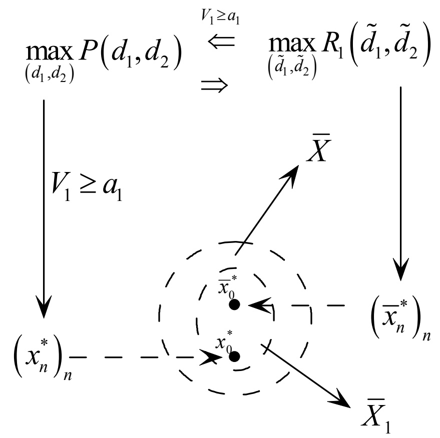

Figure 6

We note $d_n^1 = d_1, \left(d_n^2, d_n^3, ..., d_n^m\right) = d_2$

According to theorem 1.1 from [49], we have the inequality

$$\left| \max_{d_1} \min_{d_2} P\left(d_1, d_2\right) - \max_{\tilde{d}_1} \min_{\tilde{d}_2} R_1\left(\tilde{d}_1, \tilde{d}_2\right) \right| < \varepsilon \tag{222}$$

where

$$\varepsilon = -\sum_{j_1=1}^{K_1} d_1^{j_1} \ln d_1^{j_1} \tag{223}$$

If $V_1 \geq a_1$ and ε is sufficiently low, we have the following scheme (fig.7):

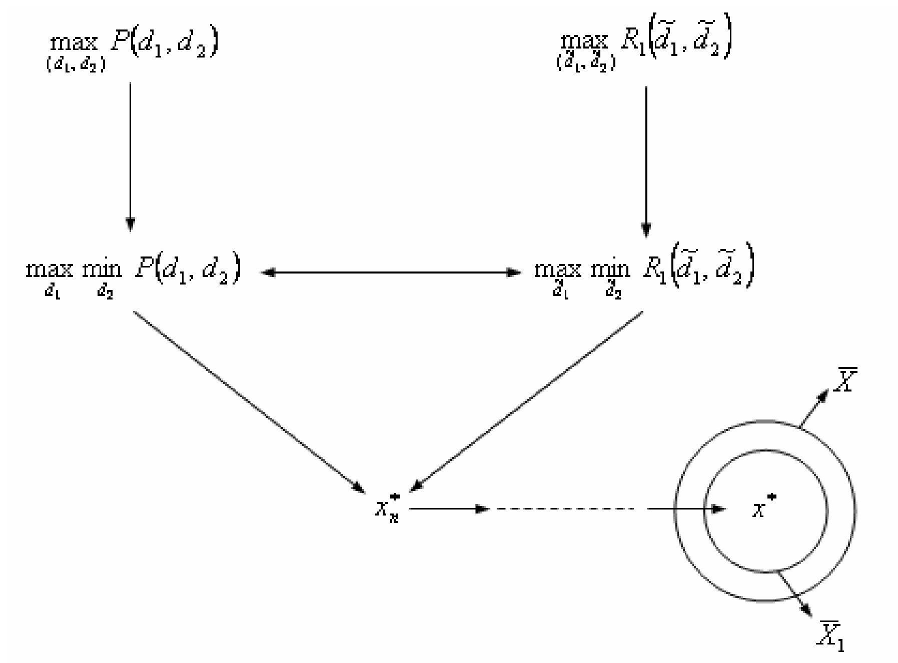

Figure 7

3.1.3 The Equalization Principle

3.1.3.1 Theoretical Considerations

The principle of equalization is a principle of optimality which characterizes a principle of cautious strategic behavior in the context of decision theory.

The results which will be presented represent a generalization of several results due to Ghermeier for a decision-making problem with two deciders and incomplete information.

We assume that the target set $\bar{X}_i$, $i = \overline{1,m}$ realizes a partition of the set $\bar{X}$. We associate with the state $x_{n+1} \in X \setminus \bar{X}$ the following elements:

a) $\tilde{P}^{n+1} = \left\{ p^{n+1} = \left(p_1^{n+1}, p_2^{n+1}, ..., p_n^{n+1}\right) \right\}, \exists d_n \in D_n, p_j^{n+1} = P_{f_n(x_n, d_n)} \left\{ \bar{x} \in \bar{X} \middle| u_j\left(\bar{x}\right) \geq a_j \right\}, j = \overline{1,m}$.

From the condition that the sets $\bar{X}_i$, $i = \overline{1,m}$ should realizes an unfolding of $\bar{X}$, the equality $\sum_{j=1}^{m} p_j^{n+1} = 1$ results directly.

b) the functionals $\tilde{u}_j^{n+1} : [0,1] \to \mathbb{R}$, continuous and monotonously ascending, $j = \overline{1,m}$, $\tilde{u}_j^{n+1}$ are interpreted depending on the partial utility associated with decider j in the state x_{n+1}, $j \in M$; $u_i^{n+1} : \tilde{P}^{n+1} \to \mathbb{R}$, $u_i^{n+1}\left(p\right) = \tilde{u}_i^{n+1}\left(p_i\right)$.

c) the antagonistic game $\Gamma_{n+1} = \left\{ \tilde{P}^{n+1} \times M, u_i^{n+1}, i \in M \right\}$. For greater convenience, in calculation we write $\tilde{P}^{n+1} = P$, $u_j^{n+1} = u_j$, $p_j^{n+1} = p_j$, $\tilde{u}_j^{n+1} = \tilde{u}_j$, $j = \overline{1,m}$.

We consider the problems:

$$(P_1)\ \max_{p \in P} \min_{1 \le i \le n} u_i(p) \tag{224}$$

$$(P_2)\ \min_{p \in P} \max_{1 \le i \le n} u_i(p) \tag{225}$$

and let us take $\{P_0\}_{opt}$, $\{P^0\}^{opt}$ the set of the solutions to these problems.

Theorem 3.3 (The Principle of Equalization) [29], [49]

The following properties are defined:

1) If $u_i(0) \le u_{i+1}(0)$, $i = \overline{1,m-1}$, then there is $p \in P$ and $k \le m-1$ so that the following conditions should be fulfilled:

a1) $\min\limits_{1 \le i \le k} u_k(p) \le u_{k+1}(0)$;

a2) $p_j > 0$, $j = \overline{1,k}$, $p_j = 0$, $j = \overline{1,k-1}$, $u_i(p) = u_k(p)$, $1 \le i \le k-1$.

2) If $u_i(1) \ge u_{i+1}(1)$, $i = \overline{1,m-1}$, then there is $k' \ge i$ and $p \in P$ so that the following conditions should be fulfilled:

b1) $\min\limits_{p \in P} u_{k'}(p) \ge u_{k'+1}(1)$;

b2) $p_j = 0$, $j = \overline{1,k'-1}$, $p_j > 0$, $j = \overline{k',m}$, $u_i(p) = u_{k'}(p)$, $i = \overline{k'+1,m}$.

Theorem 3.4 [47], [49]

If the following conditions are met:

a1) $k' \le i \le k$

a2) deciders $j \in M \setminus \{i\}$ adopt either the minmax criterion or the minmax one.

Then the following properties are defined:

1) for any $p_0 \in \{P_0\}_{opt} \cup \{P_0\}^{opt}$, there is $p' \in \{P_0\}_{opt} \cup \{P_0\}^{opt}$ so that $p_1' > p_i^0$;

2) for any $p \in \tilde{P}$, there is $p'' \in \tilde{P}$ so that $p_i'' \ge p_i$, if $\min\limits_{k' \le i \le m} \max\limits_{p^0 \in \{P^0\}_{opt}} u_i(p_i^0) = \max\limits_{1 \le i \le k} \min\limits_{p^0 \in \{P_0\}_{opt}} u_i(p_i^0)$. (226)

Theorem 3.5 [47], [49]

If the following conditions are fulfilled:

a1) $k' \le i \le k$

a2) deciders $j \in M \setminus \{i\}$ adopt either the minmax criterion or the minmax one.

Then the following properties take place:

3) for any $p_0 \in \{P_0\}_{opt} \cup \{P_0\}^{opt}$, there is $p' \in \{P_0\}_{opt} \cup \{P_0\}^{opt}$ so that $p_1' > p_i^0$;

4) for any $p \in \tilde{P}$, there is $p'' \in \tilde{P}$ so that $p_i'' \ge p_i$, if $\min\limits_{k' \le i \le m} \max\limits_{p^0 \in \{P^0\}_{opt}} u_i(p_i^0) = \max\limits_{1 \le i \le k} \min\limits_{p^0 \in \{P_0\}_{opt}} u_i(p_i^0)$. (227)

Remark 3.5

Theorems 3.4 and 3.5 generalize a result from [29]. From theorem 3.5, it results that if in a certain position, decider i obtains a favorable situation and the other deciders keep on acting cautiously, strategically speaking (in the sense of using the minmax or maxmin criteria of optimality), then decider i can maintain his favorable situation or even improve the situation in the next position according to the following diagram (fig.8):

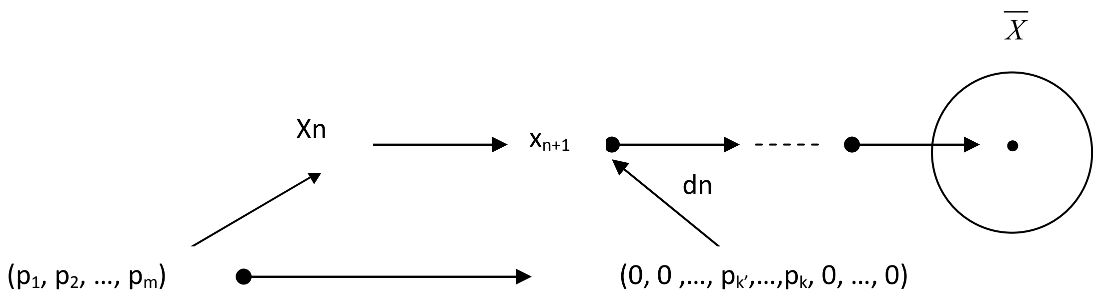

Figure 8

Remark 3.9

From theorem 3.4 and from [47] and [49] it results directly that a necessary condition for i to obtain and further on maintain a favorable situation in the sense of probability is the coalition of i in the sense of probability and a cautious strategic behavior of the deciders not forming a coalition.

3.1.3.2 The Determination and Interpretation of Optimal Solutions

Let us suppose the existence of a portofolio that contains n actives, for which, the following analytical expressions of the profitability of these actives are known:

$$r_i : R_+ \to R_+, r_i(s) = a_i s + b_i, i = \overline{1,k} \quad (228)$$

The weights $p_1, p_2, \ldots, p_k$ of these actives in the portofolio structure are also known (obvious $p_i \geq 0, i = \overline{1,k}, \sum_{i=1}^{k} p_i = 1$). The average profitability of the active portofolio is marked with $\bar{r}$.

$\bar{r} : R_+^k \to R$ and it is defined as follows:

$$\bar{r}(s_1, s_2, \ldots, s_k) = \sum_{i=1}^{k} p_i r_i(s) \quad (229)$$

Form the economic point of view, it is required that a supplementary condition should be met: $\sum_{i=1}^{k} s_i = B$, B being a fixed value and also usually signifying a budget.

The problem may be solved with the help of the financial arbitrage model. Basically, this implies the solving of the following optimization problem:

$$(P)\begin{cases} \max\limits_{s} \min\limits_{p} \sum\limits_{j=1}^{k} p_j r_j(s_j) \\ \sum\limits_{j=1}^{k} s_j = B \\ \sum\limits_{j=1}^{k} p_j = 1 \\ p_j \geq 0, s_j \geq 0, j = \overline{1,k} \end{cases} \quad (230)$$

or, in an equivalent form [23], [29], [47]:

$$(P) \max_{s} \min_{p} \sum_{i=1}^{k} p_i r_i(s_i), \text{ where} \quad (231)$$

$$s = (s_1, s_2, \ldots, s_k), \sum_{j=1}^{k} s_j = B, s_j \geq 0, j = \overline{1,k}$$

$$p = (p_1, p_2, \ldots, p_k), \sum_{j=1}^{k} s_j = 1, p_j \geq 0, j = \overline{1,k}$$

Basically, the solving of the problem (P) implies the determination of the absolute maximum value for the existence of financial extraction portfolio, independent of its structure (from the point of view of money extraction weight).

Generally, the solving of the problem (P) is considered to be difficult; however, the easiest way to find the solution is using the equalization principle [23], [29]. Without restricting the general framework of the problem, we shall number the efficiency functions $r_1, r_2, \ldots, r_k$ so the following condition should be met:

$$r_1(0) \leq r_2(0) \leq \ldots \leq r_k(0)$$

Practically we need to analyze the following situations:

1. if $r_1(0) = r_i(0), i = \overline{2,n}$, then the optimal solution $s^* = (s_1^*, s_2^*, \ldots, s_k^*)$ that we are searching for, can be determined as solution of the following algebraic system:

$$\begin{cases} r_1(s_1) = r_2(s_2) \\ r_1(s_1) = r_3(s_3) \\ \vdots \\ r_1(s_1) = r_n(s_n) \\ s_1 + s_2 + \ldots + s_n = B \end{cases} \tag{232}$$

Particular cases

Considering the fact that each function can be conveniently approximated by means of a first or a second degree polynomial, we shall analyze the following situations:

1.1. $r_i(s_i) = a_i s_i + b\,,\quad i = \overline{1,k}$ (233)

It is obvious that in this case, $r_1(0) = r_2(0) = \cdots = r_k(0)$

The system (232) can be written:

$$\begin{cases} a_1 s_1 + b = a_2 s_2 + b \\ a_1 s_1 + b = a_3 s_3 + b \\ \vdots \\ a_1 s_1 + b = a_k s_k + b \\ s_1 + s_2 + \cdots + s_k = B \end{cases}$$

The optimum solution $s^* = (s_1^*, s_2^*, \ldots, s_k^*)$ can be determined after performing some relatively easy calculations:

$$\begin{cases} s_1^* = \dfrac{B}{a_1 A} \\ s_2^* = \dfrac{B}{a_2 A} \\ \vdots \\ s_k^* = \dfrac{B}{a_k A} \end{cases} \quad \text{, where } A = \sum_{j=1}^{k} \frac{1}{a_j} \tag{234}$$

In this case, $r_1(s_1^*) = r_2(s_2^*) = \cdots = r_k(s_k^*) = \dfrac{B}{A} + b$ (235)

1.2. $r_i(s_i) = a_i s_i^2 + b_i s_i + c\,,\quad i = \overline{1,k}$ (236)

Because $r_1(0) = r_2(0) = \cdots = r_k(0)$, the optimum solution can be determined by solving the following system:

$$\begin{cases} a_1 s_1^2 + b_1 s + c = a_2 s_2^2 + b_2 s_2 + c \\ a_1 s_1^2 + b_1 s + c = a_3 s_3^2 + b_3 s_3 + c \\ \vdots \\ a_1 s_1^2 + b_1 s + c = a_k s_k^2 + b_k s_k + c \\ s_1 + s_2 + \cdots + s_k = B \end{cases} \tag{237}$$

It is relatively difficult to solve the above system; therefore, the solution that we are searching for has the following form:

$$\begin{cases} s_1^* = \dfrac{1}{a_1}\left(\dfrac{A+\sum\limits_{j=1}^{k}\dfrac{b_j-b_1}{a_j}}{\sum\limits_{j=1}^{k}\dfrac{1}{a_j}}\right) \\ s_2^* = \dfrac{1}{a_2}\left(\dfrac{A+\sum\limits_{j=1}^{k}\dfrac{b_j-b_2}{a_j}}{\sum\limits_{j=1}^{k}\dfrac{1}{a_j}}\right) \\ \vdots \\ s_k^* = \dfrac{1}{a_k}\left(\dfrac{A+\sum\limits_{j=1}^{k}\dfrac{b_j-b_k}{a_j}}{\sum\limits_{j=1}^{k}\dfrac{1}{a_j}}\right) \end{cases} \tag{238}$$

Remark 3.7

The following equalities are verified, no matter whether $s^* = \left(s_1^*, s_2^*, \ldots, s_k^*\right)$ represents the solutions of the system (232), or the solutions of particular cases 1.1, 1.2:

a) $r_1\left(s_1^*\right) = r_2\left(s_2^*\right) = \cdots = r_k\left(s_k^*\right)$ (239)

b) $\max\limits_{s}\min\limits_{p}\sum\limits_{i=1}^{k} p_i r_i\left(s_i\right) = r_i\left(s_i^*\right), \quad i = \overline{1,k}$ (240)

2. if the condition $r_1(0) \neq r_i(0), i = \overline{2,n}$ cannot be met, then the following algebraic (k-1) systems are successively solved:

$$\begin{cases} r_1\left(s_1\right) = r_2\left(s_2\right) \\ s_1 + s_2 = B \end{cases}, \begin{cases} r_1\left(s_1\right) = r_2\left(s_2\right) \\ r_1\left(s_1\right) = r_3\left(s_3\right) \\ s_1 + s_2 + s_3 = B \end{cases}, \ldots, \begin{cases} r_1\left(s_1\right) = r_2\left(s_2\right) \\ r_1\left(s_1\right) = r_3\left(s_3\right) \\ . \\ . \\ . \\ r_1\left(s_1\right) = r_n\left(s_n\right) \\ s_1 + s_2 + \ldots + s_n = B \end{cases} \tag{241}$$

We shall find the solutions $\left(\overline{s}_1^1, \overline{s}_2^1\right), \left(\overline{s}_1^2, \overline{s}_2^2, \overline{s}_3^2\right), \ldots, \left(\overline{s}_1^{k-1}, \overline{s}_2^{k-1}, \ldots, \overline{s}_k^{k-1}\right)$ of these systems and we shall also determine the n index with the property:

$$r_1\left(\overline{x}_1^n\right) = \max\left\{r_1\left(\overline{s}_1^1\right), r_1\left(\overline{s}_1^2\right), \ldots, r_1\left(\overline{s}_1^{k-1}\right)\right\} \tag{242}$$

In this case, the optimal solution $s^* = (s_1^*, s_2^*, \ldots, s_k^*)$ that we are searching for has the following property:

$$\begin{cases} s_1^* \neq 0 \\ s_2^* \neq 0 \\ \vdots \\ s_n^* \neq 0 \end{cases} \qquad \begin{cases} s_{n+1}^* = 0 \\ s_{n+2}^* = 0 \\ \vdots \\ s_k^* = 0 \end{cases} \tag{243}$$

and, consequently, $p_{n+1} = p_{n+2} = \ldots = p_n = 0$.

Therefore, only the first n components of the portfolio are economically important, the others are insignificant because their weights are equal to zero.

Particular cases

The efficiency functions have the following form

$$r_i(s_i) = a_i s_i + b_i, i = \overline{1,k} \tag{244}$$

In this case, we have to solve the following (k-1) linear algebraic systems:

$$\begin{cases} r_1(s_1) = r_2(s_2) \\ r_1(s_1) = r_3(s_3) \\ \vdots \\ r_1(s_1) = r_j(s_j) \\ s_1 + s_2 + \ldots + s_j = B \end{cases} \Rightarrow \begin{cases} a_1x_1 - a_2x_2 = b_2 - b_1 \\ a_1x_1 - a_3x_3 = b_3 - b_1 \\ \vdots \\ a_1x_1 - a_nx_n = b_n - b_1 \\ x_1 + x_2 + \ldots + x_n = B \end{cases}, j = \overline{2,k} \tag{245}$$

After performing calculations, we find the solutions of the *(k-1)* systems, as follows:

$$\begin{cases} s_1^* = \dfrac{\dfrac{B}{a_1} + \sum\limits_{j=1}^{n} \dfrac{b_j - b_1}{a_1 a_j}}{\sum\limits_{j=1}^{n} \dfrac{1}{a_j}} = \dfrac{B + \sum\limits_{j=1}^{n} \dfrac{b_j}{a_j} - b_1 \sum\limits_{j=1}^{n} \dfrac{1}{a_j}}{a_1 \sum \dfrac{1}{a_j}} \\ s_2^* = \dfrac{\dfrac{B}{a_2} + \sum\limits_{j=1}^{n} \dfrac{b_j - b_2}{a_2 a_j}}{\sum\limits_{j=1}^{n} \dfrac{1}{a_j}} = \dfrac{B + \sum\limits_{j=1}^{n} \dfrac{b_j}{a_j} - b_2 \sum\limits_{j=1}^{n} \dfrac{1}{a_j}}{a_2 \sum \dfrac{1}{a_j}} \\ \vdots \\ s_n^* = \dfrac{\dfrac{B}{a_n} + \sum\limits_{j=1}^{n} \dfrac{b_j - b_n}{a_n a_j}}{\sum\limits_{j=1}^{n} \dfrac{1}{a_j}} = \dfrac{B + \sum\limits_{j=1}^{n} \dfrac{b_j}{a_j} - b_n \sum\limits_{j=1}^{n} \dfrac{1}{a_j}}{a_n \sum \dfrac{1}{a_j}} \end{cases} \quad n = \overline{2,k} \tag{246}$$

or, in a concentrated form:

$$s_t^* = \frac{\dfrac{B}{a_t} + \sum\limits_{j=1}^{n} \dfrac{b_j - b_t}{a_t a_j}}{\sum\limits_{j=1}^{n} \dfrac{1}{a_j}}, t = \overline{1,n}, n = \overline{2,k} \tag{247}$$

If $s_1^{*,1}$, $s_1^{*,2}$, ..., $s_1^{*,k-1}$ represents the first component of the solutions for systems *1,2, ..., k-1,* after an immediate calculation we get:

$$\begin{cases} r_1\left(s_1^{*,1}\right) = \dfrac{B + \dfrac{b_1}{a_1} + \dfrac{b_2}{a_2}}{\dfrac{1}{a_1} + \dfrac{1}{a_2}} \\ r_1\left(s_1^{*,2}\right) = \dfrac{B + \dfrac{b_1}{a_1} + \dfrac{b_2}{a_2} + \dfrac{b_3}{a_3}}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3}} \\ \vdots \\ r_1\left(s_1^{*,k-1}\right) = \dfrac{B + \dfrac{b_1}{a_1} + \dfrac{b_2}{a_2} + \ldots + \dfrac{b_k}{a_k}}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_k}} = \dfrac{B + \sum\limits_{j=1}^{k} \dfrac{b_k}{a_M}}{\sum\limits_{j=1}^{j} \dfrac{1}{a_j}} \end{cases} \tag{248}$$

We calculate the *t* index with the property:

$$\frac{B+\frac{b_1}{a_1}+\frac{b_2}{a_2}+\ldots+\frac{b_t}{a_t}}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_t}}=max\left\{\frac{B+\frac{b_1}{a_1}+\frac{b_2}{a_2}}{\frac{1}{a_1}+\frac{1}{a_2}},\frac{B+\frac{b_1}{a_1}+\frac{b_2}{a_2}+\frac{b_3}{a_3}}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}},\ldots,\frac{B+\frac{b_1}{a_1}+\frac{b_2}{a_2}+\ldots+\frac{b_k}{a_k}}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_k}}\right\} \quad (249)$$

Based on [23], the optimal solution of the problem (P) verifies the requirement: $s^*_{t+1} = s^*_{t+2} = \ldots = s^*_k = 0$, therefore the weights $p_{t+1}, p_{t+2}, \ldots, p_k$ are insignificant.

Economic interpretation

We use the following notations:

$$m=min\left\{\frac{B+\frac{b_1}{a_1}+\frac{b_2}{a_2}}{\frac{1}{a_1}+\frac{1}{a_2}},\frac{B+\frac{b_1}{a_1}+\frac{b_2}{a_2}+\frac{b_3}{a_3}}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}},\ldots,\frac{B+\frac{b_1}{a_1}+\frac{b_2}{a_2}+\ldots+\frac{b_k}{a_k}}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_k}}\right\}\Rightarrow$$

$$\Rightarrow m=\min_j\left\{\frac{B+\sum_{i=1}^{j}\frac{b_i}{a_i}}{\sum_{i=1}^{j}\frac{1}{a_i}}\right\} \quad ,\ j=\overline{1,k} \quad (250)$$

$$M=max\left\{\frac{B+\frac{b_1}{a_1}+\frac{b_2}{a_2}}{\frac{1}{a_1}+\frac{1}{a_2}},\frac{B+\frac{b_1}{a_1}+\frac{b_2}{a_2}+\frac{b_3}{a_3}}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}},\ldots,\frac{B+\frac{b_1}{a_1}+\frac{b_2}{a_2}+\ldots+\frac{b_k}{a_k}}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_k}}\right\}\Rightarrow \quad (251)$$

$$\Rightarrow M=\max_j\left\{\frac{B+\sum_{i=1}^{j}\frac{b_i}{a_i}}{\sum_{i=1}^{j}\frac{1}{a_j}}\right\} \quad ,\ j=\overline{1,k}$$

It is obvious that each value given by the equality (248) belongs to the interval: $[m,M]$.

From economic point of view, it is extremely important that we can find the index t which has the following property:

$$M=\frac{B+\frac{b_1}{a_1}+\frac{b_2}{a_2}+\cdots+\frac{b_t}{a_t}}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_t}} \quad (252)$$

In this case, only the first t actives are important within the portfolio structure, because for the other $k-t$ actives, the corresponding weights are null, namely:

$$p_{t+1}=p_{t+2}=\cdots=p_k=0$$

We shall use the following notations:

$$\overline{B}=B+\frac{b_1}{a_1}+\frac{b_2}{a_2}+\cdots+\frac{b_t}{a_t} \quad (253)$$

$$A=\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_t} \quad (254)$$

Therefore, $M=\frac{\overline{B}}{A}$

Obviously, the following conditions must be met:

$$\frac{\overline{B}}{A} > \frac{\overline{B} + \frac{b_{t+1}}{a_{t+1}}}{A + \frac{1}{a_{t+1}}} \quad , \qquad \frac{\overline{B}}{A} > \frac{\overline{B} - \frac{b_{t-1}}{a_{t-1}}}{A - \frac{1}{a_{t-1}}} \tag{255}$$

In order simplify our analysis (without restricting its generality), we shall assume that a_{t-1} , a_{t+1} are positive values.

After performing calculations, we get the following results, starting form the above inequalities:

$$\frac{\overline{B}}{A} > b_{t+1} \quad , \qquad \frac{\overline{B}}{A} < b_{t-1} \tag{256}$$

Therefore, we shall have:

$$b_{t-1} > M > b_{t+1} \tag{257}$$

In the case, $t = 2$ the optimal solution of the problem (P) is the following:

$$\begin{cases} s_1^* = \dfrac{\frac{A}{a_1} + \frac{b_2 - b_1}{a_1 a_2}}{\frac{1}{a_1} + \frac{1}{a_2}} \\ s_2^* = \dfrac{\frac{A}{a_2} + \frac{b_1 - b_2}{a_1 a_2}}{\frac{1}{a_1} + \frac{1}{a_2}} \\ \vdots \\ s_t^* = 0, t = \overline{3,k} \end{cases} \tag{258}$$

As a result, the weights $p_3 = p_4 = \ldots = p_k = 0$, therefore

$$\max_s \min_p \sum_{i=1}^{k} r_i(s_i) p_i = \max_s \min_p \sum_{i=1}^{n} r_1(s_1^*) p_i$$

besides

$$r_1(s_1^*) \sum_{i=1}^{n} p_i = r_1(s_1^*) \sum_{i=1}^{2} p_i = r_1(s_1^*) = \frac{B + \frac{b_1}{a_1} + \frac{b_2}{a_2}}{\frac{1}{a_1} + \frac{1}{a_2}} \tag{259}$$

3.1.3.3 The Solution of a Ruination Problem

Let us consider a sequential decision problem in which the deciders involved are coalited in two coalitions C_1, C_2 having the final state sets $\overline{X}_1$, $\overline{X}_2$, which form a partition for $\overline{X}$: $\overline{X}_1 \cup \overline{X}_2 = \overline{X}$, $\overline{X}_1 \cap \overline{X}_2 = \varnothing$.

We also consider the model of the following market phenomenon: in their struggle for supremacy in taking hold of a certain commodity market, the deciders from C_1, intending to eliminate the deciders from C_2 which control the market, want in a first stage to take hold of at least one strategic point of the existing k in this market. Having once penetrated the commodity market, the deciders in C_1 will try the complete elimination of the deciders in C_2 by ruining them.

We interpret the decision process of the first stage as a game made up of k simultaneous periods. We assume that in this stage, the capital of each coalition is A and B respectively; each decider from C_2 can ruin m_j monetary units of the capital A in the game j, $j = \overline{1,k}$.

Let D_1^j, D_2^j be the set of the strategies of C_1 and C_2 respectively, in the game j, $j = \overline{1,k}$, for any $\left(d_1^1, d_1^2, \ldots, d_1^k\right) \in \prod_{j=1}^{k} D_1^j$, $\left(d_2^1, d_2^2, \ldots, d_2^k\right) \in \prod_{j=1}^{k} D_2^j$ we shall have: $d_1^j, d_2^j \geq 0$, $j = \overline{1,k}$, $\sum_{j=1}^{j} d_1^j = A$, $\sum_{j=1}^{j} d_2^j = B$.

We introduce the utility function $u : \prod_{j=1}^{k} D_1^j \times \prod_{j=1}^{k} D_2^j \to \mathbb{R}$:

$$u\left(d_1^1, d_1^2, ..., d_1^k, d_2^1, d_2^2, ..., d_2^k\right) = \sum_{j=1}^{k} min\left(m_j d_2^1 - d_1^j, 0\right) \quad (260)$$

Let us calculate the guaranteed optimum strategy for C_2 (maxmin strategy) as well as the maxmin value of the non-cooperative game between C_1 and C_2.

We shall use the results given by [29]:

$$V_2 = \max_{d_2} \min_{d_1} \left\{ \sum_{i=1}^{k} min\left(m_i d_2^i - d_1^i, 0\right) \right\} = \max_{d_2} \min_{1 \le i \le k} \left(m_i d_2^i - A\right) \quad (261)$$

By introducing partial utility function $\tilde{u}_i$, $\tilde{u}_i : D_1^i \to \mathbb{R}$, $\tilde{u}_i\left(d_2^i\right) = m_i d_2^i - A$, we shall have $\tilde{u}_i(0) = -A = u_1(0)$ hence it results that among the optimum strategies there will be strategies of the form $\left(d_2^j, 0, ..., 0\right)$ (where j is determined from the condition $m_j d_2^j - A = \min_{1 \le i \le k}\left(m_i d_2^i - A\right)$ so that $V_2 = u_j\left(d_2^j\right)$. It will result directly that the guaranteed optimum (simple) strategy for C_2 will be $d_2^j = \frac{B}{m_i} \sum_{i=1}^{k} \frac{1}{m_i}$, the maxmin value being

$$V_2 = min\left(\frac{B}{\sum_{i=1}^{k} \frac{1}{m_i}} - A, 0\right) \quad (262)$$

For the demonstration of the guaranteed optimum strategy for C_1 (minmax strategy) as well as of the minmax value we shall first observe that the u efficiency function is convex in $d_2 = \left(d_2^1, d_2^2, ..., d_2^k\right)$ as so the V_1 minmax value will be equal to the value of the game [29], [47]:

$$V_1 = min\left\{\min_{1 \le i \le k}\left(m_i B - A\right), 0\right\} \quad (263)$$

The minmax (mixed) strategy will be:

$$d_1 = \frac{1}{m_j} \sum_{i=1}^{k} \frac{1}{m_i},\ j = \overline{1,k} \quad (264)$$

as for any $d_2 \in \prod_{i=1}^{k} D_2^i$ we have:

$$\sum_{i=1}^{k} \frac{1}{m_i \sum_{j=1}^{k} \frac{1}{m_j}} max\left(A - m_i, d_2^i, 0\right) \ge max\left(\sum_{i=1}^{k} \frac{A - m_i d_2^i}{m_i \sum_{j=1}^{k} \frac{1}{m_j}}\right) = max\left(A - \frac{B}{\sum_{i=1}^{k} \frac{1}{m_j}}, 0\right) = V_1 \quad (265)$$

Remark 3.7

As a result of the concavity of the u functional is relation to $d_2 = \left(d_2^1, d_2^2, ..., d_2^k\right) \in B^k$ the V value of the game between the two coalitions will be equal to V_2 and consequently a decision-making behavior for C_2 which is based on keeping decisions secret does not favor this coalition. It is very important for C_2 to obtain additional information on the strategic behavior of C_1.

Remark 3.8

The optimum solution of C_1 consist in the concentration of the forces in a single game (in the j_0 game in which the condition $m_{j_0} = \min_{1 \le j \le k}\left\{m_j\right\}$ is realized), keeping the secret about the game in which it concentrates its forces. If C_2 has no information on C_1, it has to distribute its forces uniformly.

After the first stage, the remaining capital reserves being $A_1, B_1 \subset \mathbb{N}$, the second stage, the ruining stage proper, takes place an a particular sequential process:

$$X = \{a,b\}, a,b \in \mathbb{N}, a+b = A_1 + B_1$$
$$X_0 = \{A_1, B_1\}$$
$$\bar{X} = \{A_1 + B_1, 0\} \cup \{0, A_1 + B_1\}$$

If $x_n \in X$, $x_n = \left(a_1^n, a_2^n\right)$ we shall have:

$$x_{n+1} = f_n\left(x_n, d_1^n, d_2^n\right) = \left(a_1^{n+1}, a_2^{n+2}\right) \tag{266}$$

where:

$$\left(a_1^{n+1}, a_2^{n+2}\right) \in \left\{\left(a_1^n + 1, a_2^n - 1\right), \left(a_1^n - 1, a_2^n + 1\right)\right\}, \forall \left(d_1^n, d_2^n\right) \in D_1^n \times D_2^n$$

Remark 3.9

The sequential process described before consists in a series of null sums; a coalition means its having to concede to the winning coalition a monetary unit out of the available capital.

In this stage, there arises the problem of determining the mean duration of the decision-making process as well as the probabilities of getting ruined for the coalitions if it is known that the probability of mining the game for the C_1 coalition in the x_n state is $p = constant$, $n \in \mathbb{N}$.

It results that $\left(a_1^n\right)_n$ is a Markov chain homogenous with the states $0,1,2,...,C = A_1 + B_1$ and with the passing matrix [47]:

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & q & 0 & p \\ 0 & 0 & 0 & & 0 & 0 & 1 \end{pmatrix}, q = 1 - p$$

The potential matrix R is given by:

$$R = \left[I - \begin{pmatrix} 0 & p & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & 0 & 0 & p \\ 0 & 0 & 0 & & 0 & q & 0 \end{pmatrix} \right]^{-1}$$

the elements of this matrix being:

$$r_{(i,j)} = \begin{cases} \dfrac{1}{(2p-1)\left[\left(\dfrac{p}{q}\right)^C - 1\right]} \begin{cases} \left[\left(\dfrac{p}{q}\right)^j - 1\right]\left[\left(\dfrac{p}{q}\right)^{A-i} - 1\right] & , j \le i \\ \left[\left(\dfrac{p}{q}\right)^i - 1\right]\left[\left(\dfrac{p}{q}\right)^{C-i} - \left(\dfrac{p}{q}\right)^{j-1}\right] & , j > i \end{cases} & , p \ne \dfrac{1}{2} \\ \dfrac{2}{C} \begin{cases} j(C-i) & , j \le i \\ i(C-j) & , j > i \end{cases} & , p = \dfrac{1}{2} \end{cases}$$

The mean duration of the decision-making process will be:

$$D_m = \sum_{l=1}^{C-1} r_{(A,l)} = \begin{cases} \dfrac{1}{2p-1}\left[\dfrac{\left(\dfrac{p}{q}\right)^C - \left(\dfrac{p}{q}\right)^{B_1}}{\left(\dfrac{p}{q}\right)^C - 1} - A_1\right], p \neq \dfrac{1}{2} \\ A_1 B_1 \qquad , p = \dfrac{1}{2} \end{cases}$$

The ruining probability of the C_1 coalition is given by $P_r^{C_1}$, where:

Besides, if $p \neq \frac{1}{2}$

$$D_n = \frac{p^{B_1}\left(p^{A_1} - q^{A_1}\right)}{(p-q)\left(p^C - q^C\right)} - A_1 \tag{267}$$

If $p \neq \frac{1}{2}$, $P_r^{C_1} = \dfrac{\dfrac{p}{q} + \left(\dfrac{p}{q}\right)^2 + \ldots + \left(\dfrac{p}{q}\right)^{B_1}}{\dfrac{p}{q} + \left(\dfrac{p}{q}\right)^2 + \ldots + \left(\dfrac{p}{q}\right)^{B_1} + \ldots + \left(\dfrac{p}{q}\right)^C} = \dfrac{\dfrac{p}{q}\dfrac{\left(\dfrac{p}{q}\right)^{B_1} - 1}{\dfrac{p}{q} - 1}}{\dfrac{p}{q}\dfrac{\left(\dfrac{p}{q}\right)^C - 1}{\dfrac{p}{q} - 1}} = \dfrac{\left(\dfrac{p}{q}\right)^{B_1} - 1}{\left(\dfrac{p}{q}\right)^C - 1}$

If $p = \frac{1}{2}$, $P_r^{C_1} = \dfrac{B_1}{C} = \dfrac{C - A_1}{C} = 1 - \dfrac{A_1}{C}$

Besides

$$P_r^{C_1} = \begin{cases} \dfrac{\left(\dfrac{p}{q}\right)^{B_1} - 1}{\left(\dfrac{p}{q}\right)^C - 1}, p \neq \dfrac{1}{2} \\ 1 - \dfrac{A_1}{C} \qquad , p = \dfrac{1}{2} \end{cases} \tag{268}$$

and the ruining probability of the C_2 coalition will be:

$$P_r^{C_2} = 1 - P_r^{C_1}$$

3.2 Cooperative and Partial Cooperative Cases

3.2.1 Requisite Conditions of Coalization in the Maximum Probability Sense

The forming of coalitions in cooperative decision-making, competitive-conflicting processes (or partially competitive-conflicting ones) is a problem that is still insufficiently studied.

The realization of a $C \subset M$ coalition (M being the set of deciders) is largely dependent on the concept of characteristic function based in its turn on the concept of guaranteed optimum profit in the zero-sum two-players game between C and $M \setminus C$. The $M \setminus C$ coalition is artificially formed and the deciders belonging to it are supposed to the interests of the deciders in C.

This paragraph will present the contribution of the author to the introduction of a more general concept of coalition than the one based on the concepts of characteristic function and compensation.

The concept of cooperation which will be introduced will use two elements of reference coalition operator and the excess functional.

The coalition operator will associate with every $\alpha \in [0,1]$ a set (written $T\alpha$) of coalitions, each with the probability of achieving the decision process in one of the positions of the associated fixed target set, equal to α.

The concept of excess will occur both at the moment of the coalition formation and at the moment of the distribution of the final gains. In this case, the excess will be connected with the generalization of the notions of imputation from the theory of cooperative games in Neumann's sense [57] and of his immediate extensions.

For the beginning, let us consider $\mathcal{B}$, Γ – algebra of subsets in X ($X \neq \emptyset$) and $H = \{\gamma_i\}_{i \in \mathbb{N}}$ a numerable family of submeasures on $\mathcal{B}$.

To each submeasure $\gamma_i \in H$ we will associate its extension $\gamma_i^* : \mathcal{P}(X) \to [0,\infty)$ defined as follows:

$$\gamma_i^*(E) = inf\{\gamma_i(A) | E \subseteq A \in \mathcal{B}\}, \forall E \in \mathcal{P}(X) \tag{269}$$

It is known [20], [47], γ_i^* is a submeasure on Γ – algebra $\mathcal{P}(X)$, the $(\mathcal{P}(X),\Delta,\cap)$ triplet is a Γ – ring on Γ – algebra $\mathcal{P}(X)$ and if Ω_H is the subset family of $\mathcal{P}(X)$ on form:

$$\mathcal{V}_{K,\varepsilon} = \{E \in \mathcal{P}(X) | \gamma_i^*(E) < \varepsilon, i \in K\} \tag{270}$$

where $K \subset M$, K – finite, $\varepsilon > 0$, then there is a single topology Γ_H on $\mathcal{P}(X)$ and thus $(\mathcal{P}(X),\Delta,\cap,\Gamma_H)$ is a topological ring (written as $H(\mathcal{P}(X))$), Ω_H being a base of proximities of 0 for Γ_H, $H(\mathcal{B})$ will be the $\mathcal{B}$ topological subring of $H(\mathcal{P}(X))$.

It was shown in [20] that there is a finite submeasure γ on $\mathcal{B}$, so that $H \equiv \gamma$ that is H is absolutely continuous with respect to γ and γ is also absolutely continuous with respect to H. Noting $\gamma(\mathcal{B}) = H(\mathcal{B})$, $\gamma(\mathcal{B})$ is a complete semimetric space, where the semimetric perimeter is:

$$d(E,F) = \gamma(E\Delta F), \forall E,F \in \mathcal{B} \tag{271}$$

Let $\mathcal{B}_0 \subset \mathcal{B}$ be the class of measurable γ sets. It is known from [20] that $\mathcal{B}_0$ is a Γ – algebra, the restriction of γ in $\mathcal{B}_0$ is a complete and positive measure and $\mathcal{B}_0$ with the topology induced from $\gamma(\mathcal{P}(X))$ is a completely separable semimetric space with the same semimetric parameter „d", defined by the γ measure on $\mathcal{B}_0$. The equivalent relation „$\sim$" will be introduced on $\mathcal{B}_0$, so that:

$$E \sim F \Leftrightarrow \gamma(E\Delta F) = 0 \tag{272}$$

The set of equivalence classes will be noted with $\mathcal{B}_0(\gamma)$ and it makes up a Γ – complete and separable metric algebra related to the usual operations among the classes of equivalent sets. Let us consider $E \in \mathcal{B}_0(\gamma)$ and $x,y \in E$. As $x = \gamma(y)$, abbreviating through $\gamma(E)$, $\gamma_i(E)$, $i \in \mathbb{N}$ we will understand $\gamma(x)$ and $\gamma_i(x)$, $i \in \mathbb{N}$, $\forall x \in E$ respectively.

Let $\mathcal{E}_{\mathcal{B}_o(\gamma)}$ be the class of the Borellian sets from the $\mathcal{B}_0(\gamma)$ metric space and $\tilde{P}$ a measure of probability defined on $\mathcal{E}_{\mathcal{B}_o(\gamma)}$. Suppose $A \subset \mathbb{R}$ and

$$\mathcal{A}_A = \{E \in \mathcal{B}_0(\gamma) | \gamma_i(E) \in A, \forall \gamma_1 \in H\} \tag{273}$$

Theorem 3.6 [20]

We have the following properties:

1. $\forall \varepsilon > 0, \exists C_\varepsilon \subset \mathcal{B}_0(\gamma)$, compact:

$$\tilde{P}(\mathcal{B}_0(\gamma) \setminus C_\varepsilon) < \frac{\varepsilon}{2} \tag{274}$$

2. $\forall A \subset \mathbb{R}, \mathcal{A}_A \in \mathcal{E}_{\mathcal{B}_0(\gamma)}$, the subsets A_1 and A_2 exist from $\mathbb{R}$:

   a) $A_1 \subseteq A \subseteq A_2$

   b) $\tilde{P}\{E \in \mathcal{B}_0(\gamma) | \gamma_1(E) \in A_2 \setminus A_1 = 0\}, \forall i \in \mathbb{N}$

We shall further on consider the problem of sequential decision described in (36), in which we shall modify according to the condition of the existence of several deciders.

In this situation $\Gamma-$ algebra $\mathcal{B}$ will be $\Gamma-$ algebra $\mathcal{B}_x$ generated by the trace of the topology of the space X on $\overline{X}$.

Let $x\in X\setminus\overline{X}$ an intermediate position in the space of the positions. In this position, the elements γ_i^* of the H family will be defined by means of the probability measure P_x as follows:

$$\gamma_i^*\left(E\right)=P_x\left\{\overline{x}\in E\left|u_i\left(\overline{x}\right)\geq a_i\right.\right\},\forall E\in\mathcal{B}_{\overline{X}},i\in M \tag{275}$$

If M is finite, $M=\left\{1,2,...,m\right\}$ we shall define γ_i^* for $i>m$ as follows:

$$\gamma_i^*\left(E\right)=\frac{1}{m}\sum_{j=1}^{m}P_x\left\{\overline{x}\in E\left|u_j\left(\overline{x}\right)\geq a_j\right.\right\},\forall E\in\mathcal{B}_{\overline{X}} \tag{276}$$

We shall note (for the sake of facility and when there is no risk if confusion) $P=P_x$ and $\gamma_i^*=\gamma_i$, $i\in M$.

Definition 3.1

Suppose $A\subset\left[0,1\right]$; the condition $\mathcal{A}_A\in\mathcal{E}_{\mathcal{B}_0(\gamma)}$ will be called a condition of the first order of coalition in the sense of maximum probability associated to A and M or in short, the condition of the first order associated to A and M.

On the basis of the theorem, it results that if for the set $A\subset\left[0,1\right]$ the condition of the first order is fulfilled, there exists the Borelian sets A_2 from $\left[0,1\right]$ so that following properties should be verified:

$1)\ A_1\subseteq A\subseteq A_2$

$$2)\ \tilde{P}\left\{E\in\mathcal{B}_0\left(\gamma\right)\left|\gamma_j\left(E\right)\in A_2\setminus A_1,\forall j\in M\right.\right\}= \tag{277}$$

$$=\tilde{P}\left\{E\in\mathcal{B}_0\left(\gamma\right)\left|P\left\{\overline{x}\in E\left|u_j\left(\overline{x}\right)\geq a_j\right.\right\}\in A_2\setminus A_1,\forall j\in M\right.\right\}=0$$

Let us analyze the following situations:

a) a_i is the certain maximum profit of the decider $i\in M$.

In this case, if we note $\overline{X}_M=\bigcap_{i=1}^{\infty}\overline{X}_i$, for $A=\left\{1\right\}$ and $X_M\neq\varnothing$ from theorem 3.1, it follows immediately that we shall have:

$$\tilde{P}\left\{E\in\mathcal{B}_0\left(\gamma\right)\left|\overline{X}_M\subset E\right.\right\}=\tilde{P}\left\{E\in\mathcal{B}_0\left(\gamma\right)\left|P\left\{\overline{x}\in E\left|u_j\left(\overline{x}\right)\geq a_j\right.\right\}=1,\forall j\in M\right.\right\}\neq 0 \tag{278}$$

In other words, the end of the decision-making process is achieved with a non-null probability in one of the position in $\overline{X}_M$.

b) Let us suppose that $\overline{X}_i=\overline{X}_j$, $i,j\in M$, $i\neq j$ and suppose:

$$\alpha=P\left\{\overline{x}\in\overline{X}\left|u_j\left(\overline{x}\right)\geq a_j\right.\right\},j\in M \tag{279}$$

In that case, for $A=\left\{\alpha\right\}$, from theorem 4.1 it results that we shall have:

$$\tilde{P}\left\{E\in\mathcal{B}_0\left(\gamma\right)\left|\overline{X}_j\subset E,j\in M\right.\right\}=\tilde{P}\left\{E\subset\overline{X}\left|P\left\{\overline{x}\in E\left|u_j\left(\overline{x}\right)\geq a_j\right.\right\}=\alpha\right.\right\}\neq 0 \tag{280}$$

That means that if the community of interests of the deciders runs very high (in the case under consideration they have the same target set) then the class of the Borelian sets containing the $\overline{X}_i$ set is on a non-null $\tilde{P}$ measure and consequently the deciders are interested in a strategic behavior achievement of the end of the decision-making process set. The concept of coalition witch will be introduced is based on this result.

Definition 3.2

Suppose $\alpha\in\left(0,1\right]$. We say that $C\in\mathcal{P}\left(M\right)$ is a coalition of the order α (and we note by α the order of $\left(C\right)$) in the sense of maximum probability if the following conditions are fulfilled:

a) the conditions of the first order for $A=\left[\alpha,1\right]$ and C is verified;

b) $\alpha = sup\left\{\alpha \in (0,1] \left| P\left\{\overline{x} \in \overline{X} \left| \sum_{i\in B} u_i(\overline{x}) \geq \sum_{i\in B} a_i \right.\right\} \geq \alpha \geq \max_{i\in B}\left\{P\left\{\overline{x} \in \overline{X} \,\middle|\, u_i(\overline{x}) \geq a_i\right\}\right\}\right.\right\}$, $\forall B \subseteq C$, B finite.

The b) condition will be called the condition of the second order of coalition in the sense of maximum probability associated to A and C or in short, the condition of the second order associated to A and C.

Remark 3.10

This definition express the fact that when condition 1 is fulfilled for A and C, whoever the deciders of coalition C might be, trough transfer of utility they increase their probability of obtaining together a profit at least as high as the sum of the associated ceiling values.

Definition 3.3

$C \in \mathcal{P}(M)$ is a coalition formed in the sense of maximum probability if there is $\beta \in (0,1]$ so that:

$$\beta = ord(C) \tag{281}$$

Remark 3.11

If B and C are coalitions formed in the sense of maximum probability and $B \subseteq C$, then $ord(B) \leq ord(C)$. The set of coalitions that can be formed in sense of maximum probability will be noted with $\mathcal{W}$. We suppose $\mathcal{W} \neq \varnothing$.

We shall associate to the measurable space $(\mathcal{W}, \mathcal{P}(\mathcal{W}))$ the measure of probability $\overline{P}$ with which we shall define the applications $\overline{\gamma}_i : \mathcal{P}(M) \to \mathbb{R}$ given by:

$$\gamma_i(A) = \begin{cases} \begin{cases} \overline{P}\left\{C \in \mathcal{W} \,\middle|\, i \in C \subset A\right\}, i \in M & , if\ M = \mathbb{N} \\ \begin{cases} \overline{P}\left\{C \in \mathcal{W} \,\middle|\, i \in C \subset A\right\}, i = \overline{1,m} & , if\ A \neq \varnothing \\ \dfrac{1}{m}\sum_{i=1}^{m} \overline{P}\left\{C \in \mathcal{W} \,\middle|\, i \in C \subset A\right\}, i > m & , if\ M = \{1,2,...,m\} \end{cases} \end{cases} \\ 0 & , if\ A = \varnothing \end{cases} \tag{282}$$

Proposition 3.1

$\overline{\gamma}_i$ is a submeasure, $\forall i \in M$.

Remark 3.11

It can be shown that $\overline{\gamma}_i$ is really a measure, $i \in M$, but in order to organize $\mathcal{P}(M)$ as a topological ring [20] it is enough to show it is a submeasure.

Proof

For any $i \in M$, the following properties must be shown:

1) $\overline{\gamma}_i(\varnothing) = 0$

2) $\overline{\gamma}_i(A) \leq \gamma_i(B), \forall A, B \subseteq M, A \subseteq B$

3) $\overline{\gamma}_i(A \cup B) \leq \overline{\gamma}_i(A) + \overline{\gamma}_i(B), \forall A, B \subseteq M$

Property 1) follows from definition of $\overline{\gamma}_i$.

We demonstrate property 2).

Suppose $A, B \subseteq M$, $A \subseteq B$. The following possibilities occur:

i) there is no coalition in B, formed in the sense of maximum probability; if such is the case, nor are there in A coalitions formed in the sense of maximum probability and consequently we shall have:

$$\overline{\gamma}_i(A) = \overline{\gamma}_i(B) = 0$$

ii) there is $C \subset B$, $C \not\subset A$, $ord(C) = \alpha$.

In this case we have:

$$\overline{\gamma}_i(A) = 0 < \alpha = \overline{\gamma}_i(C) \leq \overline{\gamma}_i(B)$$

iii) there are coalitions in A formed in sense of maximum probability. If such is this case, they will also exist in B and consequently according to remark 3.11 the following inequality takes place:

$$\overline{\gamma}_i(A) \leq \overline{\gamma}_i(B)$$

Property 2) has been demonstrated.

Property 3) results immediately if the equality:

$$A \cup B = A \cup (B \setminus A)$$

is taken into account.

As $A \cap (B \setminus A) = \varnothing$, $\overline{\gamma}_i(A \cup B) = \overline{\gamma}_i(A \cup (B \setminus A)) = \overline{\gamma}_i(A) + \overline{\gamma}_i(B \setminus A) \le \overline{\gamma}_i(A) + \overline{\gamma}_i(B)$.

We have shown that $\overline{\gamma}_i$ is a submeasure whatever $i \in M$ might be.

3.2.2 The Excess of Coalitions Formed in the Maximum Probability Sense

We shall associate with the family $H_1 = \{\gamma_i\}_{i \in M}$ the submeasure $\overline{\gamma}$, absolutely continuous in relation to H_1 and finite.

Thus $\overline{\gamma}(\mathcal{P}(M))$ becomes a complete semimetric space with the semimetric $d : \overline{\gamma}(\mathcal{P}(M)) \times \overline{\gamma}(\mathcal{P}(M)) \to \mathbb{R}$, given by:

$$d(E,F) = \overline{\gamma}(E \Delta F), \forall E,F \in \overline{\gamma}(\mathcal{P}(M)) \tag{283}$$

We assume below that the following hypothesis is verified if for any $i \in M$, $\overline{P}\{C \in \mathcal{W} | i \in C \subset E \setminus F\} = 0$, $\overline{P}\{C \in \mathcal{W} | i \in C \subset F \setminus E\} = 0$, then $E = F$ $(E,F \in \overline{\gamma}(\mathcal{P}(M)))$.

Remark 3.12

From the way in which submeasures $\overline{\gamma}_i$, $i \in M$ were defined, it results that d is a metric (as the following condition is verified: $\forall E,F \in \overline{\gamma}(\mathcal{P}(M))$, $d(E,F) = 0 \Rightarrow E = F$). Thus $\overline{\gamma}(\mathcal{P}(M))$ becomes a $\Gamma-$ metrical and complete algebra.

Definition 3.4

The following operator is called coalition operator:

$$T : [0,1] \to \mathcal{P}(\overline{\gamma}(\mathcal{P}(M))), T\alpha = \begin{cases} \{C \in \overline{\gamma}(\mathcal{P}(M)) | ord(C) = \alpha\}, & \alpha \neq 0 \\ \{\{i\}_{i \in M}\} & , \alpha = 0 \end{cases} \tag{284}$$

Definition 3.5

The following functional is called excess associated with the coalition operator: $\tilde{R} : [0,1] \times \overline{\gamma}(\mathcal{P}(M)) \to \mathbb{R}$, given by:

$$\tilde{R}(\alpha, C) = \sum_{j \in C} \tilde{z}_j^\alpha - \partial(C) \tag{285}$$

where $\partial(C)$ represents the certain maximum gain of the C coalition.

$$z_j^\alpha = \begin{cases} a_j^\alpha & , if\ i \in T\alpha \neq \varnothing \\ \partial(\{j\}), & if\ T\alpha = \varnothing \end{cases} \tag{286}$$

a_j^α represents the ceiling associated to the decider $j \in C$, if $T\alpha \neq \varnothing$ and $C \in T\alpha$ (we assume that $\forall \alpha \in (0,1]$ and $\forall C \in T\alpha$, $\sum_{j \in C} z_j^\alpha < \infty$).

$\partial(\{j\})$ represents the certain maximum profit of the decider $j \in C$.

Remark 3.13

For fixed $(\alpha, C) \in [0,1] \times \overline{\gamma}(\mathcal{P}(M))$, the value $\tilde{R}(\alpha, C)$ will be called excess of order α associated with T and it represents a generalization of the notion of excess from the games theory.

For $T\alpha \neq \varnothing$ it can be noticed that $\tilde{R}(\alpha, C) \ge 0$, $\forall C \in T\alpha$.

With the help of the d metric from the space $\overline{\gamma}(\mathcal{P}(M))$ we shall construct Hausdorff's metric:

$$\tilde{d} : \mathcal{P}(\overline{\gamma}(\mathcal{P}(M))) \times \mathcal{P}(\overline{\gamma}(\mathcal{P}(M))) \to \mathbb{R},$$

$$\tilde{d}(A,B) = max\{\rho(A,B), \rho(B,A)\}, \forall A,B \in \mathcal{P}(\overline{\gamma}(\mathcal{P}(M))) \tag{287}$$

where

$$\rho(A,B)=\sup_{x\in A}\inf_{y\in B} d(x,y) \tag{288}$$

Further on, we assume that for any $\alpha\in(0,1]$, $T\alpha\neq\varnothing$.

Let $\varepsilon>0$ be sufficiently small (but fixed) and $\tilde{P}$ a measure of probability defined on $\mathcal{E}_{\overline{\gamma}(\mathcal{P}(M))}$.

On the basis of theorem 3.6, the compact $C_\varepsilon\subset\overline{\gamma}(\mathcal{P}(M))$ so that $\tilde{P}\left(\overline{\gamma}(\mathcal{P}(M))\setminus C_\varepsilon\right)<\varepsilon$.

Let us take:

$$T^\varepsilon:[0,1]\to\mathcal{P}(C_\varepsilon), T^\varepsilon\alpha=\begin{cases}\left\{\{C\}\in C_\varepsilon \middle| ord(C)=\alpha\right\}, & \alpha\neq 0\\ \left\{\{i\}_{i\in M}\right\} & ,\alpha=0\end{cases} \tag{289}$$

$$\tilde{R}^\varepsilon=\tilde{R}\Big|_{[0,1]\times C_\varepsilon}$$

$$R_1,R_1^\varepsilon:[0,1]\to\mathbb{R}, R_1(\alpha)=\inf_{i\in T\alpha}\tilde{R}(\alpha,i), R_1^\varepsilon(\alpha)=\inf_{i\in T^\varepsilon\alpha}\tilde{R}(\alpha,i) \tag{290}$$

$$R_2,R_2^\varepsilon:[0,1]\to\mathbb{R}, R_2(\alpha)=\sup_{i\in T\alpha}\tilde{R}(\alpha,i), R_2^\varepsilon(\alpha)=\sup_{i\in T^\varepsilon\alpha}\tilde{R}(\alpha,i) \tag{291}$$

It should be noted that functionals R_1, R_2 associate with every $\alpha\in[0,1]$ represent the inferior margin and the superior margin, respectively, of the α order excess corresponding to T.

A problem in connection with the actual realization of the coalitions in the sense of maximum probability is whether the decider taking part in the decision process know the interval within which the excess varies.

With a view to solving this problem, we shall formulate two theorems referring to the reaching of the margins of the excess.

Theorem 3.7

The following results take place:

1) If $\tilde{R}^\varepsilon$ is s.c.i., T^ε is closed and for any $\alpha\in[0,1]$, $T^\varepsilon\alpha$ is compact, then there is $\alpha_*\in[0,1]$ so that $R_1^\varepsilon(\alpha_*)=\min_{\alpha\in[0,1]}R_1^\varepsilon(\alpha)$.

2) If $\tilde{R}^\varepsilon$ is s.c.i., T^ε is s.c.s. and for any $\alpha\in[0,1]$, $T^\varepsilon\alpha$ is compact, then there is $\alpha_*\in[0,1]$ so that $R_1^\varepsilon(\alpha_*)=\min_{\alpha\in[0,1]}R_1^\varepsilon(\alpha)$.

Proof

To avoid ambiguity, we shall specify each time the case in which the semicontinuity is the one for univocal applications of the one for multivocal ones.

1) Let us take $(\alpha_n)_n\subset[0,1]$, $\lim_n\alpha_n=\alpha_0$.

From the fact that $\tilde{R}^\varepsilon$ is s.c.i. (as a univocal application) and $T^\varepsilon\alpha$ is compact for any $\alpha\in[0,1]$, it results that there is $i_n^0\in T\alpha$ so that:

$$\inf_{i_n\in T^\varepsilon\alpha_n}\tilde{R}^\varepsilon(\alpha_n,i_n)=\min_{i_n\in T^\varepsilon\alpha_n}\tilde{R}^\varepsilon(\alpha_n,i_n)=\tilde{R}^\varepsilon(\alpha_n,i_n^0) \tag{292}$$

As $R_1^\varepsilon(\alpha_n)=\inf_{i_n\in T^\varepsilon\alpha_n}\tilde{R}^\varepsilon(\alpha_n,i_n)$, the following equality results from (292):

$$R_1^\varepsilon(\alpha_n)=\tilde{R}^\varepsilon(\alpha_n,i_n^0) \tag{293}$$

If we consider the sequence $(i_n^0)_n$, $i_n^0\in T^\varepsilon\alpha_n$, as C_ε is compact, it results that the sequence $(i_n^0)_n$ will contain a highly convergent subsequence $(i_{n_k})_k$.

Let us take $i_0=\lim_k i_{n_k}$. (For convenience, we shall use the sequence $(i_n)_n$ instead of the subsequence $(i_{n_k})_k$).

We shall have:

$$\lim_n \alpha_n = \alpha_0, \lim_n i_n^0 = i_0, i_n^0 \in T^{\varepsilon}\alpha_n, \forall n \in \mathbb{N} \tag{294}$$

As T^{ε} is closed, it results on the basis of definition 3.5 and of (289) that $i_0 \in T^{\varepsilon}\alpha_0$.

$\tilde{R}^{\varepsilon}$ being s.c.i. (as a univocal application) we have:

$$\underline{\lim}_n \tilde{R}^{\varepsilon}\left(\alpha_n, i_n^0\right) \geq \tilde{R}^{\varepsilon}\left(\alpha_0, i_0\right) \geq \min_{i \in T^{\varepsilon}\alpha_0} \tilde{R}^{\varepsilon}\left(\alpha_0, i\right) = R_1^{\varepsilon}\left(\alpha_0\right) \tag{295}$$

From (293) and (295) the following inequality results directly:

$$\underline{\lim}_n R_1^{\varepsilon}\left(\alpha_n\right) \geq R_1^{\varepsilon}\left(\alpha_0\right)$$

which proves that R_1^{ε} is s.c.i. (as a univocal application) and, consequently, there is $\alpha_* \in [0,1]$ so that:

$$R_1^{\varepsilon}\left(\alpha_*\right) = \min_{\alpha \in [0,1]} R_1^{\varepsilon}\left(\alpha\right) \tag{296}$$

2) In order to demonstrate this property it is enough to show that R_1^{ε} is s.c.i. (as a univocal application). Let $\left(\alpha_n\right)_n \subset [0,1]$, $\lim_n \alpha_n = \alpha_0$.

Just as in the previous point, it can be shown that there is $i_n^0 \in T^{\varepsilon}\alpha_n$ so that:

$$R_1^{\varepsilon}\left(\alpha_n\right) = \min_{i_n \in T^{\varepsilon}\alpha_n} \tilde{R}^{\varepsilon}\left(\alpha_n, i_n\right) = \tilde{R}^{\varepsilon}\left(\alpha_n, i_n^0\right) \tag{297}$$

Considering the subsequence $\left(\alpha_{n_k}\right)_k \subset \left(\alpha_n\right)_n$ which has the properties [47]:

$$\underline{\lim}_n R_1^{\varepsilon}\left(\alpha_n\right) = \lim_k R_1^{\varepsilon}\left(\alpha_{n_k}\right), \lim_k i_{n_k}^0 = i_0 \tag{298}$$

$T^{\varepsilon}\alpha$, $\forall \alpha \in [0,1]$ being closed and T^{ε} s.c.s. it results directly that $i_0 \in T^{\varepsilon}\alpha_0$.

Consequently, the following equality occurs:

$$\tilde{R}^{\varepsilon}\left(\alpha_0, i_0\right) \geq R_1^{\varepsilon}\left(\alpha_0\right) \tag{299}$$

Taking into account (294) and the fact that $\tilde{R}^{\varepsilon}$ is s.c.i. (as a univocal application), we obtain:

$$\underline{\lim}_n R_1^{\varepsilon}\left(\alpha_n\right) = \lim_k \tilde{R}^{\varepsilon}\left(\alpha_{n_k}, i_{n_k}^0\right) \geq \tilde{R}^{\varepsilon}\left(\alpha_0, i_0\right) \geq \tilde{R}_1^{\varepsilon}\left(\alpha_0\right) \tag{300}$$

so that $R_2^{\varepsilon}\left(\alpha_*\right) = \min_{\alpha \in [0,1]} R_2\left(\alpha\right)$.

Corollary 3.7.1.

If T^{ε}, $\tilde{R}^{\varepsilon}$ are continuous, it can be show that R_1^{ε} is also continuous.

Demonstration is immediate.

Indeed, C_{ε} being compact, in order to demonstrate the continuity of R_1^{ε} it is sufficient to show that it is s.c.i. and s.c.s.

The inferior semicontinuity of R_1^{ε} results directly from theorem 3.7 (property 2) and the superior semicontinuity is immediate.

Remark 3.14

Generally, the conditions of inferior semicontinuity and of superior semicontinuity are not sufficient to ensure continuity in the case of multivocal applications. If X, Y are metric spaces, Y is also compact, then the multivocal application $T_1 : X \to \mathcal{P}(Y)$ is continuous only if it is s.c.i. and s.c.s.

Theorem 3.8

If for $\alpha \in [0,1]$, T_{α}^{ε} is compact, then the following results take place:

1) if T^{ε} is closed (as a multivocal application), $\tilde{R}^{\varepsilon}$ is s.c.s. (as a univocal application), then there is $\alpha^* \in [0,1]$ so that $R_2^{\varepsilon}\left(\alpha^*\right) = \max_{\alpha \in [0,1]} R_2^{\varepsilon}\left(\alpha\right)$;

2) if T^{ε} is continuous (as an multivocal application), $\tilde{R}^{\varepsilon}$ is continuous (as a univocal application), then there is $\alpha_* \in [0,1]$ so that $R_2^{\varepsilon}\left(\alpha_*\right) = \min_{\alpha \in [0,1]} R_2^{\varepsilon}\left(\alpha\right)$, $R_2^{\varepsilon}\left(\alpha^*\right) = \max_{\alpha \in [0,1]} R_2^{\varepsilon}\left(\alpha\right)$.

Proof

Let us show that R_2^{ε} is s.c.s. (as a univocal application).

Let us take $(\alpha_n)_n \subset [0,1]$, $\lim_n \alpha_n = \alpha_0$.

Considering the fact that $T^\varepsilon \alpha_n$ is compact, $\forall n \in \mathbb{N}$ there is $i_n^0 \in T^\varepsilon \alpha_n$, $n \in \mathbb{N}$ so that the following equalities take place:

$$R_2^\varepsilon(\alpha_n) = \sup_{i_n \in T^\varepsilon \alpha_n} \tilde{R}^\varepsilon(\alpha_n, i_n) = \max_{i_n \in T^\varepsilon \alpha_n} \tilde{R}^\varepsilon(\alpha_n, i_n^0) \tag{301}$$

Similarly to the previous theorem, from the compactness of C_ε and T^ε being closed (as a multivocal application) it results that there is $i_0 \in T^\varepsilon \alpha_0$, $\left(i_{n_k}^0\right)_k = \left(i_n^0\right)_n$ so that:

$$\lim_k i_{n_k}^0 = i_0$$

(For the sake of convenience, we shall use the sequence $\left(i_n^0\right)_n$ instead of the sequence $\left(i_{n_k}^0\right)_k$).

On the basis of the s.c.s. of R we shall have:

$$\overline{\lim_n} \tilde{R}^\varepsilon(\alpha_n, i_n) \le \tilde{R}^\varepsilon(\alpha_0, i_0) \le \max_{i \in T^\varepsilon \alpha_n} \tilde{R}^\varepsilon(\alpha_0, i) = R_2^\varepsilon(\alpha_0) \tag{302}$$

From (301) and (302) it results directly that:

$$\overline{\lim_n} R_2^\varepsilon(\alpha_n) \le R_2^\varepsilon(\alpha_0)$$

which means that R_2^ε is s.c.s. and consequently property 1 takes place.

3) In order to demonstrate property 2, we can either proceed as in the previous theorem (property 2) or we can demonstrate the continuity of R_2^ε directly (in fact we shall do so).

Let us take $(\alpha_n)_n \subset [0,1]$, $\lim_n \alpha_n = \alpha_0$.

From the fact that $T^\varepsilon \alpha_n$ is compact, $\forall n \in \mathbb{N}$ it results that there is $i_n^0 \in T^\varepsilon \alpha_n$ so that we shall have:

$$R_2^\varepsilon(\alpha_n) = \sup_{i_n \in T^\varepsilon \alpha_n} \tilde{R}^\varepsilon(\alpha_n, i_n) = \max_{i_n \in T^\varepsilon \alpha_n} \tilde{R}^\varepsilon(\alpha_n, i_n) = \tilde{R}^\varepsilon(\alpha_n, i_n^0) \tag{303}$$

T^ε being continuous, it means that it will be s.c.s. and closed (as a multivocal application) and consequently there is $i_0 \in T\alpha$, so that $\lim_n i_n^0 = i_0$.

Passing to the limit in (303) we obtain:

$$\lim_n R_2^\varepsilon(\alpha_n) = \lim_n \tilde{R}^\varepsilon(\alpha_n, i_n^0) = \tilde{R}^\varepsilon(\alpha_0, i_0) = \max_{i \in T^\varepsilon \alpha_0} \tilde{R}^\varepsilon(\alpha_0, i) = R_2^\varepsilon(\alpha_0)$$

and so R_2^ε is continuous, which demonstrates property 2.

Remark 3.15

Theorems 3.7 and 3.8 specify the conditions of existence of the excess margins. From a practical point of view, the deciders knowledge of the interval $\left[R_1^\varepsilon(\alpha_*), R_2^\varepsilon(\alpha^*)\right]$ is of an utmost importance as the possession of this information can influence them in their choice for the coalition in the sense of maximum probability.

Theorem 3.9

If for any $\alpha \in [0,1]$, the set $T\alpha$ contains only one coalition, then the following properties take place:

1) $\forall \alpha, \beta \in [0,1]$, $2\alpha \ge \beta \ge \alpha$, we have the inclusion $T\alpha \setminus T\beta \subset T(\alpha - \beta)$;

2) let us take $C \in \mathcal{P}(M)$; if there is $\alpha \in [0,1]$ so that $C = T\alpha$, T being continuous in the topology generated by Hausdorff's metric, and $\gamma(\mathcal{P}(M))$ is compact in this topology, then the following equalities take place:

$$\alpha = \inf\{\beta \in [0,1] \| C \subset T\beta\} = \sup\{\delta \in [0,1] \| T\delta \subseteq C\}$$

Proof

1) If $\alpha = \beta = 0$, from the definition of T it results that $T\alpha \setminus T\beta = T(\beta - \alpha) = \varnothing$. Suppose $\alpha, \beta \ne 0$. Let us take $i_0 \in T\alpha \setminus T\beta$. It results then that $i_0 \in T\alpha$, $i_0 \notin T\beta$ and then there is $P_{i_0} \in \mathbb{R}$, $P_{i_0} = P\left\{\bar{x} \in \bar{X} \middle| u_{i_0}(\bar{x}) \ge a_{i_0}\right\}$ so that the following inequalities are verified:

$$P_{i_0} \geq \alpha;\, P_{i_0} < \beta \tag{304}$$

Property 1) can be demonstrated trough reduction ad absurdum.

Supposing that $i_0 \notin T(\beta - \alpha)$, the following inequalities takes place:

$$P_{i_0} > \beta - \alpha \tag{305}$$

From (304) and (305) and the inequalities $2\alpha \geq \beta \geq \alpha$, we have:

$$3\alpha \geq \beta + \alpha \geq 2\alpha \geq P_{i_0} + \alpha > \beta + \alpha \tag{306}$$

From the inequalities (306) it results that $\beta \geq P_{i_0}$, but that contradicts inequality $P_{i_0} > \beta$ from (306) hence the supposition is false.

So, $i_0 \in T\beta$ and then $i_0 \in T(\beta - \alpha)$.

2) Let us take $\alpha_1 = inf\left\{\beta \in [0,1] \middle| C \subseteq T\beta\right\}$. That means that there is a sequence $(\beta_n)_n \subset [0,1]$ so that $\alpha_1 = \lim_n \beta_n$, $C \subseteq T_{\beta_n}$.

From the compactness of $\gamma(\mathcal{P}(M))$, it result that there is a subsequence $(\beta_{n_k})_k \subset (\beta_n)_n$ and $\beta_0 \in [0,1]$ so that:

$$T\beta_0 = \lim_k T\beta_{n_k}$$

T being continuous, it means that $T\beta_0 = T\left(\lim_k \beta_{n_k}\right)$, and $C \subseteq T\beta_0$ so $\alpha_1 = \beta_0 \geq \alpha$ and considering the way in which α_1 was defined it results directly that $\alpha_1 = \alpha$. Similarly, the following equality can be demonstrated:

$$\alpha = sup\left\{\delta \in [0,1] \middle| T\delta \subseteq C\right\}$$

3.2.3 The Case of Finite Coalitions

A basic result of this subchapter is the fact that in the case when compensation is allowed and the coalitions are finite, the concept of coalition in the sense of maximum probability is more general than the concept of coalition in the sense of characteristic function.

The other results refer to the necessary conditions of coalition in the sense of maximum probability, to the existence of non-ordinary coalitions as well as to certain properties of the mixed strategies of two deciders for which the problem of coalition is interpreted as a game problem.

The existence of non-ordinary coalitions will be demonstrated under the conditions of the generalization of the notion of imputation from the theory of cooperative games in the Neumann sense and of his immediate extensions.

3.2.3.1 The Properties of Coalitions Formed in the Maximum Probability Sense

From what has been shown so far ,obviously the underlying element in the realization of the coalition in the sense of maximum probability is represented by the fact that through the transfer of utilities, deciders increase their probability of realizing the end of the decisional process in one of the positions of the fixed target set.

Let us take coalition $C = \{i_1, i_2, \ldots, i_n\} \subset M$.

Theorem 3.10

If the following conditions are realized:

1) $P\left\{\bar{x} \in \bar{X} \middle| u_{i_j}(\bar{x}) \geq a_{i_j}\right\} \in (0,1),\, j = \overline{1,n}$;

2) for any $A, B \subset C$, the sets of the type:

a) $M_1 = \left\{\bar{x} \in \bar{X} \middle| \sum_{i\in A} u_i(\bar{x}) \geq \sum_{i\in A} a_i\right\}$, $M_1' = \left\{\bar{x} \in \bar{X} \middle| \sum_{i\in B} u_i(\bar{x}) \geq \sum_{i\in B} a_i\right\}$;

b) $M_2 = \left\{\bar{x} \in \bar{X} \middle| \sum_{i\in A} u_i(\bar{x}) \geq \sum_{i\in A} a_i\right\}$, $M_2' = \left\{\bar{x} \in \bar{X} \middle| \sum_{i\in B} u_i(\bar{x}) < \sum_{i\in B} a_i\right\}$,

$$M_2'' = \left\{ \overline{x} \in \overline{X} \,\middle|\, \sum_{i \in A \cup B} u_i\left(\overline{x}\right) \geq \sum_{i \in A \cup B} a_i \right\}$$

c) $M_3 = \left\{ \overline{x} \in \overline{X} \,\middle|\, \sum_{i \in A} u_i\left(\overline{x}\right) < \sum_{i \in A} a_i \right\}$, $M_3' = \left\{ \overline{x} \in \overline{X} \,\middle|\, \sum_{i \in B} u_i\left(\overline{x}\right) \geq \sum_{i \in B} a_i \right\}$,

$$M_3'' = \left\{ \overline{x} \in \overline{X} \,\middle|\, \sum_{i \in A \cup B} u_i\left(\overline{x}\right) \geq \sum_{i \in A \cup B} a_i \right\}$$

are independent in relation to P, then a necessary condition of coalition in the sense of maximum probability of the deciders in C is the realization of the condition:

$$P\left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_j}\left(\overline{x}\right) \geq a_{i_j} \right\} \geq \frac{1}{2},\ j = \overline{1,n}$$

Proof

In order to demonstrate this theorem, we shall start from the idea that the realization of the coalition C can be done through the successive coalition of the deciders: a first decider from C forms a coalition with another, then the newly formed coalition coalites with a third decider from C and so on until coalition C is formed. As $i_1, i_2 \in C$ we shall have:

$$P\left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_1}\left(\overline{x}\right) + u_{i_2}\left(\overline{x}\right) \geq a_{i_1} + a_{i_2} \right\} \geq max\left\{ P\left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_1}\left(\overline{x}\right) \geq a_{i_1} \right\}, P\left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_2}\left(\overline{x}\right) \geq a_{i_2} \right\} \right\}$$

Let us take the sets:

$$A = \left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_1}\left(\overline{x}\right) + u_{i_2}\left(\overline{x}\right) \geq a_{i_1} + a_{i_2} \right\}$$

$$A_1 = \left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_1}\left(\overline{x}\right) \geq a_{i_1} \right\} \cap \left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_2}\left(\overline{x}\right) \geq a_{i_2} \right\}$$

$$A_2 = \left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_1}\left(\overline{x}\right) < a_{i_1} \right\} \cap \left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_2}\left(\overline{x}\right) > a_{i_2} \right\} \cap \left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_1}\left(\overline{x}\right) + u_{i_2}\left(\overline{x}\right) \geq a_{i_1} + a_{i_2} \right\}$$

$$A_3 = \left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_1}\left(\overline{x}\right) > a_{i_1} \right\} \cap \left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_2}\left(\overline{x}\right) < a_{i_2} \right\} \cap \left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_1}\left(\overline{x}\right) + u_{i_2}\left(\overline{x}\right) \geq a_{i_1} + a_{i_2} \right\}$$

From the way in which the sets A, A_1, A_2, A_3 have been made, the following properties are verified:

$$A = A_1 \cup A_2 \cup A_3,\ A_i \cap A_j = \emptyset, i, j = \overline{1,3}, i \neq j \tag{307}$$

We shall write:

$$P_{i_1} = P\left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_1}\left(\overline{x}\right) \geq a_{i_1} \right\}$$

$$P'_{i_1} = P\left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_1}\left(\overline{x}\right) > a_{i_1} \right\}$$

$$P_{i_2} = P\left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_2}\left(\overline{x}\right) \geq a_{i_2} \right\}$$

$$P'_{i_2} = P\left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_2}\left(\overline{x}\right) > a_{i_2} \right\}$$

$$P_{1,2} = P\left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_1}\left(\overline{x}\right) + u_{i_2}\left(\overline{x}\right) \geq a_{i_1} + a_{i_2} \right\}$$

From (307) and from conditions 2 from the enunciation of the theorem we have:

$$P_{1,2} = P_{i_1} P_{i_2} + \left(1 - P_{i_1}\right) P'_{i_2} P_{1,2} + \left(1 - P_{i_2}\right) P'_{i_1} P_{1,2}$$

Hence we obtain:

$$P_{1,2} = \frac{P_{i_1} P_{i_2}}{1 - \left(1 - P_{i_1}\right) P'_{i_2} + \left(1 - P_{i_2}\right) P'_{i_1}} \tag{308}$$

As the coalition is realized in the sense of maximum probability we shall have:

$$P_{1,2} \geq max\left\{ P_{i_1}, P_{i_2} \right\}$$

From $P_{1,2} \geq P_{i_1}$, considering condition 1 from the enunciation of the theorem and the fact that $P_{i_1} \geq P'_{i_1}$, $P_{i_2} \geq P'_{i_2}$, it results that:

$$\begin{gathered}P_{i_1}P_{i_2} \ge P_{i_1}\left[1-\left(1-P_{i_1}\right)P'_{i_2}-\left(1-P_{i_2}\right)P'_{i_1}\right] \Rightarrow \\ \Rightarrow P_{i_2} \ge 1-\left(1-P_{i_2}\right)P'_{i_1}-\left(1-P_{i_1}\right)P'_{i_2} \ge 1-\left(1-P_{i_1}\right)P_{i_2}-\left(1-P_{i_2}\right)P_{i_1} = 1-P_{i_1}+2P_{i_1}P_{i_2}-P_{i_2} \Rightarrow \\ \Rightarrow 1-P_{i_1} \le 2P_{i_2}\left(1-P_{i_1}\right)\end{gathered} \tag{309}$$

From (309) we obtain $P_{i_2} \ge \frac{1}{2}$ (the simplification was possible as $P_{i_1} \in (0,1)$).

Similarly from condition $P_{1,2} \ge P_{i_2}$ it results that $P_{i_1} \ge \frac{1}{2}$.

As $i_3 \in C$, there is $P_{1,2,3} \in (0,1)$ so that:

$$\begin{gathered}P_{1,2,3} = P\left\{\bar{x} \in \bar{X} \,\middle|\, u_{i_1}(\bar{x})+u_{i_2}(\bar{x})+u_{i_3}(\bar{x}) \ge a_{i_1}+a_{i_2}+a_{i_2}\right\} \ge \\ \ge max\left\{P_{1,2}, \left\{\bar{x} \in \bar{X} \,\middle|\, u_{i_3}(\bar{x}) \ge a_{i_3}\right\}\right\} \ge max\left\{P_{i_1}, P_{i_2}, P_{i_3}\right\}\end{gathered} \tag{310}$$

where:

$$P_{i_3} = P\left\{\bar{x} \in \bar{X} \,\middle|\, u_{i_3}(\bar{x}) \ge a_{i_3}\right\}$$

Following the former procedure through which we have constructed the sets A, A_1, A_2, A_3, considering condition 1 from the enunciation of the theorem and (310), we shall be led (after a calculation similar to the previous one) to the inequalities:

$$1-P_{i_3} \le 2P_{i_2}\left(1-P_{i_3}\right) \Rightarrow P_{1,2} \ge \frac{1}{2}$$

$$1-P_{1,2} \le 2P_{i_3}\left(1-P_{1,2}\right) \Rightarrow P_{i_3} \ge \frac{1}{2}$$

Following this procedure, after $n-1$ stages we shall reach the following conclusions:

$$P_{1,2,\dots,n-1} = P\left\{\bar{x} \in \bar{X} \,\middle|\, \sum_{i \in C \setminus \{i_n\}} u_i(\bar{x}) \ge \sum_{i \in C \setminus \{i_n\}} a_i\right\} \ge \frac{1}{2}$$

$$P_{i_n} = P\left\{\bar{x} \in \bar{X} \,\middle|\, u_{i_n}(\bar{x}) \ge a_{i_n}\right\} \ge \frac{1}{2}$$

We have shown that:

$$P_{i_j} \ge \frac{1}{2},\ j=\overline{1,n},\ P\left\{\bar{x} \in \bar{X} \,\middle|\, \sum_{j \in C} u_j(\bar{x}) \ge \sum_{j \in C} a_j\right\} \ge \frac{1}{2} \tag{311}$$

Consequently, provided that the requirements 1 and 2 from the enunciation of the theorem are observed, deciders can form a coalition in the sense of maximum probability if $P_{i_j} \ge \frac{1}{2}$, $j=\overline{1,n}$.

Remark 3.16

From theorem 3.10 results that this concept of coalition advantages these deciders with relatively great chances of fulfillment of their proposed objectives.

Theorem 3.11

If the deciders of the $M=\{1,2,\dots,m\}$ set form a coalition in the sense of maximum probability and the following conditions are verified:

C$_1$) $P\left\{\bar{x} \in \bar{X} \,\middle|\, u_i(\bar{x}) \ge a_i\right\} = 1$ (312)

C$_2$) $P\left\{\bar{x} \in \bar{X} \,\middle|\, u_j(\bar{x}) \ge a_j\right\} \le 1, \forall j \in M \setminus \{i\}$ (313)

then coalition M can also be formed in the sense of the characteristic function, compensation being allowed.

Proof

1) If $P\left\{\bar{x} \in \bar{X} \,\middle|\, u_j(\bar{x}) \ge a_j\right\} = 1$, for any $j \in M$, then the application $v : \mathcal{P}(M) \to \mathbb{R}$, defined as:

$$\nu(S)=\begin{cases}\sum_{i\in S}a_i\,, & S\neq\varnothing\\ 0 & ,S=\varnothing\end{cases}$$

is a characteristic function.

Indeed, from the way in which application ν has been defined, the equality $\nu(\varnothing)=0$ takes place. As M is a coalition formed in the sense of maximum probability, on the basis of definition 3.2, the following equalities occur:

$$ord(M)=1, ord(M_1)=1, \forall M_1\subset M$$

Let us take $M_1,M_2\subset M$, $M_1\cap M_2=\varnothing$.

$$\nu(M_1\cup M_2)=\sum_{i\in M_1\cup M_2}a_i=\sum_{i\in M_1}a_i+\sum_{j\in M_2}a_j=\nu(M_1)+\nu(M_2) \tag{314}$$

From the equality $\nu(\varnothing)=0$ and (3.126) it results directly that ν is a characteristic function.

2) We assume that $P\left\{\overline{x}\in\overline{X}\,\middle|\,u_j(\overline{x})\geq a_j\right\}<1$, $\forall j\in M\setminus\{i\}$.

From $P_i=P\left\{\overline{x}\in\overline{X}\,\middle|\,u_i(\overline{x})\geq a_i\right\}=1$, it results directly that for any $M_1\subset M$ which contains i, we have:

$$P_{M_1}=P\left\{\overline{x}\in\overline{X}\,\middle|\,\sum_{i\in M_1}u_i(\overline{x})\geq\sum_{i\in M_1}a_i\right\}=1$$

Let us take the application $\nu:\mathcal{P}(M)\to\mathbb{R}$, given by:

$$\nu(S)=\begin{cases}min\left\{\sum_{i\in S}a_i, sup\left\{V\in\mathbb{R}\,\middle|\,P\left\{\overline{x}\in\overline{X}\,\middle|\,\sum_{i\in S}u_i(\overline{x})\geq V\right\}=1\right\}\right\}, & S\neq\varnothing\\ 0 & ,S=\varnothing\end{cases} \tag{315}$$

From the definition of ν, the property $\nu(\varnothing)=0$ results directly.

Let us take $M_1,M_2\subset M$, $M_1\cap M_2=\varnothing$.

If one of the M_1, M_2 subsets contains i (we assume that M_1 had this property) then we have:

$$\begin{aligned}\nu(M_1\cup M_2)&=\sum_{j\in M_1\cup M_2}a_j=\sum_{j\in M_1}a_j+\sum_{j\in M_2}a_j\geq\\ &\geq\sum_{j\in M_1}a_j+sup\left\{V\in\mathbb{R}\,\middle|\,P\left\{\overline{x}\in\overline{X}\,\middle|\,\sum_{j\in M_2}u_j(\overline{x})\geq V\right\}=1\right\}=\nu(M_1)+\nu(M_2)\end{aligned} \tag{316}$$

If none of the M_1 and M_2 sets contains i, considering the fact that the deciders from M form a coalition in the sense of probability, we shall have to analyze the following situations:

1) $i\notin M_1$, $i\notin M_2$, the $M_1\cup M_2$ coalition will reach its target set with the probability:

$$P=\left\{\overline{x}\in\overline{X}\,\middle|\,\sum_{i\in M_1\cup M_2}u_i(\overline{x})\geq\sum_{i\in M_1\cup M_2}a_i\right\}=1$$

In this case, we shall have:

$$\nu(M_1)=V_1=sup\left\{V\in\mathbb{R}\,\middle|\,P\left\{\overline{x}\in\overline{X}\,\middle|\,\sum_{i\in M_1}u_i(\overline{x})\geq V\right\}=1\right\} \tag{317}$$

$$\nu(M_2)=V_2=sup\left\{V\in\mathbb{R}\,\middle|\,P\left\{\overline{x}\in\overline{X}\,\middle|\,\sum_{i\in M_2}u_i(\overline{x})\geq V\right\}=1\right\} \tag{318}$$

$$\nu(M_1\cup M_2)=a_1+a_2\geq V_1+V_2=\nu(M_1)+\nu(M_2) \tag{319}$$

2) $i\notin M_1$, $i\notin M_2$, the $M_1\cup M_2$ coalition can reach its target set with the probability:

$$P = \left\{ \overline{x} \in \overline{X} \;\middle|\; \sum_{i \in M_1 \cup M_2} u_i(\overline{x}) \geq \sum_{i \in M_1 \cup M_2} a_i \right\} < 1$$

In this case, we shall have:

$$\nu(M_1) = V_1$$
$$\nu(M_2) = V_2$$
$$\nu(M_1 \cup M_2) = V_{1,2} = \sup\left\{ V \in \mathbb{R} \;\middle|\; P\left\{ \overline{x} \in \overline{X} \,\middle|\, u_i(\overline{x}) \geq V \right\} = 1 \right\} \tag{320}$$

From $V_{1,2} = V_1 + V_2$ the following equality results:

$$\nu(M_1 \cup M_2) = \nu(M_1) + \nu(M_2) \tag{321}$$

Therefore, ν is a characteristic function.

The case where except decider i, there are also other deciders which fulfill condition C_2), through equality (but not all), is treated similarly.

Definition 3.6

It is called imputation of order α ($\alpha \in (0,1)$) any element $z \in \mathbb{R}^n$ which verifies the condition: if $C \in T\alpha$, $C = \{i_1, i_2, ..., i_n\}$ then $z_{i_k} > a_{i_k}$, $k = \overline{1,n}$ (we have supposed that for any $\alpha \in (0,1)$ and $C \in T\alpha$, $card\, C = n$ if C is not ordinary or total).

We shall call imputation of order 1, the imputation introduced in §1.2.3.

Let Z_α be the set of imputations of order α, associated with the set of coalitions of order α, $\alpha \in (0,1)$; we assume the M set of being finite.

Application $e_\alpha : T\alpha \times Z_\alpha \to \mathbb{R}$, given by:

$$e_\alpha(C,z) = \begin{cases} \alpha \sum_{k=1}^{n} a_{i_k} - \sum_{k=1}^{n} \alpha_{i_k} z_{i_k}, \alpha \in (0,1) \\ -\nu(C) + \sum_{k=1}^{n} z_{i_k} \qquad , \alpha = 1 \end{cases} \tag{322}$$

is called excess of order α associated with $T\alpha$ (we have assumed $C = \{i_1, i_2, ..., i_n\}$, $z = \left(z_{i_1}, z_{i_2}, ..., z_{i_n}\right)$, $\alpha_{i_k} = P\left\{ \overline{x} \in \overline{X} \,\middle|\, u_{i_k}(\overline{x}) \geq a_{i_k} \right\}$, $k = \overline{1,n}$).

Theorem 3.12.

If there is $z^* \in Z_\alpha$ so, that the following condition is realized:

$$\max_{z \in Z_\alpha} \min_{C \in T\alpha} e_\alpha(C,z) = \min_{C \in T\alpha} e_\alpha(C,z^*) \tag{323}$$

then there are no ordinary coalitions in T.

Proof

If $\mathcal{C} = \{C_1, C_2, ..., C_k\}$ is the set of coalitions in $T\alpha$ which realizes the condition:

$$\min_{C \in T\alpha} e_\alpha(C,z) = e_\alpha(C_i, z), i = \overline{1,k}$$

then condition (323) can be written under the equivalent form [25]:

$$\max_{C \in T\alpha} \min_{C \in \mathcal{C}} \left(\sum_{i \in C} \left(z_i^* - z_i \right) \right) = 0 \tag{324}$$

We assume $\alpha \neq 1$.

We first show that $\mathcal{C}$ contains no ordinary coalitions.

Indeed, if there were $j \in M$ so that $\{j\} \in \mathcal{C}$, then there would always be a $z \in Z_\alpha$ imputation so that:

$$e_\alpha(j,z) > e_\alpha(j,z^*) \tag{325}$$

Taking into account the way in which the e_α functional from (325) has been defined, we obtain:

$$\alpha a_j - \alpha_j z_j > \alpha a_j - \alpha_j z_j^* \Rightarrow z_j^* - z_j > 0$$

and so condition (324) is not verified.

That means that in $\mathcal{C}$ there are no ordinary coalitions.

We show, by reduction ad absurdum, that there are no ordinary coalitions in $T\alpha \setminus \mathcal{C}$ either.

If there were $j \in T\alpha$, $j \notin C_i$, $i = \overline{1,k}$, it would mean that for any $C \in \mathcal{C}$ (we assume $C = \{i_1, i_2, ..., i_n\}$), $C \cup \{j\} \in \mathcal{C}$ and consequently:

$$e_\alpha\left(C \cup \{j\}, z^*\right) > e_\alpha\left(C, z^*\right) \tag{326}$$

Performing the calculation in (326), we shall have:

$$\alpha\sum_{k=1}^{n} a_{i_k} + \alpha a_j - \sum_{k=1}^{n} \alpha_{i_k} z_{i_k} - \alpha_j z_j > \alpha\sum_{k=1}^{n} a_{i_k} - \sum_{k=1}^{n} \alpha_{i_k} z_{i_k}^* \tag{237}$$

From (327), we obtain the inequality $\alpha a_j > z_j^* \alpha_j$ and so $a_j > z_j^*$ which is impossible, z_j^*, being an imputation of α order.

That means that decider j belongs to a coalition and consequently there are no ordinary coalitions in $T\alpha$.

We assume $\alpha = 1$.

We show by reduction ad absurdum, that in $T\alpha$ there are no ordinary coalitions.

If $\{j\} \in T\alpha$, $j \notin C_i$, $i = \overline{1,k}$, then $e_\alpha\left(C \cup \{j\}, z^*\right) > e_\alpha\left(C, z^*\right)$, $\forall C \in \mathcal{C}$, $C = \{i_1, i_2, ..., i_n\}$ and so:

$$-v\left(C \cup \{j\}\right) + \sum_{k=1}^{n} z_{i_k}^* + z_{j_k}^* > v(C) + \sum_{k=1}^{n} z_{i_k}^* \tag{328}$$

From (328) we have the inequality:

$$v\left(C \cup \{j\}\right) < v(C) + v\left(\{j\}\right)$$

and therefore, the property of superadditivity of v is not verified. So, in $T\alpha \setminus \mathcal{C}$ there are no ordinary coalitions.

If $\{j\} \in T\alpha$ existed, then $z \in Z_\alpha$ would exist so that $z_j^* - v\left(\{j\}\right) > z_j - v\left(\{j\}\right)$, so $z_j^* > z_j$ and consequently condition (328) is not verified.

That means that for $\alpha = 1$ there are no ordinary coalitions in $T\alpha$, either.

Corollary 3.12.1.

If the e_α functional has saddle points on $T\alpha \times Z_\alpha$, then there are no ordinary coalitions in $T\alpha$.

3.2.3.2 Choosing the Probability of Coalization

An important problem in forming coalitions in the sense of maximum probability is that of settling the ceiling for each decider. Even if the transfer of payments is allowed, for high ceilings chosen by decision makers, it is possible for the intersection of the target sets to be void and so, the coalition cannot be realized.

We make the following hypothesis: if coalition $C = \{1, 2, ..., n\}$ is realized in the sense of the maximum probability, its formation is realized through the successive coalition of the deciders (a first decider forms a coalition with another, then this coalition will form a new one with another decider and so on until coalition C is formed). Just as in the case of non-cooperative games, we shall interpret each intermediary coalition as a decider and, for this reason, we can interpret the problem of the formation of a partial coalition as a game problem with two deciders $J = \left([0,a],[0,b],F\right)$ with the pay function:

$$F(x,y) = \begin{cases} L(x,y) \ , 0 \le x \le y \le 1 \\ M(x,y), 0 \le y \le x \le 1 \end{cases} \tag{329}$$

where strategy (x,y) represents the pairs of probabilities (of realization of the target sets chosen by two deciders) on the basis of which the formation of an intermediary coalition is attempted.

We assume that $L(x,x) < M(x,x)$, $\forall x \in [0,1]$, L, M are continuously increasing in respect with y, decreasing in respect with x, Lipschitzian of C constant and that there exists $H : [0,1] \to \mathbb{R}$ so that $H(0) = F(0,0)$, $H(1) = F(1,1)$. [25]

Let $n \in \mathbb{N}$ be sufficiently great and $\Delta_n = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, +1\right\}$ partition of the segment $[0,1]$ built with the help of n.

We also make the hypothesis that the strategies of the two deciders will be elements of Δ_n (even if the two deciders choose a finite set of strategies whose elements do not belong to Δ_n, an n^* can be chosen and Δ_{n^*} can be built accordingly, so that the sets of strategies of the two deciders should be included in Δ_{n^*}).

Let us consider the discrete game where the utility function is $\overline{F}(i,j) = F\left(\frac{i-1}{n}, \frac{j-1}{n}\right)$, $i,j = \overline{1,n+1}$

On the basis of the properties of L and M , this game will have equilibrium poits in mixed strategies.

Let $p = (p_1, p_2, ..., p_n)$, $q = (q_1, q_2, ..., q_n)$ be any point of equilibrium.

Theorem 3.13 [16], [25]

If $A \in \mathbb{R}$ and $x_1 \in [0,1]$, the following inequality takes place $M(x,x) - L(x,x) \geq A$ for any $x \in [0, x_1]$, and the inequalities form below are verified:

$$p_i \leq \frac{C}{n} \frac{1}{A - \frac{C}{n}}, i = 3,4,...,i_0$$

where $i_0 = \begin{cases} [x_1, n] + 1, if\ x_1 < 1 \\ n \qquad ,if\ x_1 = 1 \end{cases}$ ($[a]$ represents the whole part of $a \in \mathbb{R}$).

Theorem 3.14 [16], [25]

If $A \in \mathbb{R}$ and $y_1 \in [0,1]$ and the inequality is verified: $M(y,y) - L(y,y) \geq A$, $\forall y \in [0, y_1]$, then $q_j < \frac{C}{n} \frac{1}{A - \frac{C}{n}}$, $j = \overline{3, j_0}$, $q_2 \leq \frac{4C}{An}$, where $j_0 = \begin{cases} [n, y_1] + 1, if\ y_1 < 1 \\ n \qquad ,if\ y_1 = 1 \end{cases}$.

Remark 3.17

Theorems 3.13 and 3.14 show in fact that for a sufficiently great n in the mixed strategies of the two deciders, only the first two components and the components of indices higher than i_0 and j_0 respectively are significant. In other words, any optimum strategy of the first decider has $n - i_0 + 2$ significant components and any optimum strategy for the second decider has $n = j_0 + 2$ significant components or $n - j_0 + 1$ significant components.

From a practical point of view, these significant components of the point of equilibrium (p,q) imply the use of either the very small simple strategies (which usually have high ceilings), or of the very great simple strategies (which usually have low ceilings).

Remark 3.18

We have the following starting elements:

- $a_1, a_2, \ldots, a_n$ represent the ceilings associated with the decision makers that take part in the decisional process
- $p_1, p_2, \ldots, p_n$ refer to the probabilities of achieving these limits, namely $p_i = P\left\{\overline{x} \in \overline{X} : u_i\left(\overline{x}\right) \geq a_i\right\}$

According to the coalition in the context of maximum probability it is obvious that $p_i \geq \frac{1}{2}$ (i.e. this coalition criterion applies only in situations when all decision makers stand good chances of getting the proposed minimum gain).

It is important to know which are the decision makers that decide, in the end, the coalition forming.

Since this criterion allows the transfer of utility, information on probabilities $p_1, p_2, \ldots, p_n$ can be obtained from the condition that the weighted sums of the modules of deviations of gains for the decision-makers in relation to the gain of each decision maker should be a fixed constant.

Denoting this constant with k , the conditions are expressed algebraically as follows:

$A \cdot P = K$ where the matrices A,P and K are the following:

$$A = \begin{pmatrix} |a_1 - a_1||a_1 - a_2|\ldots|a_1 - a_n| \\ |a_2 - a_1||a_2 - a_2|\ldots|a_2 - a_n| \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ |a_n - a_1||a_n - a_2|\ldots|a_n - a_n| \end{pmatrix}$$

$$P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \quad , \quad K = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

It is obvious that we have a matrix equation equivalent to the following linear algebraic system:

$$\begin{cases} 0 \cdot p_1 + |a_1 - a_2|\, p_2 + \ldots + |a_1 - a_n|\, p_n = k \\ |a_2 - a_1|\, p_1 + 0 \cdot p_2 + \ldots + |a_2 - a_n|\, p_n = k \\ \vdots \\ |a_n - a_1|\, p_1 + |a_n - a_2|\, p_2 + \ldots + |a_n - a_n|\, p_n = k \end{cases} \tag{330}$$

For the sake of calculations easiness (and without affecting the final result) we suggest $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n$

Under the circumstances, the algebraic system from above becomes:

$$\begin{cases} 0 \cdot p_1 + (a_1 - a_2)\, p_2 + \ldots + (a_1 - a_n)\, p_n = k \\ (a_1 - a_2)\, p_1 + 0 \cdot p_2 + \ldots + (a_2 - a_n)\, p_n = k \\ \vdots \\ (a_1 - a_n)\, p_1 + (a_2 - a_n)\, p_2 + \ldots + 0 \cdot p_n = k \end{cases} \tag{331}$$

The solution of this system is the following:

$$p_1 = p_n = \frac{k}{a_1 - a_n} \quad , \qquad p_2 = p_3 = \cdots = p_{n-1} = 0$$

Therefore, only two decision makers are important in making the coalition, the probabilities of achieving their objectives are equal. Practically only the decision maker with the higher ceiling and the one with the lowest ceiling are involved in forming the coalition. In other words, only the decision maker with the best possible chances and the decision maker with the poorest chances are actually making the coalition, as a first step. During the next step, if the coalition is made by other successive coalitions, decision makers 1 and *n* shall be taken into account (they are already considered to be acting as a partial coalition) and they shall interact with one of the decision makers 2,3,…,n-1. This process continues until the entire coalition is completed.

Remark 3.19

If the matrix K takes the form $\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$, then the system (x) becomes:

$$\begin{cases} 0 \cdot p_1 + (a_1 - a_2)\, p_2 + \ldots + (a_1 - a_n)\, p_n = k_1 \\ (a_1 - a_2)\, p_1 + 0 \cdot p_2 + \ldots + (a_2 - a_n)\, p_n = k_2 \\ \vdots \\ (a_1 - a_n)\, p_1 + (a_2 - a_n)\, p_2 + \ldots + 0 \cdot p_n = k_n \end{cases} \tag{332}$$

This system is extremely difficult to solve.

If $n = 3$, the solution of the system:

$$\begin{cases} 0 \cdot p_1 + (a_1 - a_2) p_2 + (a_1 - a_3) p_3 = k_1 \\ (a_1 - a_2) p_1 + 0 \cdot p_2 + (a_2 - a_3) p_3 = k_2 \\ (a_1 - a_3) p_1 + (a_2 - a_3) p_2 + 0 \cdot p_3 = k_3 \end{cases} \quad (333)$$

is the following:

$$\begin{cases} x_1 = \frac{1}{\Delta}\left(k_3 (a_1 - a_2) + k_2 (a_1 - a_3) - k_1 (a_2 - a_3)\right)(a_2 - a_3) \\ x_2 = \frac{1}{\Delta}\left(k_1 (a_2 - a_3) + k_3 (a_1 - a_2) - k_2 (a_1 - a_3)\right)(a_1 - a_3) \\ x_3 = \frac{1}{\Delta}\left(k_2 (a_1 - a_3) + k_1 (a_2 - a_3) - k_3 (a_1 - a_3)\right)(a_1 - a_3) \end{cases} \quad (334)$$

where

$$\Delta = 2(a_1 - a_2)(a_1 - a_3)(a_2 - a_3) \quad (335)$$

In the particular case $k_1 = k_2 = k_3$, if we use the notation k for their common value, then

$$x_1 = x_3 = \frac{k}{a_1 - a_3}\ ,\ x_2 = 0 \quad (336)$$

Remark 3.20

From behavioral point of view, if one of the decision makers knows that reaching the target set is done with a certain probability between the lowest and the highest probability of the participating decision makers, it is preferable to have a waiting policy. Practically he will wait until the decisional process will lead to a situation where his chance of reaching the target set is the best or the poorest.

Remark 3.21

The problem gets complicated if the ceilings $a_1, a_2, \ldots, a_n$ are not fixed and therefore the target sets $\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n$ vary during the decisional process.

3.2.3.3 The Entropic Solution of a Cooperative Game

The cooperative games occupy a special place in the conflictual model theory, by their very content; determining the solution of a cooperative game is, in the last analysis, a distribution problem, whose solution must be optimal for all decision-makers. Several solution (Shapely value, negotiation set, stable sets a.s.c.) are remarked in the games theory, as the notion of optimum is linked to the use of an optimality criterion; each of the above mentioned solutions presents obvious drawbacks, connected either to the lack of generality of the obtained solution, or to the very conceptual framework of approaching the problem.

We present further on a concept of solving cooperative games, which leads finally to a non-linear programming problem.

Let us suppose that coalition C is formed n decision-makers: quantities V_i and a_i are attached to each decision-maker i, $i = \overline{1,n}$ and their significances are as follows:

- V_i – maximin utility, i.e. the certain gain value
- a_i – coalition utility.

There are following possibilities, in this situation, if we denote by X_i the total gain of decision-maker i:

- $X_i \le a_i$, if the total gain $X \le \sum_{i=1}^{n} a_i$, $i = \overline{1,n}$ (fig.9)
- $X_i > a_i$, if the total gain $X > \sum_{i=1}^{n} a_i$, $i = \overline{1,n}$. (fig.10)

We consider the non-determination function – Watanabe's inaccuracy:

$$F(x,\alpha) = -x^{\alpha} \ln x - (1-x)^{\alpha} \ln(1-x)^{\alpha} + \frac{1-\alpha}{\alpha}\left[x^{\alpha} + (1-x)^{\alpha} - 1\right], \alpha \in (0,1) \quad (337)$$

Definition 3.6

The solution of the cooperative game is the vector $\left(X_1, X_2, \ldots, X_n\right)$ which minimizes the total non-determination:

$$\tilde{F}\left(x_1, x_2, \ldots, x_n, \alpha\right) = \sum_{i=1}^{n} F\left(x_i, \alpha\right) \tag{338}$$

where $x_1, x_2, \ldots, x_n$ are built according to the procedure indicated in [47]:

$$x_i = \frac{X_i - V_i}{a_i - V_i}, i = \overline{1,n} \tag{339}$$

Remark 3.22

The significance of the above introduced solution is connected to the fact that, as the decision-makers make up a coalition with a certain organization degree W, it is rational to distribute to each decision-maker a gain such that the deviations from the distributed gains and the theoretical ones give a minimal total inaccuracy, a fact which can be achieved with the help of function $\tilde{F}$.

On the other hand, as we shall see, function $\tilde{F}$ can be well approximated through function:

$$F^*\left(x_1, x_2, \ldots, x_n, \alpha\right) = \frac{1 + 3n\alpha - 3\alpha}{\alpha} \sum_{i=1}^{n} H_i + \frac{n}{\alpha} \tag{340}$$

where H_i is the gain non-determination for the player i, $i = \overline{1,n}$.

Proposition 3.2

a) The approximate solution of the cooperative game, in the case $X \le \sum_{i=1}^{n} a_i$, is given by:

$$X_i = x_i^0\left(a_i - V_i\right) + V_i, i = \overline{1,n}$$

where $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$ is the solution of the problem:

$$\min F^*\left(x_1, x_2, \ldots, x_n, \alpha\right)$$
$$\begin{cases} \sum_{i=1}^{n} C_i x_i = Y \\ x_i \ge 0, i = \overline{1,n} \end{cases} \tag{341}$$

b) The approximate solution of the cooperative game, in the case $X > \sum_{i=1}^{n} a_i$, is given by:

$$X_i = \frac{a_i - V_i}{x_i^0} + V_i, i = \overline{1,n} \tag{342}$$

where $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$ is the solution of the problem:

$$\min F^*\left(x_1, x_2, \ldots, x_n, \alpha\right)$$
$$\begin{cases} \sum_{i=1}^{n} \frac{C_i}{x_i} = Y \\ x_i > 0, i = \overline{1,n} \end{cases} \tag{343}$$

We denoted by $C_i = a_i - V_i$, $Y = X - \sum_{i=1}^{n} V_i$.

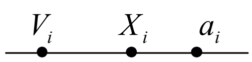

Figure 9

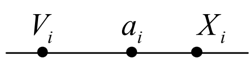

Figure 10

Proof

Case **a)** corresponds to the distribution in fig.9.

Denoting by $x_i = \dfrac{X_i - V_i}{a_i - V_i} \Rightarrow x_i\left(a_i - V_i\right) = X_i - V_i$ as $a_i - V_i = C_i \Rightarrow \sum_{i=1}^{n} C_i x_i = X - \sum_{i=1}^{n} V_i$ and as $X - \sum_{i=1}^{n} V_i = Y \Rightarrow \sum_{i=1}^{n} C_i x_i = Y$, the constraints of problem (341) result immediately.

Case **b)** corresponds to the distribution in Fig.10.

Denoting $x_i = \dfrac{a_i - V_i}{X_i - V_i} \Rightarrow X_i - V_i = \dfrac{a_i - V_i}{x_i} \Rightarrow Y = \sum_{i=1}^{n} \dfrac{C_i}{x_i}$, the constraints of problem (2), result immediately.

Remark 3.23

Case a) corresponds to the situation when the decision-makers of the coalition obtain smaller gains than the proposed ones, and case b) corresponds to the situation when the decision-makers obtain higher gains than the proposed ones. These two situation are unique and they result by comparing quantities X and $\sum_{i=1}^{n} a_i$.

Both cases lead to a non-linear programming problem, which easily to solve in comparison with the above mentioned difficulties, associated to the solution of a cooperative game.
Let us prove the following proposition, before proving that F^* is a good approximation of $\tilde{F}$.

Proposition 3.3

$$F\left(x,\alpha\right) = \frac{1-\alpha}{\alpha} e^{H(1-\alpha)} - \frac{\partial}{\partial\alpha} e^{H(1-\alpha)} - \frac{1-\alpha}{\alpha}$$

H being the entropy in Renyi's sense:

$$H\left(x,\alpha\right) = \begin{cases} \dfrac{1}{1-\alpha} ln\left[x^\alpha + \left(1-x\right)^\alpha\right] & ,\alpha \neq 1 \\ -x\,ln\,x - \left(1-x\right)ln\left(1-x\right), \alpha = 1 \end{cases} \tag{344}$$

Proof

$$H'_\alpha = \frac{1}{\left(1-\alpha\right)^2} ln\left[x^\alpha + \left(1-x\right)^\alpha\right] + \frac{1}{1-\alpha}\left[\frac{x^\alpha\, ln\,x + \left(1-x\right)^\alpha ln\left(1-x\right)}{x^\alpha + \left(1-x\right)^\alpha}\right] =$$
$$= \frac{1}{1-\alpha}\left[\frac{x^\alpha\, ln\left(1-x\right)^\alpha ln\left(1-x\right)}{x + \left(1-x\right)}\right] + H$$

$$\begin{aligned} F\left(x,\alpha\right) &= \frac{1-\alpha}{\alpha} e^{H(1-\alpha)} + He^{H(1-\alpha)} - H'_\alpha\left(1-\alpha\right)e^{H(1-\alpha)} - \frac{1-\alpha}{\alpha} = \\ &= \frac{1-\alpha}{\alpha} e^{H(1-\alpha)} - \frac{\partial}{\partial\alpha} e^{H(1-\alpha)} - \frac{1-\alpha}{\alpha} \end{aligned} \tag{345}$$

quod erat demonstrandum.

We shall denote now by H_i the gain entropy of decision-maker i (it is obvious that H_i shall be Rênyi's entropy).

$$\tilde{F}\left(x_1,x_2,...,x_n,\alpha\right) = \sum_{i=1}^{n} F\left(x_i,\alpha\right) = \frac{1-\alpha}{\alpha}\sum_{i=1}^{n} e^{H_i(1-\alpha)} - \frac{\partial}{\partial\alpha}\sum_{i=1}^{n} e^{H_i(1-\alpha)} - \sum_{i=1}^{n}\frac{1-\alpha}{\alpha} \tag{346}$$

(after an immediate calculation).

Remark 3.24

We shall consider further down $\alpha < \dfrac{1}{2}$, from the considerations exposed in [48], and we shall neglect the products between overunitary powers of α and H_i, as well as the overunitary powers of H_i, $i = \overline{1,n}$. We shall have then:

$$\sum_{i=1}^{n} e^{H_i(1-\alpha)} \sum_{i=1}^{n} \frac{e^{H_i}}{e^{\alpha H_i}} = \sum_{i=1}^{n} \frac{1+\frac{H_i}{1!}+\frac{H_i^2}{2!}+\ldots}{1+\frac{\alpha H_i}{1!}+\frac{\alpha^2 H_i^2}{2!}+\ldots} \tag{347}$$

According to remark 3.24, relation (347) can be approximated through:

$$\sum_{i=1}^{n} \frac{1+H_i}{1+\alpha H_i}$$

$$\sum_{i=1}^{n} e^{H_i(1-\alpha)} \approx \sum_{i=1}^{n} \frac{n+\sum_{i=1}^{n} H_i + (n-1)\alpha \sum_{i=1}^{n} H_i}{1+\alpha \sum_{i=1}^{n} H_i} \tag{348}$$

We take into account (348) and the remark 3.24 for determining $\frac{\partial}{\partial \alpha} \sum_{i=1}^{n} e^{H_i(1-\alpha)}$.

Finally, we obtain:

$$\frac{\partial}{\partial \alpha} \sum_{i=1}^{n} e^{H_i(1-\alpha)} \approx \frac{\partial}{\partial \alpha} \frac{n+\sum_{i=1}^{n} H_i + (n-1)\alpha \sum_{i=1}^{n} H_i}{1+\alpha \sum_{i=1}^{n} H_i} = \frac{(1-\alpha)\left(\sum_{i=1}^{n} H_i\right)'_{\alpha} - \sum_{i=1}^{n} H_i - \left(\sum_{i=1}^{n} H_i\right)^2}{1+2\alpha \sum_{i=1}^{n} H_i}$$

$F^*(x_1, x_2, \ldots, x_n, \alpha)$ can be, at last, determined, by neglecting all terms wherein there appear products between the powers of α and the subunitary quantities $\sum_{i=1}^{n} H_i$, $\left(\sum_{i=1}^{n} H_i\right)'_{\alpha}$ (excepting those products which contain, besides the mentioned factors, also the powers of n). We obtain finally, after an immediate calculation:

$$F^*(x_1, x_2, \ldots, x_n, \alpha) = \frac{(1+3n\alpha-3\alpha)\sum_{i=1}^{n} H_i}{\alpha} - \frac{n}{\alpha}$$

We can draw immediately the conclusion from the form of F^*, that minimizing F^* is equivalent to minimizing the sum of the non-determinations of the gain of the n players; in other words expressed, $(x_1^0, x_2^0, \ldots, x_n^0)$ is the solution of the given problem in the context of the constraints of cases a) or b), and so we have:

$$\sum_{i=1}^{n} H(x_i^0, \alpha) = min \sum_{i=1}^{n} H(x_i, \alpha) \tag{349}$$

Remark 3.25

Taking into account that the organization degree in Watanabe's sense is $W = \sum_{i=1}^{n} H_i - \mathcal{H}$, $\mathcal{H}$ being the non-determination of the aggregate formed by the n decision-makers, in the case when we know the organization degrees of the system formed by n gamblers, and the optimal distribution determining problem becomes:

Case a)	Case b)
$min\left[\mathcal{H}(x_1, x_2, \ldots, x_n, \alpha) + W\right]$	$min\left[\mathcal{H}(x_1, x_2, \ldots, x_n, \alpha) + W\right]$
$\begin{cases} \sum_{i=1}^{n} C_i x_i = Y \\ x_i \geq 0, i = \overline{1,n} \end{cases}$	$\begin{cases} \sum_{i=1}^{n} \frac{C_i}{x_i} = Y \\ x_i > 0, i = \overline{1,n} \end{cases}$

We are also led to two non-linear programming problems, whose numerical solving can be made with the help of known methods which, in their turn, can be transposed into an algorithm; this algorithm can be easily programmed by using the computer.

One of the most difficult problems in connection with the use of this concept of distribution of individual gains is the specification of the α parameter. From the calculation easiness point of view, the most convenient choice is $\alpha = 1$, but the choice of this parameter must take into account the real framework of the problem under consideration.

Application

Three banking units from different countries cooperate in realizing a common target with the following weights: 0.25; 0.35; 0.40 of the whole investment.

Initial calculations show the possibility that the three banking units should obtain a guaranteed profit of 50 million monetary units, the individual contributions being of 10, 14, 16 million monetary units.

Unfavorable and unexpected circumstances in the international situation have led to a total profit of 42 million monetary units. How much will get each banking unit?

Solution

The data of the problem in the context of the given model are the following:

$$V_1 = 10, V_2 = 14, V_3 = 16$$
$$a_1 = 12, a_2 = 18, a_3 = 20$$
$$x_1 = \frac{X_1 - 10}{2}, x_2 = \frac{X_2 - 14}{4}, x_3 = \frac{X_3 - 16}{4}$$

From $\alpha = 1$, we are led to the following non-linear programming problem:

$$\min\left(-\sum_{i=1}^{3}\left(x_i \ln x_i + (1 - x_i)\ln(1 - x_i)\right)\right)$$
$$\begin{cases} \sum_{i=1}^{3} C_i x_i = B \\ x_i \geq 0, i = \overline{1,3} \end{cases} \tag{350}$$

where:

$$C_1 = a_1 - V_1 = 2, C_2 = a_2 - V_2 = 4, C_3 = a_3 - V_3 = 4$$
$$B = 42 - \sum_{i=1}^{3} V_i = 2$$

Problem (350) becomes:

$$\min\left(-\sum_{i=1}^{3}\left(x_i \ln x_i + (1 - x_i)\ln(1 - x_i)\right)\right)$$
$$\begin{cases} x_1 + 2x_2 + 2x_3 = 1 \\ x_1, x_2, x_3 \geq 0 \end{cases} \tag{351}$$

The solution of the problem (351) is:

$$x_1 = 0{,}12\ ;\ x_2 = x_3 = 0{,}22$$

Accordingly, the profits of each banking unit will be:

$$x_1 = 10{,}14\ ;\ x_2 = 14{,}88\ ;\ x_3 = 16{,}88$$

It should be noted that if it had been admitted as cooperation conditions the repartition of profits, above the investments made, proportional to the contribution of each banking unit, the solution of this problem would have been:

$$x_1^* = 10{,}5\ ;\ x_2^* = 14{,}7\ ;\ x_3^* = 16{,}8$$

By comparison with this solution which favors considerably the first and the last banking unit, the solution (x_1, x_2, x_3) favors the participation of the second banking unit.

Let us also calculate the Shapley solution [77] (X_1^*, X_2^*, X_3^*).

$$X_1^* = 10 + \overline{X}_1,\ X_2^* = 14 + \overline{X}_2,\ X_3^* = 16 + \overline{X}_3 \tag{352}$$

where:

$$\overline{X}_i = \sum_{\substack{T \subset C \\ i \in T}} \frac{(i-1)!(n-i)!}{n!}\left[\nu(T) - \nu(T \setminus \{i\})\right], i = \overline{1,3}$$

where T has the property:

$$\nu(C_1) = \nu(C_1 \cap T), \forall C_1 \subset C$$

For:

$$\nu(\{1\}) = \nu(\{2\}) = \nu(\{3\}) = \nu(\varnothing) = 0$$

$$\nu(\{1,2\}) = \frac{1}{3}, \nu(\{2,3\}) = 1, \nu(\{1,3\}) = -\frac{1}{3}, \nu(\{1,2,3\}) = 2$$

we shall have:

$$X_1^* = 10{,}33\ ;\ X_2^* = 15\ ;\ X_3^* = 16{,}16$$

It should be noticed that the differences between the three solutions are minor.

3.3 The Analysis of Coalitions Stability

The partial results regarding this problem have been presented in paragraph 3.2.3.2.

With the help of fixed ceilings $a_1, a_2, \ldots, a_n$ one can immediately determine the probability of reaching this limits: $p_i = P\left\{\overline{x} \in \overline{X} : u_i\left(\overline{x}\right) \geq a_i\right\}$, as well as the target sets $\overline{X}_i$ of each decision maker: $\overline{X}_i = \left\{\overline{x} \in \overline{X} : u_i\left(\overline{x}\right) \geq a_i\right\}$, $i = \overline{1,m}$.

For each decision maker, a field of probability and entropy associated with achieving the target set is being built properly:

$$1 \to (p_1, 1-p_1)\ ;\ H_1 = -\left(p_1 \ln p_1 + (1-p_1)\ln(1-p_1)\right)$$

$$2 \to (p_2, 1-p_2)\ ;\ H_2 = -\left(p_2 \ln p_2 + (1-p_2)\ln(1-p_2)\right)$$

$$\vdots$$

$$m \to (p_m, 1-p_m)\ ;\ H_m = -\left(p_m \ln p_m + (1-p_m)\ln(1-p_m)\right)$$

For the sake of calculations easiness, we assume that the target sets $\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_m$ form a partition of the target set $\overline{X}$.

In this case, $\sum_{i=1}^{m} p_i = 1$ and the degree of organization of the decision makers set $M = \{1, 2, \ldots, m\}$ becomes:

$$W = -\sum_{i=1}^{m} (1-p_i)\ln(1-p_i)$$

Since $W = -\sum_{i=1}^{m} \frac{1-p_i}{p_i} \ln(1-p_i)$, we shall apply further on the principle of equalization in which the efficiency function r_i is defined by the following equality:

$$r_i(s_i) = \frac{1-s_i}{s_i}\ ,\ i = \overline{1,m}$$

Obviously, r_i is monotonically increasing since $r_i' > 0$.

Further on, the following systems will be solved:

$$\begin{cases} r_1(s_1)=r_2(s_2) \\ s_1+s_2=1 \end{cases}, \quad \begin{cases} r_1(s_1)=r_2(s_2) \\ r_1(s_1)=r_3(s_3) \\ s_1+s_2+s_3=1 \end{cases}, \quad \begin{cases} r_1(s_1)=r_2(s_2) \\ r_1(s_1)=r_3(s_3) \\ r_1(s_1)=r_4(s_4) \\ s_1+s_2+s_3+s_4=1 \end{cases} \quad \dots \quad \begin{cases} r_1(s_1)=r_2(s_2) \\ r_1(s_1)=r_3(s_3) \\ \vdots \\ s_1+s_2+\dots+s_m=1 \end{cases}$$

The solutions of these systems and the efficiency function values are as follows:

$$\begin{cases} s_1^*=\frac{1}{2} \\ s_2^*=\frac{1}{2} \end{cases}, r_1(s_1^*)=ln\frac{1}{2} \quad ; \quad \begin{cases} s_1^*=\frac{1}{3} \\ s_2^*=\frac{1}{3} \\ s_3^*=\frac{1}{3} \end{cases}, r_1(s_1^*)=2\,ln\frac{2}{3}$$

$$\begin{cases} s_1^*=\frac{1}{4} \\ s_2^*=\frac{1}{4} \\ s_3^*=\frac{1}{4} \\ s_4^*=\frac{1}{4} \end{cases}, r_1(s_1^*)=3\,ln\frac{3}{4} \tag{353}$$

$$\vdots$$

$$\begin{cases} s_1^*=\frac{1}{m} \\ s_2^*=\frac{1}{m} \\ \vdots \\ s_m^*=\frac{1}{m} \end{cases}, r_1(s_1^*)=(n-1)ln\frac{n-1}{n} \tag{354}$$

We have $ln\frac{1}{2}=-0,69$; $ln\frac{4}{9}=-0,81$; $ln\frac{27}{64}=-0,86$; $ln\frac{256}{625}=-0,87$; ... ; $\lim_n (n-1)ln\frac{n-1}{n}=\lim_n ln\left(\frac{n-1}{n}\right)^{n-1}=ln\lim_n\left(1-\frac{1}{n}\right)^{n-1}=ln\,e^{-1}=-1$

The natural number t which has the property:

$$ln\left(\frac{t-1}{t}\right)^{t-1}=max\left\{ln\frac{1}{2},ln\frac{4}{9},\dots ln\left(\frac{n-1}{n}\right)^{n-1}\right\} \text{ is } t=2.$$

Obviously, the optimum number of decision makers in a coalition equals two. Therefore, this is a theoretical answer to the fact that in practice the number of decision makers in a coalition does not exceed three, in principle.

Remark 3.26

For $n>3$, the sequence $(a_n)n$, in which the general term is $a_n=ln\left(\frac{n-1}{n}\right)^{n-1}$, slowly converge to -1.

Moreover, the previous result is consistent with the findings of the last part of paragraph 3.2.3.2. which states that the number of decision makers in a coalition is equal to two.

Remark 3.27

If the weighted Guiaşu entropies (presented in Chapter 1) had been taken into account instead of the unweighted entropies $H_1, H_2, \dots H_4$, then the degree of organization would have been:

$$W = -\sum_{i=1}^{m}(1-p_i)ln(1-p_i) + \sum_{i=1}^{m} p_i u_i$$

In this case, one can apply the equalization principle with efficiency functions as follows:

$$r_i(s_i) = \frac{s_i - 1}{s_i} ln(1-s_i) + u_i \quad , i = \overline{1,m}$$

In the event that $u_i = s_i$ (as is the case of informational energy), the systems admit the following solutions:

$$\begin{cases} r_1(s_1) = r_2(s_2) \\ s_1 + s_2 = 1 \end{cases} \Rightarrow \begin{cases} s_1^* = \dfrac{1}{2} \\ s_2^* = \dfrac{1}{2} \end{cases} ; r_1(s_1^*) = -ln\frac{1}{2} + \frac{1}{2} \tag{355}$$

$$\begin{cases} r_1(s_1) = r_2(s_2) \\ r_1(s_1) = r_3(s_3) \\ s_1 + s_2 + s_3 = 1 \end{cases} \Rightarrow \begin{cases} s_1^* = \dfrac{1}{3} \\ s_2^* = \dfrac{1}{3} \\ s_3^* = \dfrac{1}{3} \end{cases} ; r_1(s_1^*) = -2\,ln\frac{2}{3} + \frac{2}{3} \tag{356}$$

$\vdots$

$$\begin{cases} r_1(s_1) = r_2(s_2) \\ r_1(s_1) = r_3(s_3) \\ \vdots \\ r_1(s_1) = r_m(s_m) \\ s_1 + s_2 + \ldots + s_m = 1 \end{cases} \Rightarrow \begin{cases} s_1^* = \dfrac{1}{m} \\ s_2^* = \dfrac{1}{m} \\ \vdots \\ s_m^* = \dfrac{1}{m} \end{cases} ; r_1(s_1^*) = -(m-1)ln\frac{m-1}{m} + \frac{1}{m} \tag{357}$$

t is the index searched, for which the maximum M is achieved:

$$M = max\left\{\frac{1}{2} - ln\frac{1}{2}; \frac{1}{3} - 2\,ln\frac{2}{3}; \ldots; \frac{1}{m} - (m-1)ln\frac{m-1}{n}\right\}$$

where the index t which verifies the property:

$$\frac{1}{t} - (t-1)ln\frac{t-1}{t} = M$$

equals 2 and therefore the optimum number of decision makers at a certain time during the decisional process is the same as in the previous case.

Remark 3.28

If the degree of organization of the system is defined with the help of Guiasu entropy then the efficiency function r_i is defined as follows:

$$r_i(s_i) = ln\frac{u_i}{s_i}$$

In this case of the above system, the corresponding solutions are:

$$\begin{cases} r_1(s_1) = r_2(s_2) \\ s_1 + s_2 = 1 \end{cases} \Rightarrow \begin{cases} s_1^* = \dfrac{u_1}{u_1 + u_2} \\ s_2^* = \dfrac{u_2}{u_1 + u_2} \end{cases} ; r_1(s_1^*) = ln(u_1 + u_2) \tag{358}$$

$$\begin{cases} r_1(s_1) = r_2(s_2) \\ r_1(s_1) = r_3(s_3) \\ s_1 + s_2 + s_3 = 1 \end{cases} \Rightarrow \begin{cases} s_1^* = \dfrac{u_1}{u_1 + u_2 + u_3} \\ s_2^* = \dfrac{u_2}{u_1 + u_2 + u_3} \\ s_3^* = \dfrac{u_3}{u_1 + u_2 + u_3} \end{cases} ; r_1(s_1^*) = ln(u_1 + u_2 + u_3) \tag{359}$$

$$\vdots$$

$$\begin{cases} r_1(s_1) = r_2(s_2) \\ r_1(s_1) = r_3(s_3) \\ \vdots \\ r_1(s_1) = r_m(s_m) \\ s_1 + s_2 + \ldots + s_m = 1 \end{cases} \Rightarrow \begin{cases} s_1^* = \dfrac{u_1}{u_1 + u_2 + \ldots + u_m} \\ s_2^* = \dfrac{u_2}{u_1 + u_2 + \ldots + u_m} \\ \vdots \\ s_m^* = \dfrac{u_m}{u_1 + u_2 + \ldots + u_m} \end{cases} ; r_1(s_1^*) = ln\left(\sum_{i=1}^{m} u_i\right) \tag{360}$$

Obviously, in this case the determination of index t having the property:

$$ln\left(\sum_{i=1}^{t} u_i\right) = max\{ln(u_1 + u_2), ln(u_1 + u_2 + u_3), \ldots, ln(u_1 + u_2 + \ldots + u_m)\}$$

depends on the efficiency values $u_1, u_2, \ldots, u_m$.

If $u_1, u_2, \ldots, u_m \geq 0$, then $t = m$, thus all deciders form a coalition.

If there is a negative efficiency among $u_1, u_2, \ldots, u_m$ then the index t cannot be determined with certainty.

For example, if $u_1 = 3, u_2 = 2, u_3 = -1, u_4 = -2, u_5 = 3$, then there are two indices, $t = 2$, $t = 5$ for which we have the equalities:

$$ln(u_1 + u_2) = ln(u_1 + u_2 + u_3 + u_4 + u_5) =$$
$$max\{ln(u_1 + u_2), ln(u_1 + u_2 + u_3), ln(u_1 + u_2 + u_3 + u_4), ln(u_1 + u_2 + u_3 + u_4 + u_5)\}$$

3.4 Bibliographical Notes and Comments

The first paragraph of this chapter is an extension of some results from [49]. It emphasizes the principle of equalization, a less known optimality principle owed to Ghermeier [29]. A central result related to the use of this principle within the decision theory is that a decision maker can improve the decisional process without adopting a strategic risk behavior in case other decision makers adopted a cautious behavior (in maxmin or minmax sense).

The coalition criterion is based on the notion of characteristic function and on the possibility of utility transfer. In fact, if one keeps the notations established by the probability theory, where the events are incompatible and independent the following equalities occur:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(P(A \cap B) + P(B \cap C) + P(C \cap A)) + P(A \cap B \cap C)$$

The following equality results from calculations:

$$2P(A \cup B \cup C) = P(A \cup B) + P(A \cup C) + P(B \cup C) + P(A \cap B) + P(A \cap C) + P(B \cap C) + 2P(A \cap B \cap C)$$

Therefore, it seems logical that decision makers be interested in making together an income that would increase individual gains. Obviously, this way of making a coalition does not exclude the problem of distributing the final gains (an open question in cooperative game theory).

The degree of generality of this coalition concept has been studied and the notions of gain repartition and excess have been expanded (the term excess was first introduced in [47]).

The problem of coalition stability analysis is less studied although it is of great importance from practical point of view. The explanation derives from the different ways of forming (the concept of coalitions).

The analysis of coalition stability has been approached from algebraic and entropic point of view. From algebraic point of view, the analysis was made based on a result from [25] related to the games on a square unit and it was extended to an algebraic solution of the problem so that the weighted sum of the gain deviation of the players should be constant. The entropic solution has an interesting approach; it is consistent with the algebraic solution, adding that in a coalition (with players organized according to the probability of reaching the target set) only the top three decision makers are important.

The entropic solution of the distribution of the final gain is actually the solution of an optimization problem. The given example shows that it is slightly different from the solution formulated by Shapley. But generally, we cannot say that it is the best or the least good solution than other solutions of a cooperative game. There are different points of view regarding the coalition formation [47], [63], [77] and, consequently, there are different results and interpretations.

The multitude of different theorizations of the way of making up coalitions and of distributing the gains among deciders can be accounted for through the different way in which the notion of cooperation is formulated.

The Neumann – Morgenstern theory and its immediate extensions accept the notions of characteristic function and of imputation as notions of great importance.

Generally, the transfer of utilities is accepted within these theories. A weak point of view of this theory is the following idea: if M is the set of deciders, the formation of the coalition $C \subset M$ is indispensably linked with the formation of the coalition $M \setminus C$ and of the zero sum game between C and $M \setminus C$.

The interpretation of the decisional process as a zero sum two players game has obvious calculation advantages but it doesn't correspond altogether to real situations. Firstly, the formation of the C coalition doesn't necessarily baring about the formation of the $M \setminus C$ coalition. Secondly, such a decisional model leads, theoretically, to low gains for deciders.

According to Shapley [59], the formation of coalitions is based upon the notion of characteristic function and upon up the total coalition. Each decider is associated with a value, which represents his negotiating power (i.e. what he can claim from the gain of the total coalition). In comparison with the Neumann – Morgenstern theory, Shapley's theory has the advantage of dealing with the determination of the concrete value of the profit of each decider.

Caplow [11] associates with each decider taking part in the decisional process a certain weight (called the deciders force). According to Caplow's theory, the coalition which is made up must dispose of a least half of the total force (in case that several coalitions comply with this requirement, the coalition with the least total force is made up). A weak point of view of this theory is represented by the total coalition (which according to this point of view shouldn't be made up).

The theory elaborated by Vickray [86] is based on the notions of characteristic function and imputation, the existence of a set of imputations is assumed which corresponds to an accepted standard of behavior and depending on this set of imputations a classification is made of all the sets of imputations on the basis of their stability.

Aumann ad Maschler [3] were not concerned with the conditions in which a coalition i is actually made up; starting from the fact that formation of the coalition is an accomplished fact, they elaborated a theory concerning the determination of the solution of the game and which is based on the notion of configuration. According to Aumann ad Mascher the target of the players in a coalition is not to obtain the maximum gain each (in this case there will never be an agreement), but to reach a certain stability which is also supposed to express the force of each player.

Other concepts of coalition and distribution of the profits can be found in [44] and [63].

The concept of coalition introduced in this chapter belongs to the author and as it follows from the results of this chapter it is of a more general nature than the concept of coalition based on the notions of characteristic function and compensation. Besides, if the conditions of the corollary 3.121 are observed, there will be no ordinary coalitions and all coalitions made up in the sense of maximum probability will have equal excesses.

APPLICATIONS OF MINMAX EQUALITIES AND EQUILIBRIUM POINTS IN THE CONTEXT OF ANALIZING THE OPERATION SAFETY OF THE SYSTEM

From probabilistic point of view, these quantitative characteristics are in fact statistical indicators associated with the random variable characterizing the evolution of the system until the first failure or between two consecutive failures.

4.1 Optimal Minmax Analysis of Failure Moments for a System

4.1.1 The Probability of Safety Operation within a period of Time and Around a Fixed Moment

Let us consider T the time until the failure occurs (obviously T is a random variable and F is the corresponding distribution function).

There are two ways of describing the reliability of a system in a given period of time [12], [90]:

a) during period (0, t).

The probability of system failure during this period is determined with the help of the distribution function F defined by the equality:

$$F(t)=P(T<t) \tag{361}$$

If $R=1-F$ is the function representing the reliability of the system, then the probability of functioning without failure at a point during period (0, t) is given by

$$R(t)=P(T\geq t)=1-F(t) \tag{362}$$

b) during period $(t_1,t_2),\, t_1>0$.

The determination of the probability of failure and of operation without failure during period (t_1,t_2), is possible using conditional probability formula:

$$F(t_1,t_2)=\frac{P(t_1\leq T<t_1+t_2)}{P(T\geq t_1)}=\frac{F(t_1+t_2)-F(t_1)}{R(t_1)} \tag{363}$$

$$R(t_1,t_2)=\frac{P(T>t_1+t_2)}{P(T\geq t_1)}=\frac{R(t_1+t_2)}{R(t_1)} \tag{364}$$

The graphs of functions *F* and *R* are shown in figure 11:

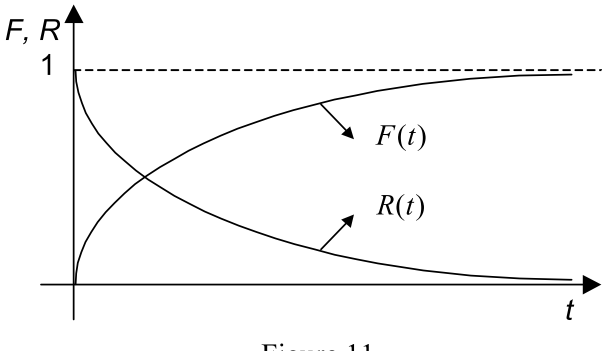

Figure 11

Practically, it arises the problem of "local" behavior around a fixed time *t*. From analytical point of view, this behavior of the physical system analyzed can be described by using two indicators [12], [55]:

a) the density of probability

This indicator describes system behavior around the time *t* regardless of the behavior of the system until that time.

The density of probability is denoted by f and it is defined as:

$$f(t)=\lim_{\Delta t\to o}\frac{P(t\leq T<t+\Delta t)}{\Delta t} \tag{365}$$

If we take into consideration the characteristics of the distribution function and the definition of the function derivative at a certain point, the equality (365) becomes:

$$f(t)=\lim_{\Delta t\to 0}\frac{F(t+\Delta t)-F(t)}{\Delta t}=F'(t) \tag{363}$$

It must be noted that the density of probability at time *t* was defined as the ratio between the probability of failure during period $(t,t+\Delta t)$ and the length of this period, when $\Delta t\to 0$.
By direct calculation we may immediately get the equality:

$$F(t)=\int_0^t f(x)dx,\quad R(t)=1-\int_0^t f(x)dx \tag{367}$$

b) the failure rate

This indicator describes the behavior around time *t* on condition of proper functioning until that time.
The failure rate is denoted by *z* and it is defined as follows:

$$z(t)=\lim_{\Delta t\to 0}\frac{F(t+\Delta t)-F(t)}{R(t)\Delta t} \tag{368}$$

Performing calculations in (368) we get:

$$z(t)=\frac{f(t)}{R(t)}=\frac{F'(t)}{R(t)}=-\frac{R'(t)}{R(t)} \tag{369}$$

From the last equalities, we can deduce the relationship between the reliability of the system, the density of probability and the failure rate:

$$R(t)=e^{-\int_0^t z(x)dx},\quad f(t)=z(t)e^{-\int_0^t z(x)dx} \tag{370}$$

By representing both probability density and the rate of failure in the same reference system, we obtain the graph from figure 12:

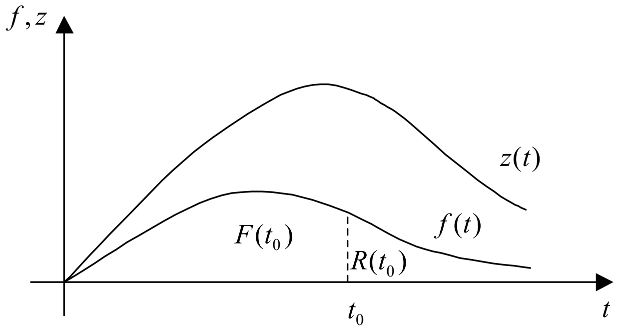

Figure 12

4.1.2 The Determination of Failure Moments of a System

The aim of this chapter is to present a benchmark for operational safety analysis of a system. Actual knowledge of the moments of failure of a system is useful in order to present both theoretical results, which make the subject of the next chapter and an economic analysis related to the minimum consumption of electricity for some types of activities used in mining.
Failure times are calculated differently depending on the diversity of situations that may be encountered under practical conditions.

4.1.2.1 Minmax Optimal Determination of Failure Moment for the Global Statistic Model

The results presented in this paragraph characterize the global statistical model (and thus the structure of the system is not taken into account, while the analysis is performed without taking into consideration the possibilities of renewing the system) and they turn out to be important for further analysis of the problem studied.
It starts from some statistical considerations: from experimental data and through verifying some important laws of probability theory, we can determine both the average operation time m and the dispersion D (or equivalently, the standard square deviation σ).
The efficiency function adopted is the average operation time (Ghermeier [29])

$$f(x,p(t))=\int_0^x p(t)\,dt+m\,p(x) \tag{371}$$

where: x – represents the failure moment of the system;

p – represents the failure function of the system (characterized by the average value m and dispersion D).

Remark 4.1

Obviously, the following conditions are necessary:

$$p(0)=1,\ \int_0^\infty p(t)\,dt=m,\ 2\int_0^\infty t\,p(t)\,dt=m^2+D \tag{372}$$

Actually, the efficiency function introduced by Ghermeier derives from studying a system during operation, for which duplication is possible.

Without restricting the generality of the problem discussed below and (mostly) without influencing the final result, the efficiency function can be practically used for any system for which we analyze the operational safety (Ghermeier [29]).

Since we are interested in the maximum safe operation time of system analyzed, we are led to solving the following optimization problem [53]:

(P) $\quad \max_x \min_{p(t)} f(x,p(t))$

Remark 4.2

The interpretation of this problem is as follows: because practically there is an infinite number of failure curves that have an average value m and the dispersion D (m and D are known quantities), we are interested in determining the maximum safe operation time of the system analyzed, regardless of system failure (that is to say, regardless of the operating conditions of the system). As we are interested in adopting decisions according to which the operating conditions of a system must be, if not optimal, at least continuously improved, it is extremely important to know the minimal extremity failure curve (fig.13).

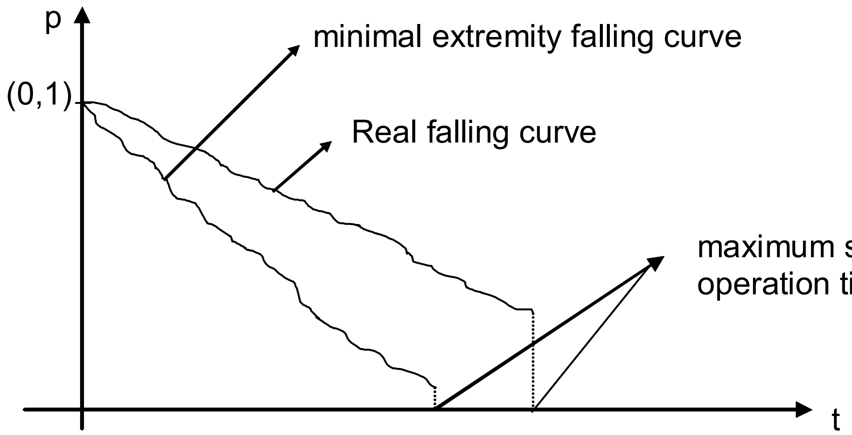

Figure 13

The solution of problem $\max_x \min_{p(t)} f(x,p(t))$ is found by using a method which is characteristic to the game theory [29], [53] and it yields the following results:

1. The optimum safe operation time x_0 is the solution of the equation below:

$$(x-m)^4-2Dx(m-x)+D^2=0 \tag{373}$$

There are the following possibilities:

1.1 $m^2<D$; in this case, equation (373) does not have a real solution

1.2 $D<m^2<\frac{7}{4}D$; in this case, equation (373) has two real roots x_1^*,x_2^* placed in the following intervals: $x_1^*\in\left(m-\sqrt{\frac{D}{2}},m\right)$, $x_2^*\in\left(m,m+\sqrt{\frac{D}{2}}\right)$

1.3 $m^2 > \frac{7}{4}D$; in this case, equation (216) has two roots $x_1^* \in \left(0,\, m-\sqrt{\frac{D}{2}}\right)$, $x_2^* \in \left(m+\sqrt{\frac{D}{2}}, m+D\right)$.

It is obvious that practically the optimum solution x_0 sought is the smallest of the roots corresponding to cases 1.2 and 1.3.

The actual determination of the optimal solution sought x_0 is possible by using a special technique for solving algebraic equations based on the so-called "contraction principle" of Banach.

$$x_0 = m - \sqrt{\frac{-D+\sqrt{D(4m^2-3D)}}{2}} \tag{374}$$

2. the extremity failure curve p_0 has the following analytical representation:

$$p_0(t) = \begin{cases} 0 & ,t > x_0 \\ e^{-\frac{t}{m}} & ,t \le x_0 \end{cases} \tag{375}$$

Remark 4.3

Practically the system operates safely up to the moment x_0 regardless of the failure curve.

The worst system failure curve is p_0 and it is of exponential type. It can be observed that although exponential distributions are most commonly used in the theory of safety, they are often (as in the case of the analyzed system) the worst. Of course, there are situations where the use of exponential distributions is justified.

Remark 4.4

In terms of energy loss analysis, determining moment x_0 represents in fact the knowing when energy consumption reaches the maximum value.

4.1.2.2 The Case When the System Consists of Identical Distributed Components

The results obtained in the previous paragraph involves major changes in the situation when the analyzed system consists of a large number of identically distributed components (in probabilistic terms).

Such situations occur when high-capacity belt conveyors are used in mining operations (as a specific design feature is the large number of rolls).

Suppose that the system is composed of n identically distributed components and for each component we have determined the average value m and the dispersion D (p denotes the failure function).

Also, for each component we must determine maximum operating time x_0 (based on equality 373) and the maximum-type curve (given by equality 375).

We shall use notations m_n and D_n for the average value and the dispersion of the system. Calculating these quantities is extremely difficult (especially the calculation of m_n), but knowing them is essential in analyzing the operation safety and the power consumption of the system studied.

Quantity m_n is actually a solution to the following optimization problem [29]:

$$(\mathrm{P}_1) \qquad m_n = \min_{p(t)} \int_o^\infty p^n(t)\,dt \tag{376}$$

and it is calculated using Pontreaghin maximum principle.

Hence it results that:

$$m_n = \begin{cases} \frac{m}{2}\left(\frac{2n}{2n-1}\right)^n \left(\frac{m^2}{m^2+D}\right)^{n-1} & ,\ if\ D \ge \frac{m^2}{2n-1} \\ m - \sqrt{D\frac{(n-1)^2}{2n-1}} & ,\ if\ D < \frac{m^2}{2n-1} \end{cases} \tag{377}$$

It is obvious that in the case of $D \ge \frac{m^2}{2n-1}$, when *n* is high enough we get

$$m_n \approx \sqrt{e}\frac{m}{2}\left(\frac{m^2}{m^2+D}\right)^{n-1} \tag{378}$$

Calculations to determine D_n are relatively immediate; it is essential that the determination of m_n and D_n makes us aware of the safe operating time x_0^n of the system, as well as of the extremity maxmin curve p_0^n which characterizes the system:

$$x_0^n = m_n - \sqrt{\frac{-D_n + \sqrt{(4m_n^2 - 3D_n)D_n}}{2}} \tag{379}$$

$$p_0^n(t) = \begin{cases} e^{-\frac{t}{m_n}} & , t \le x_0^n \\ 0 & , t > x_0^n \end{cases} \tag{380}$$

Remark 4.5

Since the quantity x_0 can be determined only if $m^2 > D$, it is clear that the optimum time x_0^n can be determined only in the situation $m^2 > (2n-1)D$ and, therefore, the equality (379) occurs only if $m_n = m - \sqrt{D\frac{(n-1)^2}{2n-1}}$.

Remark 4.6

Although a full analysis of energy consumption is made in the next paragraph, knowing the optimum moments x_o and x_0^n leads to the following interpretations from practical point of view:

- The moment x_0 signifies the end of the period when the consumption of energy is acceptable.
- The moment x_0^n refers to the maximum downtime (when the system practically does not work), fig.14.

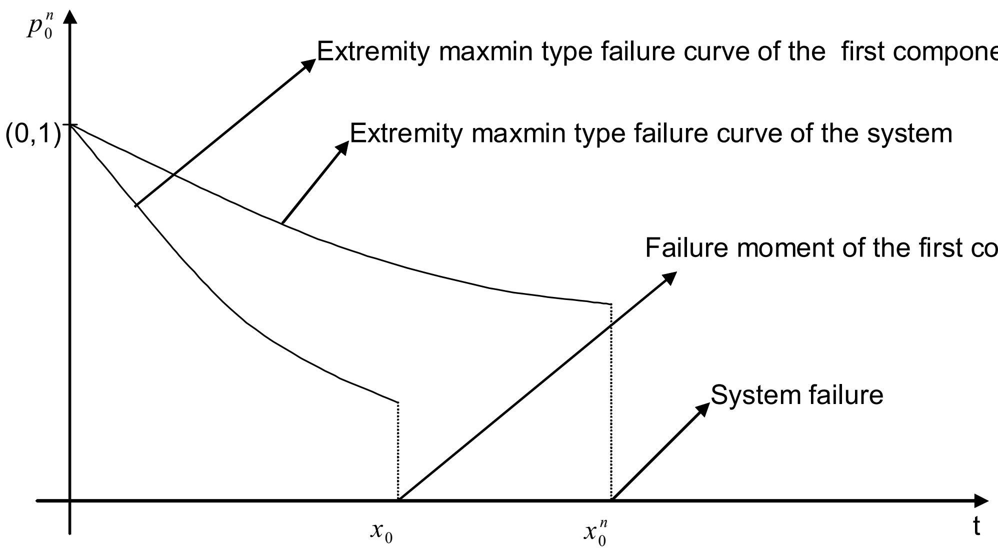

Figure 14

- The period of time in which there is an increase of electricity consumption up to the system failure

$$x_0^n - x_0 \approx \sqrt{\frac{D_n(n-1)}{2}} \tag{381}$$

The significance of the relationship (381) is extremely relevant: practically, the period of time in which the electricity consumption increases up to the failure of the system depends on two components:

- a component D_n that takes into account the operational safety of the system (D_n represents the system reliability as a specific element);
- a component n that takes into account the structure of the system.

Remark 3.7

If the components are not identically distributed, by solving a stochastic optimization problem [68] component failure functions can be approximated through a single failure function based on the following equality:

$$X = m\prod_{i=1}^{m} m_i \frac{\sum_{i=1}^{m}(-1)^{i-1}\frac{1}{m_i}X_i\prod_{j\neq i}(1-a_j)}{\sum_{i=2}^{n}\prod_{j=i}^{n}(a_j-1)} \tag{382}$$

where:

X_i is the random variable associated with the operation of the component $i,\ i=\overline{1,n}$;

m_i is the average value of variable $X_i,\ i=\overline{1,n}$;

m is the average value of variable X;

The constants $a_1,a_2,...,a_n$ are determined from the condition $M(X_i,X_i)=a_i m_i^2$, where $M(X_i,X_i)$ represent the average of the product of random variables X_i,X_i.

Remark 4.8

We are led to this result if we take into consideration the following problem:

Let us determine the random variable $\widetilde{X}$ which has the following property:

$$M\left|\widetilde{X}-X\right|^2 \le M\left|X_i-X\right|^2,\quad i=\overline{1,n} \tag{383}$$

(This represents a stochastic optimization problem which calls into requisition the method of least squares).

The error ε resulting from the approximation of X by $\widetilde{X}$ is a minimal quadratic error, which is given by the following equality:

$$\varepsilon^2 = M\left|X-\widetilde{X}\right|^2$$

The problem (1) leads us to a stochastic system which has the following solution:

$$\widetilde{X} = \frac{1}{\Gamma}\begin{vmatrix} (X_1,X_1)(X_1,X_2)\dots(X_1,X_n)X_1 \\ (X_2,X_1)(X_2,X_2)\dots(X_2,X_n)X_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ (X_n,X_1)(X_n,X_2)\dots(X_n,X_n)X_n \\ (X,X_1)(X,X_2)\dots(X,X_n)\,0 \end{vmatrix} \tag{384}$$

where: $\Gamma = \Gamma(X_1,X_2,\dots,X_n)$ is the Gramm determinant of the system, given by the following equality:

$$\Gamma = \begin{vmatrix} (X_1,X_1)(X_1,X_2)\cdots(X_1,X_n) \\ (X_2,X_1)(X_2,X_2)\cdots(X_2,X_n) \\ \dots\dots\dots\dots\dots\dots\dots \\ (X_n,X_1)(X_n,X_2)\cdots(X_n,X_n) \end{vmatrix} \tag{385}$$

The average quadratic error ε^2 can be computed as follows:

$$\varepsilon^2 = \frac{\Gamma(X_1,X_2,\dots,X_n,X)}{\Gamma(X_1,X,\dots,X_n)} \tag{386}$$

Taking into consideration the fact that the random variables $X_1,X_2,\dots,X_n$ are independent, the following equalities will take place:

$$M(X_i\cdot X_j)=M(X_i)M(X_j),\quad i,j=\overline{1,n}$$

Let us denote by $m_1,m_2,\dots,m_n$ the average values of the random variables $X_1,X_2,\dots,X_n$ and by m the average value of the random variable X.

We shall mark with $a_1,a_2,\dots,a_n$ the real numbers which have the following property:

$$M(X_iX_i)=a_i m_i^2,\quad i=\overline{1,n}$$

The random variable $\widetilde{X}$ is determined based on the equality (384). First, we shall compute the following determinant:

$$\varDelta = \begin{vmatrix} (X_1,X_1)(X_1,X_2)\ldots(X_1,X_n)\,X_1 \\ (X_2,X_1)(X_2,X_2)\ldots(X_2,X_n)\,X_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (X_n,X_1)(X_n,X_2)\ldots(X_n,X_n)\,X_n \\ (X,X_1)(X,X_2)\ldots(X,X_n)\,0 \end{vmatrix} \tag{387}$$

Let us develop this determinant by the last column:

$$\varDelta = X_1 \begin{vmatrix} (X_2,X_1)(X_2,X_2)\ldots(X_2,X_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (X_n,X_1)(X_n,X_2)\ldots(X_n,X_n) \\ (X,X_1)(X,X_2)\ldots(X,X_n) \end{vmatrix} -$$

$$-X_2 \begin{vmatrix} (X_1,X_1)(X_1,X_2)\ldots(X_1,X_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (X_n,X_1)(X_n,X_2)\ldots(X_n,X_n) \\ (X,X_1)(X,X_2)\ldots(X,X_n) \end{vmatrix} + \ldots + (-1)^{n-1} X_n \begin{vmatrix} (X_1,X_1)(X_1,X_2)\cdots(X_1,X_n) \\ (X_2,X_1)(X_2,X_2)\cdots(X_2,X_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (X,X_1)(X,X_2)\cdots(X,X_n) \end{vmatrix} =$$

$$= X_1 \begin{vmatrix} m_2m_1 \;\; a_2m_2^2 \ldots m_2m_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ m_nm_1 \;\; m_nm_2 \ldots a_n^2m_n^2 \\ mm_1 \;\; mm_2 \ldots mm_n \end{vmatrix} - X_2 \begin{vmatrix} a_1m_1^2 \;\; m_1m_2 \ldots m_1m_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ m_nm_1 \;\; m_nm_2 \ldots a_n^2m_n^2 \\ mm_1 \;\; mm_2 \ldots mm_n \end{vmatrix} + \ldots + (-1)^{n-1} X_n \begin{vmatrix} a_1m_1^2 \;\; m_1m_2 \ldots m_1m_n \\ m_2m_1 \;\; a_2m_2^2 \ldots m_2m_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ mm_1 \;\; mm_2 \ldots mm_n \end{vmatrix} \tag{388}$$

Let us perform calculations within relation (388); we shall also take into consideration the properties of determinants. After an immediate calculation we shall obtain:

$$D = m\prod_{i=1}^{n} m_i^2 \sum_{i=1}^{n}\left[(-1)^{i-1}\frac{1}{m_i}X_i\prod_{j\neq i}(1-a_j)\right] \tag{389}$$

$$\Gamma(X_1,X_2,\ldots,X_n) = \begin{vmatrix} a_1m_1^2 \;\; m_1m_2 \ldots m_1m_n \\ m_2m_1 \;\; a_2m_2^2 \ldots m_2m_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ m_nm_1 \;\; m_nm_2 \ldots a_nm_n^2 \end{vmatrix} =$$

$$= \prod_{i=1}^{n} m_i \begin{pmatrix} (a_1-1)(a_2-1)(a_3-1)\ldots(a_{n-1}-1)(a_n-1)+ \\ +(a_2-1)(a_3-1)\ldots(a_{n-1}-1)(a_n-1)+ \\ +(a_3-1)\ldots(a_{n-1}-1)(a_n-1)+ \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ +(a_{n-1}-1)(a_n-1)+ \\ +(a_n-1) \end{pmatrix} = \prod_{i=1}^{n} m_i \left[\sum_{i=1}^{n}\prod_{j=1}^{n}(a_j-1)\right] \tag{390}$$

Relations (389) and (390) yield the analytical expression of the random variable $\widetilde{X}$:

$$\widetilde{X} = m\prod_{i=1}^{n} m_i \frac{\sum_{i=1}^{n}\left[(-1)^{i-1}X_i\prod_{j\neq 1}(1-a_j)\right]}{\sum_{i=1}^{n}\prod_{j=1}^{n}(a_j-1)} \tag{391}$$

Knowing the analytical expression of the random variable $\widetilde{X}$, we can easily determine the average value $\widetilde{m}$:

$$\widetilde{m} = m\prod_{i=1}^{n} m_i \frac{\sum_{i=1}^{n}\left[(-1)^{i-1}\prod_{j\neq 1}\left(1-a_j\right)\right]}{\sum_{i=1}^{n}\prod_{j=1}^{n}\left(a_j-1\right)} \tag{392}$$

4.1.2.3 The Determination of Failure Moments by Taking into Account the Structure of the System and its Probability of Renewal

The purpose of this paragraph is to present some very important results related to the study of the safety in the system functioning and hence the study of energy losses within the system.
The results are important due to the new elements and to the extremely general conditions in which they have been obtained.

4.1.2.3.1 The Determination of the Availability of a System and the Determination of the Unavailability Depending on Each Component of a System

Practically, it starts from the consideration that the analyzed system S is composed of n subsystems $S_i, S_2, ..., S_n$ and that the failure of any of these subsystems leads to the failure of system S.
Renewals and failures of subsystems $S_i, i=\overline{1,n}$ are described by continuous random variables the probability densities of which can be defined by:

$$f_1^i(x)=a_i e^{-a_i x},\ x\geq 0;\ f_2^i(x)=r_i e^{\cdot r_i x},\ x\geq 0,\ i=\overline{1,n} \tag{393}$$

In this case, failure and renewal rates are the quantities a_i and $r_i,\ i=\overline{1,n}$, respectively.

Remark 4.9

The assumption that the probability distributions characterizing both defects and renewals of subsystems $S_i,\ i=\overline{1,n}$ are of exponential or approximately exponential type has its justification in practical reasons [12], [49].
We shall note m_1^i and m_2^i the average operation time and the renewal time, respectively, corresponding to subsystem S_i and m_1, m_2 the average operating time and renewal time corresponding to system S.
If state 0 represents the state of good working conditions of system S and if states $1,2,...,n$ represent the states of failure of the system, corresponding to the n components, system S can be associated with the following graph (figure 15).

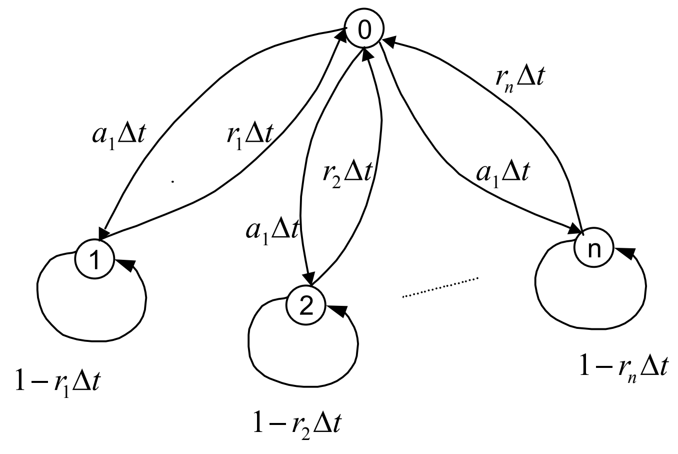

Figure 15

If we use P_0 to denote the availability of the system and P_i for its unavailability due to component i, the graph above is assigned a system of finite difference equations [12], [49]:

$$\begin{cases} P_i(t+\Delta t)=P_0(t)a_i\Delta t+P_i(t)(1-r_i\Delta t),\ i=\overline{1,n} \\ P_0(t)+\sum_{i=1}^{n}P_i(t)=1 \end{cases} \tag{394}$$

This system of equations with finite difference leads to first-order differential equations:

$$\frac{d\,P_i}{dt}+r_i\,P_i-a_iP_0=,\ i=\overline{1,n} \tag{395}$$

on the initial condition $P_0(0)=1$.

Applying Laplace transformation we get:

$$\sum_{i=1}^{n}P_i^*(s)=\frac{1}{s}-P_0^*(s) \tag{396}$$

and hence we have [49], [53], the Laplace transformation of the availability of the system

$$P_0^*(s)=\frac{1}{s\left(\sum_{k=1}^{n}\frac{a_k}{s+r_k}+1\right)} \tag{397}$$

as well as the Laplace transformation of the unavailability of the system due to component i:

$$P_i^*(s)=\frac{a_i}{s+r_i}P_0^*(s)=\frac{\frac{a_i}{s+r_i}}{s\left(1+\sum_{k=1}^{n}\frac{a_k}{s+r_k}\right)} \tag{398}$$

Given the analytical expression of Laplace transformations P_0^* and P_i^* (having rational forms), and the actual calculation P_0 and P_i is done through very difficult computation [49], [53]:

$$\begin{cases} P_i(t)=\frac{a_i}{B_i}\left(1-e^{-\frac{B_i}{A_i}t}\right),\ i=\overline{1,n} \\ P_0(t)=\frac{m_1}{m_1+m_2}-\sum_{k=1}^{n}e^{-\frac{B_i}{A_i}t} \end{cases} \tag{399}$$

where

$$\frac{a_i}{B_i}=\frac{a_i}{r_i}\cdot\frac{m_1}{m_1+m_2}$$

$$\frac{B_i}{A_i}=\frac{r_i(m_1+m_2)}{1+m_1-r_im_2}$$

4.1.2.3.2 The Determination of Failure Moments as Equilibrium Points

Failure moments $t_1,t_2,...,t_n$ of system S due to components $S_1,S_2,...,S_n$ as equilibrium points system are in fact solutions to the equations below:

$$R(t)=P_i(t),\ i=\overline{1,n} \tag{400}$$

or in an equivalent form:

$$e^{-\frac{t}{m_1}}=\frac{a_i}{B_i}\left(1-e^{-\frac{B_i}{A_i}t}\right),\ i=\overline{1,n} \tag{401}$$

Let us suppose that the following equalities take place:

$$\frac{a_1}{r_1}\frac{m_1}{m_1+m_2}>\frac{a_2}{r_2}\frac{m_1}{m_1+m_2}>\ldots>\frac{a_n}{r_n}\frac{m_1}{m_1+m_2}$$

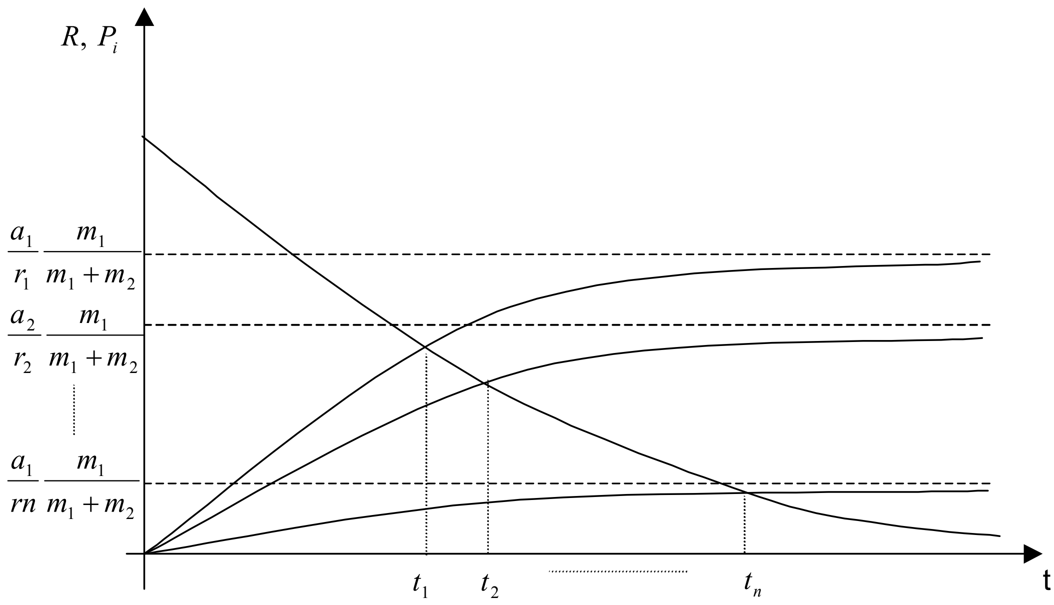

Figure 16

We note $e^{-\frac{t}{m_i}}$ which leads to solving the equation:

$$x = \frac{1}{C}\left(1 - x^D\right) \tag{402}$$

where $\quad C = \frac{B_i}{a_i} \qquad D = \frac{B_i}{A_i} m_1$

Appling the method of successive approximations, after a relatively easy calculation, we obtain the solution x^* of the equation (401)

$$x^* = \frac{1}{1 + \frac{A_i}{a_i m_i}} \tag{403}$$

which leads to the solution of the equation (402):

$$t_i = m_1\, \ell n\left(1 + \frac{A_i}{a_i m_1}\right) \tag{404}$$

Practically, the failure moments of the system are obtained by solving of an equation type (401) for each subsystem.

If we take into consideration that $A_i \approx B_i m_1$ from (404) it results:

$$t_i \approx m_1\, \ell n\left[1 + \frac{r_i}{a_i}\left(1 + \frac{m_2}{m_1}\right)\right] \tag{405}$$

If is obvious that relation (405) shows the dependence between the failure moments, reliability elements m_1, m_2 of the analyzed system and reliability elements a_i, r_i of the subsystems.

The approximate values, obtained by linearization method, $t_1, t_2, ..., t_n$ are the following [49], [53]:

$$t_i = \frac{m_1(1 + m_1 - r_i m_2)}{a_i m_1^2 + m_1 - r_i m_2 + 1}\ ,\quad i = \overline{1,n} \tag{406}$$

Remark 4.10
The determination of failure moments presents the following advantages from economic point of view:
1) interventions to the system are performed taking into consideration only the costs and not the times and costs;
2) periods of maximum energy loss coincide with moments of failure; the problem of analyzing the optimal energy consumption can be partially solved by analyzing the reliability of the system.

4.2 Optimal Analysis of the Dependence between the Energy Consumption of Systems and their Operation Safety

The paragraph approaches a very difficult problem with major technical-economic implications.
The known results from literature, apparently numerous, did not solve completely the problem, and the multitude of partial results, reflecting usually different points of view, lead to different conclusions and interpretations.
The starting reference point of this paragraph is represented by the dependency between the electrical energy consumption W and the system reliability function R [29], [49].
This dependency can be observed practically, following the data related both to the increase of the electrical energy consumption over time during electrical actions, and to the decrease of their reliability over time.
The time moments that characterize the increase of the electrical energy consumption present (analytically speaking) two components: one of them dependant of the base indicators of the reliability *m* and *D* (average value and dispersion), while the other component is dependant of the system structure. The component dependant of the system structure admits different analytical expressions, depending of the type of the studied system.
The efficiency function that will be adopted further was Ghermeier's work and admits the following analytical expression:

$$F(R,W)=\int_0^\infty\left[\int_0^x W(t)dt+mW(x)\right]dR(x) \tag{407}$$

with the following conditions:

$$\int_0^\infty R(t)dt=m,\ \ 2\int_0^\infty tR(t)d=m^2+D \tag{408}$$

The efficiency function F represents the average energy consumption during system's safe operation and is built based on the reliability function of the system, R, and on the specific consumption of electrical energy W.

4.2.1 The Determination of Failure Moments as Equilibrium Points and the Determination of Maximum Levels of Energy Consumption

The failure function t_i of the system, based on the component S_i, is determined by intersecting the reliability function *R(t)* of the system with the failure of the system due to the component S_i (401).

$$e^{-\frac{m_1}{t}}=\frac{a_i}{B_i}\left(1-e^{-\frac{B_i}{A_i}t}\right),\quad i=\overline{1,n} \tag{409}$$

The approximate solution of the equation is:

$$t_i=m_1 ln\left[1+\frac{r_i}{a_i}\left(1+\frac{m_2}{m_1}\right)\right],\quad i=\overline{1,n} \tag{410}$$

where a_i, r_i represent the failure rate, respectively the recurrence rate of the component S_i, while m_1, m_2 represent the average operating time, respectively the average recurrence time.
The equilibrium point of the system t_e is determined on the intersection between the optimal curves R_{max} and W_{min} (the curve of maximum reliability and the curve of minimum consumption).

These extreme curves represent solutions of the following optimization problem:

$$\text{(P)}\qquad \max_R \min_W F(R,W) \tag{411}$$

This problem can be solved by using Pontriaghin [29] maxim principle.

The analytical expressions of the extreme curves R_{max} and W_{min} are:

$$R_{max}(t)=\begin{cases} 1 & ,0\le t<t_1 \\ e^{-\frac{t-t_1}{m}} & ,t_1\le t\le t_2 \\ 0 & ,t>t_2 \end{cases} \tag{412}$$

$$W_{min}(t)=\begin{cases} 0 & ,0\le t<t_1 \\ \dfrac{t-t_1}{t_2-t_1}\left[1-k\left(e^{\frac{t_2}{m}}-e^{\frac{t_1}{m}}\right)\right]+k\left(e^{\frac{t}{m}}-e^{\frac{t_1}{m}}\right), & t_1\le t\le t_2 \end{cases} \tag{413}$$

where k is a constant of integration.
The following equality appears immediately:

$$R_{max}(t_1)=1,\quad W_{min}(t_2)=1$$

The graph of functions R_{max} and W_{min} are presented in fig.17.

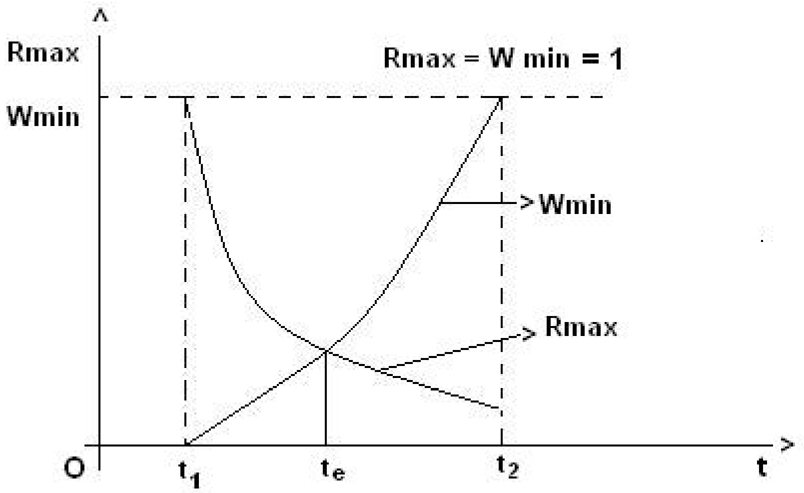

Figure 17

The searched equilibrium point is the solution of the following equation: $R_{max}(t)=W_{min}(t)$, besides

$$e^{-\frac{t-t_1}{m}}=\frac{t-t_1}{t_2-t_1}\left[1-k\left(e^{\frac{t_2}{m}}-e^{\frac{t_1}{m}}\right)\right]+k\left(e^{\frac{t}{m}}-e^{\frac{t_1}{m}}\right) \tag{414}$$

where $t_1=me^{-z}, t_2=mz+me^{-z}, z$ represents the solution of the equation $e^{-zz}+2ze^{-z}+\frac{D}{m^2}-1=0$

The last equation can only be approximated. It is most convenient to use the linearization method, but the results are influenced by the point in which Mc Laurin series are developed.
For example, if this development is carried out, then the equation is written as follows:

$$(1-z)^2+2z(1-z)=1-\frac{D}{m^2}$$

and has a positive solution $z=\frac{\sqrt{D}}{m}$.
As a consequence we have:

$$t_1=me^{-z}=m-\sqrt{D}$$
$$t_2=t_1+mz=m$$
$$t_2-t_1=mz=\sqrt{D}$$

The equilibrium point represents the solution of the equation:

$$kt^2 + t\left(D-m\right) - mD = 0$$

that is to say $t_1,t_2 = \dfrac{m-D \pm \sqrt{\left(m-D\right)^2 + 4mDk}}{2}$ (415)

It is obvious that the solution depends on the value of the integration constant k.

For example, if $k = 0$, hence it results that $t = m - D$ and if $k = 1$ then:

$$t = \frac{m-D+\sqrt{\left(m-D\right)^2 + 4mD}}{2} = \frac{m-D+m+D}{2}\text{, therefore } t = m$$

If the linearization is made at a point different from 0, one can use the result [49] according to which the approximated solution of the equation $e^{-2z} + 2ze^{-z} + 2ze^{-z} + \dfrac{D}{m^2} - 1 = 0$ is $z = -\dfrac{1}{2}ln\left(1 - \dfrac{D}{m^2}\right)$.

The approximate value of the equilibrium point is $t_e \approx m\left(1 + \dfrac{1}{eK}\right)$.

Correspondent to the found equilibrium point, it can be determined immediately both the value of the electrical energy consumption and the value of the reliability function in this point:

$$R(t_e) = W(t_e) \approx e^{-\frac{D}{2m^2-D}} \tag{416}$$

For $t_e \approx m$, it results immediately

$$R\left(t_e\right) = W\left(t_e\right) \approx 0.55$$

Practically, when the electrical energy consumption becomes $e^{-\frac{D}{2m^2-D}}$, it is obvious that this will be followed by the fast increase of the consumption and the fast decrease of the system reliability.

The moment t_e marks the time when one must intervene on the system (for increasing the technical performances, for example type BRP), or he is announced that he needs to intervene on the system in the immediate period (not higher than $t_2 - t_e$).

We notice that the moments t_0 and t_e are very close, practically

$$t_e - t_0 \approx \sqrt{\frac{-D + \sqrt{D(4m^2 - 3D)}}{2}} \tag{417}$$

Also, the real moment of system failure t_{real} is found in the interval $[t_0,t_2]$, and because $t_2 - t_0 \approx \dfrac{D^2}{2m(2m-D)}$, it is obvious that the real analysis of the interventions that can be done on the system must take into consideration the points t_0 or t_e (obviously it is preferred the point t_e because of its economic significance, and also because it allows an easier analytical expression).

An important problem related to the analysis of the optimal consumption of electrical energy is its effective calculation in the intervals $[0,t_2]$ and $[0,t_e]$ corresponding to the optimal curve W_{min}.

The knowledge of these values is essential economically, because it allows that the interventions on the system (for increasing its performances) to be done not regarding the time and cost, but only the cost (because the time moments of interventions are known).

After some difficult calculations it results:

$$\int_0^{t_2} W_{min}(t)\,dt = \int_{t_1}^{t_2}\left\{\frac{t-t_1}{t_2-t_1}\left[1 - K\left(e^{\frac{t_2}{m}} - e^{\frac{t_1}{m}}\right)\right] + K\left(e^{\frac{t}{m}} - e^{\frac{t_1}{m}}\right)\right\}dt =$$

$$= \frac{mz}{2} - \frac{K}{2}\left[mz\left(e^{z+e^{-z}} + e^{e^{-z}}\right) - 2m\left(e^{z+e^{-z}} - e^{e^{-z}}\right)\right]$$

$$\int_o^{t_e} W_{min}(t)\,dt = \int_{t_1}^{t_r}\left\{\frac{t-t_1}{t_2-t_1}\left[1 - K\left(e^{\frac{t_2}{m}} - e^{\frac{t_1}{m}}\right)\right] + K\left(e^{\frac{t}{m}} - e^{\frac{t_1}{m}}\right)\right\}dt =$$

$$= \frac{\left(1-z-e^{-z}\right)^2}{2z} - \frac{K}{2}\left\{\frac{1-z-e^{-z}}{z}\left[\left(1-z-e^{-z}\right)\left(e^{z+e^{-z}} + e^{z}\right) + 2\left(mz + me^{-z}\right)e^{e^{-z}}\right] - 2m\left(e - e^{e^{-z}}\right)\right\}$$

where z is the solution of the equation

$$e^{-2z}+2ze^{-z}+\frac{D}{m^2}-1=0 \tag{418}$$

therefore, $z=ln\frac{1}{\sqrt{a}}, a=\sqrt{1-\frac{D}{m^2}}$

It is obvious that we are interested in the ratio $R=\frac{r_1}{r_2}$, where

$$r_1=\int_0^{t_e} W_{min}(t)\,dt,\; r_2=\int_0^{t_2} W_{min}(t)\,dt. \tag{419}$$

From calculations, it results $R=\left(\frac{1-e^{-z}}{z}\right)^2$ and it is obvious that the graphic of R depends on m and D (the graphical representation of the function $R(z)=\left(\frac{1-e^{-z}}{z}\right)^2$ is shown in figure 18).

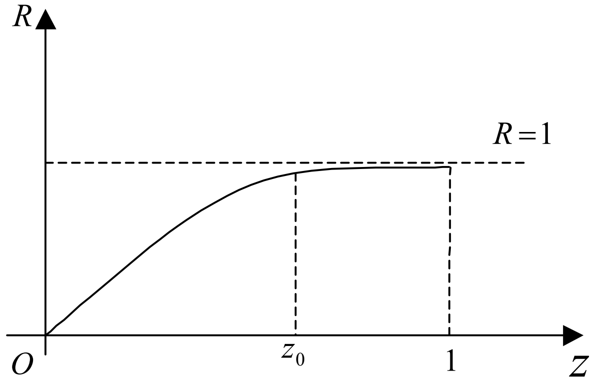

Figure 18

For $z_0=\frac{1}{\sqrt{a}}$, it results $R\approx\frac{2m^2-D}{2m^2}$, so $R\approx 1-\frac{D}{2m^2}$.

Practically, the moment t_0, when the intervention on the system is recommended, is characterized by the fact that the ratio between the consumption done until that moment and the consumption that could be done until the system failure moment t_2 is about $1-\frac{D}{2m^2}$.

In case of small dispersion (situation that is eventually not found during electrical actions in mining exploitations) we have $R \approx 1$, so the intervention moment is quite equal to the failure moment of the system.

4.2.2 The Determination of the Influence of Reducing Electric Energy Consumption and Intervention Costs upon the System

The reduction of electrical energy consumption and the calculation of costs of intervention on the system are analyzed regarding the equilibrium points (of the components and the system) and also regarding the operation safety increase acquired after interventions.

Practically, there are the following situations:

a) Interventions on the system components are done in the equilibrium point of the system (basically in the situation when the equilibrium moments of the components are about the same as the equilibrium moments of the system).

The reduction of the electrical energy consumption because of the operation safety increased in moment t_e (the moment of intervention on the system) with C_s is represented in figure 19 by the shaded area (the area of the curved trapezium ECt't_e):

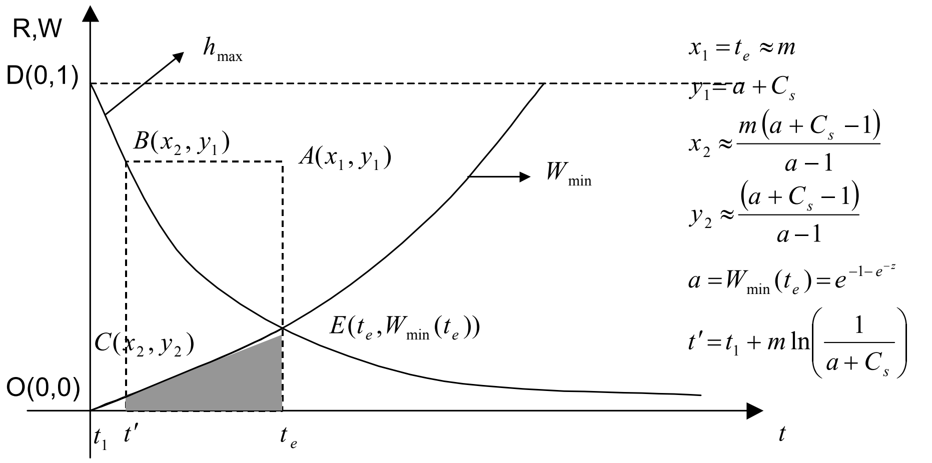

Figure 19

The reduction of the electrical energy consumption is noted with S_r and may be determined based on the following equation:

$$S_r = \int_{t'}^{t_e} \left(W_{min}(t) - t'\right) dt \tag{420}$$

where t' is the solution of the equation:

$$R_{max}(t) = W_{min}(t_e) + C_s \tag{421}$$

After a short calculation it results:

$$t' = t_1 + m \, ln\left(\frac{1}{W_{min}(t_e) + C_s}\right) \tag{422}$$

therefore,

$$S_r = \frac{C_s}{(a-1)^2}(2 - 2a - C_s)S \tag{423}$$

where S represents the area of the triangle Ot_eE.

By the same token, the reduction coefficient of electrical energy consumption - R is:

$$R = \frac{S_r}{S} = \frac{C_s}{(a-1)^2}(2 - 2a - C_s) \tag{424}$$

meaning that $R \approx 4C_s(1 - C_s)$.

The relation $R \approx 4C_s(1 - C_s)$ represents a fundamental result of this paragraph, as it expresses precisely the dependency between the reduction of electrical energy consumption and the increase of system operation safety.

The graphic of the function $R(C_s) \approx 4C_s(1 - C_s)$ on the interval [0, ½] (only here as the operation safety increase is done starting with the value $R_{max}(t_e) \approx 0.55$) is shown in figure 20.

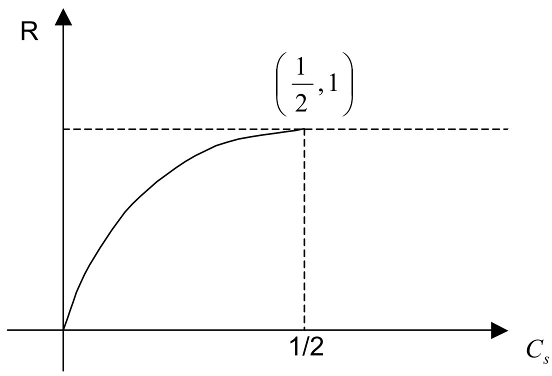

Figure 20

An important problem, related to the dependency between the reduction of the electrical energy consumption and the increase of system operation safety in the analyzed system, is the determination of the consumption reduction when the system's safety increase is one percent.

From a short calculation, we get:

$$R\left(C_s+0.01\right)-R\left(C_s\right)=0.04\left(1-2C_s\right) \tag{425}$$

and, consequently it results:

$$R(0.44+0.01)-R(0.44)=0.004<R\left(C_s+0.01\right)-R\left(C_s\right)<0.04=R(0+0.01)-R(0) \tag{426}$$

from which we can conclude that operation's safety increase by one percent ensures a linear variation of the consumption within 0,4% and 4%.

This variation of the reduction of energy consumption is obviously influenced by the increase of reliability level C_s and we observe that the reduction is higher as the one percent variation of operation safety starts from a lower value of C_s.

In conclusion, the results obtained are the following:

- the increase of operation safety:

$$C_s=e^{-\frac{t_e}{m}}\sum_{i=0}^{n}\frac{\left(\frac{t_e}{m}\right)^i}{i!} \tag{427}$$

- the reduction of electrical energy consumption:

$$R=4C_s(1-C_s) \tag{428}$$

- the cost of intervention on moment t_e:

$$C(t_e)=\frac{m_1+m_2}{m_1}\,\frac{\sum_{i=1}^{n}e^{-\frac{r_i}{a_i}}}{n} \tag{429}$$

b) Interventions on system components are done up to the equilibrium moment of the system (in case the equilibrium points of the components are spread around the equilibrium moment of the system).

In this case, the interventions (and the possibilities of grouping them) are analyzed regarding the costs.

The increase of operation safety admits the following expression:

$$C_s=\frac{n_0^2}{3n(n_0-1)}$$ (n_0 represents the number of components object of interventions)

The reduction of electrical energy consumption has the following form:

$$R(C_s)\approx\frac{4n_0(3n-n_0)}{9n^2} \tag{430}$$

c) Interventions on the n-n$_0$ components are done above the equilibrium point of the system.

In this case we recommend a global intervention on a moment close to the failure moment of the system t_2.

The purpose of studying this situation is to analyze only the interventions done on moments above the equilibrium point of the system. Basically, the intervention moments may be chosen randomly; the single requisite regarding these interventions that decision makers must comply with is not to get over the failure moment of the system t_2.

d) Calculation of the reduction of electrical energy consumption when the intervention is done in a random moment.

If C_s represents the safety gain obtained through interventions in equilibrium moments (of the components) below the system equilibrium point, we have:

$$R(t)\approx\frac{(t-m)\left[W_{min}(t_e)+\frac{a+C_s-1}{a-1}\right]}{(m_i-m)\left(1+\frac{m+t_1-m_i}{m}\right)}=\frac{(t-m)\left[2m-t-t_1+m\left(1+\frac{C_s}{a-1}\right)\right]}{(m_i-m)(2m+t_1-m_i)} \tag{431}$$

Since the ratio *R(t)* has constant denominator (as the system will not fail until the moment m_i), the analysis of the reduction of electrical energy consumption, after the intervention in the moment *t*, can be done regarding only the numerator of *R(t)*.
Considering that a ≈ 0.55, while C_s varies between 0 and 0.45, there can be established the dependency between the reduction of electrical energy consumption and the increase of system operation safety, but also important conclusions particularizing the moment *t*.
If we note

$$C=(m_i-m)(2m+t_1-m_i)\ ,$$

we shall have:

$$R(t)=\frac{1}{C}(t-m)\left(3m+\frac{mC_s}{a-1}-t_1-t\right) \tag{432}$$

The graphic of the function R is shown in figure 21.

We notice that the maximum reduction R_{max} is realized on moment $t_0=2m+\frac{mC_s}{2(a-1)}-\frac{t_1}{2}$ and has the following value:

$$R_{max}=\left(m+\frac{mC_s}{2(a-1)}-t_1\right)^2 \tag{433}$$

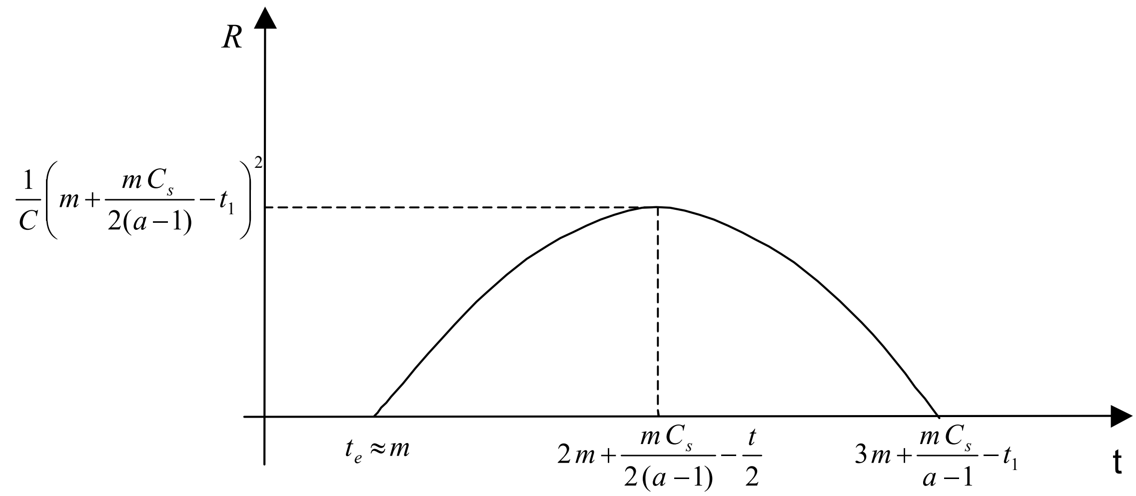

Figure 21

We can mark out immediately some important consequences:
- If the moment t_0 is done with a maximum increase of operation safety ($C_s = 0.45$), then, after a short calculation we get $R_{max}=\frac{1}{4}\cdot\frac{n-n_0}{n}$, so the maximum reduction is 25%. The maximum increase of operation safety is obtained obviously when the intervention is done in moment t_0 on all the components that were not object of intervention until the moment t_e (n_0 represents the number of the components object of intervention until the equilibrium moment t_e).
- The case C_s = 0.45 refers actually to a specific situation (when there were few interventions until the moment t_e, and no interventions in the moment t_e).

If we consider that the interventions until the equilibrium moment may lead to an increase of operation safety with maximum 33%, the intervention in the moment t_0 leads to a reduction of the consumption $R_{max}=\frac{0.052m^2}{0.4m^2}\left(\frac{n-n_0}{n}\right)$, and consequently, the maximum reduction is 13%.

The explanation of this result is that the previous successive interventions already lead to an increase of system operation safety and, as a consequence, to a reduction of electrical energy consumption.

- For a = 0.55 and C_s between 0 and 0.33, after a short calculation results that t_e is between 1.2m and 1.3m, so very close to t_2. As a result, it is recommended that the intervention to be done on all system components in a moment quite close to its failure moment.

Depending on the number of components already object of intervention and on the previous gain in operation safety, the intervention on all components in the moment t_0 (close to t_2) varies between $13\frac{n-n_0}{n}\%$ and $25\frac{n-n_0}{n}\%$.

e) Calculation of the system intervention cost

Since it is recommended the intervention in the moment t_0 on all components that were not object of intervention until the system equilibrium moment, the cost of the intervention may be calculated as average of all corresponding costs for the n-n_0 components.

The cost of intervention in the moment t_0 may be calculated as a cost of renewal type BRP.

The cost $C_i(t_0)$ of the intervention on the component S_i is given by the following equation:

$$C_i(t_0)=t_0\frac{B_i}{A_i}e^{-\frac{B_i-a_i}{A_i}t_0}-\frac{B_i}{B_i-a_i}e^{-\frac{B_i-a_i}{A_i}t_0} \tag{434}$$

After a short calculation it results:

$$C_i(t_0)\approx\frac{m_1+m_2}{m_1}e^{-\frac{r_i}{a_i}},\ i=\overline{1,n-n_0} \tag{435}$$

Therefore, the total cost $C(t_0)$ admits the following expression:

$$C(t)=\frac{m_1+m_2}{m_1(n-n_0)}\sum_{i=1}^{n-n_0}e^{-\frac{r_i}{a_i}} \tag{436}$$

We observe that this cost (that actually represents the share of intervention cost in system's purchase cost) depends explicitly both on system's reliability elements and on the component's object of intervention reliability elements.

4.2.3 The Dynamic Aspect

An important aspect, but extremely difficult to calculate, is represented by the dynamic analysis of the approached problem. Practically, the equilibrium moments evolve in time based on specific rules, and all calculations related to establishing the increase coefficients of operation safety, reduction of electrical energy consumption and evaluation of costs suffer major changes.

The analysis of the reduction of electrical energy consumption by increasing the operation safety may be detailed if we effectively calculate the coefficients C_s^1, C_s^2, …, C_s^k, …, which represent the increase coefficients of the system reliability after the interventions in the moments $t_1, t_2, \ldots, t_k, \ldots$.

Taking into consideration the system with a the serial type structure, we have

$$\begin{gathered}
C_s^1=R_2(t_1)R_3(t_1).....R_n(t_1)(1-R_1(t_1))\\
C_s^2=R_3(t_2)R_4(t_2).....R_n(t_2)(1-R_2(t_2))R_1(t_2-t_1)\\
C_s^3=R_4(t_3)R_5(t_3).....R_n(t_3)(1-R_3(t_3))R_2(t_3-t_{22})R_1(t_3-t_1)\\
\vdots\\
C_s^k=R_{k+1}(t_k)R_{k+2}(t_k)...R_n(t_k)(1-R_k(t_k))R_{k-1}(t_k-t_{k-1})\cdot R_{k-2}(t_k-t_{k-2})....R_1(t_k-t_1)
\end{gathered} \tag{437}$$

We can write these equations into a concentrated form:

$$C_s^k=\prod_{i=k}^{n}R_{i+1}(t_i)(1-R_i(t_i))\prod_{j=1}^{k-1}R_j(t_k-t_j),\ k=\overline{1,n} \tag{438}$$

Because

$$R_i(t)=e^{-\frac{t}{m_i}},\ \frac{1}{m}=\frac{1}{m_1}+\frac{1}{m_2}+...+\frac{1}{m_n}$$

after a short calculation, we obtain:

$$C_s^k = e^{-\frac{t_k}{m}+\frac{t_1}{m_1}+\frac{t_2}{m_2}+...+\frac{t_k}{m_k}}\left(1-e^{-\frac{t_k}{m_k}}\right),\ k=\overline{1,n} \tag{439}$$

If the intervention moments on the system t_1, t_2, ..., t_n are also equilibrium type, after calculations it results:

$$C_s \approx e^{-k-\frac{m_k}{m}-\frac{1}{2}} \tag{440}$$

Because $0 \le C_s^k \le 0.55,\ k=\overline{1,n}$, it results immediately the following possibilities:

a) $C_s^k = 0,55$, so $C_s^k \approx e^{-\frac{1}{2}}$ resulting $k=\frac{m_k}{m}$. Since $k \ge 1$, this mean that $m_k \ge m$.

b) $C_s^k = 0$, therefore $k=\frac{2m_k+m}{2m}$. Since $k \ge 1$, it results that $m_k \ge m/2$.

Knowing that the equilibrium point of the system t_e is around the value m, the analysis of the extreme values of C_s^k, leads to the following results:
1. In the interval $(0,t_1]$ the system operates with high safety (basically the reliability is close to 1) and the electrical energy consumption is low;
2. In the interval $(t_1,1/2\ t_e)$, there may appear failures in the system, but its renewal may be done with high safety, which leads to low energy consumption;
3. In the interval $[1/2\ t_e,\ t_e)$ any system failure because of one component leads to a low possibility to increase the reliability after the intervention on the component. It is basically the same as case b) analyzed previously.

In this case, because the reliability of the system maintains relatively low, it is obvious that we will have relatively high consumption of energy in the analyzed interval.
4. If the intervention on the system is done around the equilibrium point, then the system safety may be maximized, which leads to a minimum consumption of electrical energy.

Regarding the dynamic analysis, considering that the successive interventions on the system lead to a continuous movement of the equilibrium points towards right, the recurrence relation between two consecutive equilibrium type moments is:

$$t_e^n = t_e^{n-1}\left(1+\frac{m}{t_e^{n-1}+mC_s^n-t_1}\right) \tag{441}$$

Because $t_1 \approx m-\frac{D}{2m}$, for m very large in comparison with D, we can consider that $t_1 \approx m$.

For the sake of calculus easiness, we shall consider that $C_s^n \approx 0.5,\ n \ge 1$, because we know that the value of these coefficients is at most equal to 0.55.

Consequently, the sequence $\left(t_s^n\right)_n$, defined by the recurrence relation (441) can be written in the following form:

$$t_s^{n+1} = t_s^n\left(1+\frac{m}{t_s^n+\frac{1}{2}m-m}\right),\quad t_s^0 = m,\ n \ge 1 \tag{442}$$

Hence, we shall have:

$$t_s^1 = t_s^0\left(1+\frac{m}{t_s^0-\frac{m}{2}}\right) = m\left(1+\frac{1}{\frac{1}{2}}\right) = 3m$$

$$t_s^2 = t_s^1\left(1+\frac{m}{t_s^1-\frac{m}{2}}\right) = 3m\left(1+\frac{2}{5}\right) = \frac{21}{5}m \approx 4m$$

$$t_s^3 = t_s^2\left(1+\frac{m}{t_s^2-\frac{m}{2}}\right) = 4m\left(1+\frac{2}{7}\right) = \frac{36}{7}m \approx 5m$$

$$t_s^4 = t_s^3\left(1+\frac{m}{t_s^3-\frac{m}{2}}\right) = 5m\left(1+\frac{2}{9}\right) = \frac{55}{9}m \approx 6m$$

By means of mathematical induction method, we can prove that:

$$t_s^{n+1} = t_s^n\left(1+\frac{m}{t_s^n-\frac{m}{2}}\right) = \left(n+2\right)m\left(1+\frac{2}{2n+3}\right) \approx \left(n+2\right)m \tag{443}$$

This final relation can be considered extremely important from the perspective of determining the moments of interventions that have to be made upon the system.

In the particular case where the interventions are done only around equilibrium type moments, this recurrence relation becomes:

$$t_e^n = t_e^{n-1} + m \tag{444}$$

so $t_e^n \approx \left(n+1\right)m$; this outcome bears out the situations observed practically.

In conclusion, the maximum reduction of electrical energy consumption is

$$R_{max} = \left(m+\frac{mC_s}{2(a-1)}-t_1\right)^2,\ a\approx 0.55 \tag{445}$$

while the total cost of the intervention is

$$C(t) = \frac{m_1+m_2}{m_1(n-n_0)}\sum_{i=1}^{n-n_0} e^{-\frac{r_i}{a_i}} \tag{446}$$

The calculation of the increase coefficient of operation safety C_s gets complicated when the number of renewed components is high or when the interventions in the intermediate equilibrium moments on the system are grouped.

More precisely, if in the moment t_K is done an intervention on the component i, then the increase coefficient of operation safety C_s^K, may be easier determined by using the following relation:

$$C_s^K = R_1(t_K).....R_{i-1}(t_K)R_{i+1}(T_K)....R_n(t_K)\left(1-R_i(t_e^i)\right) \tag{447}$$

We noted with t_e^i the equilibrium moment of the component i, and with t_K the difference between the renewal moment t_K and the moment of the last renewal of the component i.

In case of individual or grouped interventions, their costs may be determined easier. For example, if in a random equilibrium moment there is done an intervention on the component i, the cost of this intervention may be approximated by the cost of a renewal type BRP:

$$K(t_e^i) = \frac{a_i r_i}{r_i - a_i}\left[e - e^{-\frac{r_i}{a_i}}\left(1+\frac{r_i}{a_i}\right)\right] \tag{448}$$

For the latest relation we noted with a_i and r_i, the failure rate, respectively the renewal rate of the component i.

4.3 Bibliographical Notes and Comments

In specialized literature [5], [12], [90], the term reliability/operating safety has the following senses: science, interdisciplinary field, general property of systems.

Regarded as a **science**, the reliability emphasizes the malfunctioning laws of technical equipment; practically, there are three theoretical possibilities of approaching these laws: probabilistically, energetically and spectrally.

Using the probabilistic methods in analyzing the operating safety of a system represents a natural consequence of the fact that the period of time during which the malfunctioning occurs is a random variable. Considering

that the probability theory includes a great number of important results, one can immediately motivate that the probabilistic approach of operating safety is the most common.
On the other hand, using the probabilistic methods in studying the reliability issue has a disadvantage: it is impossible to determine exactly when the malfunctioning occurs.
Therefore, the specialized literature also reveals other methods of analyzing the reliability of a system (it is the case of energetic and spectral theories regarding reliability). Unfortunately, the results obtained are not conclusive enough [50].
Regarded as an **interdisciplinary field**, the reliability is a relatively recent field (practically, the first important results are dated back to the 50s) [12].
The elements that encouraged the development of this distinct research field are the complexity of systems, as well as the negative effects of mass production upon the quality of products.
Actually, designing complex technical systems against the background of intense technological development from during the last decades brought forward the possibility of relatively frequent malfunctions with unwanted consequences. The need to be ensured against such unwanted events led to adopting appropriate technical measures (which can guarantee the functioning of systems over well-defined periods of time) and to setting rigorous ground rules.
Another element that encouraged the operating safety as a distinct research field is the acknowledgement of the fact that an inappropriate increase of the reliability of products may be highlighted in parallel with the development of mass production (a direct consequence of technological development especially, and of economic- social development in general).
An explanation for this failure comes from the fact that around the year 1950 a clear distinction was made between the concept of good functioning of a system and the operating safety over longer periods of time by repeated interventions (in terms of maintaining the high quality of the operating system) and some specialized studies which could show the evolution in time of the performances of the system.
This distinction could not be made earlier because the traditional disciplines (such as resistance of materials) ensured by correct implementation of safety coefficients, the fact that a system would be functioning during operation as well as over a long period of time (sometimes longer than the forecasted time).
The use of safety coefficients method implies persistent experience where there are no spectacular changes either in the type of technology used or in the materials used.
From this point of view, the new techniques (computers, telecommunications, electronics etc.) are characterized by dynamism and they don’t have enough experience to produce spectacular performances without the help of strictly specialized studies in order to ensure the proper functioning as well as the maintaining of performances over longer periods of time.
Regarded as a **general property of systems**, reliability is the property of preserving the performances of systems [12].
This point of view set forth some considerations about the future of material production and of permanent increase of reliability alongside with technological development.
The specialized literature uses two senses of the term reliability; they are: reliability in a limited sense and reliability in a broad sense.
Reliability in a limited sense refers to the classical meaning of operating safety as a property of systems to preserve their performances. Practically, the operating safety in a limited sense studies the process of performance loss up to the first malfunction.
Reliability in a broad sense includes the concepts of reliability in a limited sense and maintainability (the ability to restore the performances of a system in the long the run) [5]. It is obvious that the maintainability of the system depends on both its architecture and on the way of organizing its service: purchasing spare parts, training service engineers etc. This is why operating safety in a broad sense (also called efficiency by some authors) is more complex and it is characteristic to the study of systems subject to renewal.
Regardless of its meaning (reliability in a limited sense – for studying the operating safety of systems up to the first malfunctioning and reliability in a broad sense - for studying the operating safety of systems subject to renewal) the reliability of a system can be achieved globally or structurally.
The **global reliability** of a system is independent of its structure and it is achieved through the global statistical model. [12], [89].
The **structural reliability** of a system is achieved with the help of the so-called structural models taking into consideration the elements of the system and the relationships among them. These models admit an accurate

description of the reliability of the system (the functional model, the logical model, the failure shaft model, models based on Markov chains) but they have the following disadvantages:

a) they cannot always be implemented in the case of systems subject to renewal;

b) they yield numerous and difficult calculations.

Whatever the disadvantages may be, in the case of using structural models, one can get results which admit a full characterization of the reliability of the systems studied.

Meeting this challenge is an immediate consequence of the fact that through a structural analysis of the system reliability one can establish in fact a relation between the reliability function of the system and the reliability function of its components.

It was acknowledged that achieving the ideal reliability of a complex system is practically impossible. In other words, the idea of creating a system that does not degrade over time is unrealistic whatever the investments made to ensure maximum reliability may be.

This is why, in practical situations, the problem of establishing exactly, at a certain time, the real level of system reliability arises and according to this the operating period, the maintenance period and the safety elements must be determined correctly. The methods for accomplishing this goal are numerous and they are part of difficult calculations based on the use of modern mathematical equipment: optimization, operational calculations, integral calculations, probabilities and statistics.

At the same time, knowing the real operating data of the system, one can determine concrete technical measures (which make the object of maintainability) so that the operating safety of the system improves.

Therefore, knowing the optimum level the reliability of a system may reach represents a major problem.

In specialized literature there are several criteria for determining the optimum level of reliability; the most frequently used ones are the economic ones.

APPLICATIONS OF MINMAX EQUALITY WITHIN PROBLEMS REGARDING CAPITALIZATION OF COMPOUND INTEREST

5.1 The Capitalization Polynomial and Types of Optimum Problems

The general form of capitalization polynomial can be determined from the following elements (figure 22):

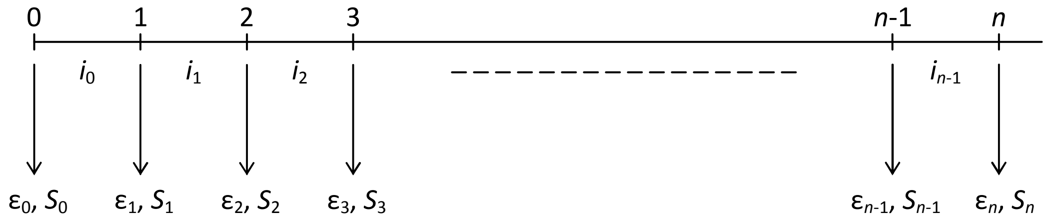

Figure 22

❖ At times $0,1,2,\dots,n$ the following sums are invested or withdrawn $S_0,S_1,\dots,S_n$; this operation is marked by the following symbols $\varepsilon_i S_i,\, i=\overline{0,n}$,

where $\varepsilon_i = \begin{cases} 1, & \text{in case money investments are made;} \\ 0, & \text{in case there are no financial transactions;} \\ -1, & \text{in case cash is withdrawn.} \end{cases}$

❖ $i_1,i_2,\dots,i_n$ represent the unit interests charged during the following intervals $[0,1),[1,2),\dots,[n-1,n)$;

❖ Always $\varepsilon_0 = 1$, and $\varepsilon_n \in \{-1,0\}$, which means that at initially there are cash investments, and at the end there are fund withdrawals or no financial operations at all.

According to the periods of time $0,1,2,\dots,n$ the following capitalizations of compound interest will take place:

$$\begin{cases} \varepsilon_0 S_0 \to \varepsilon_0 S_0(1+i_1)(1+i_2)\dots(1+i_n) \\ \varepsilon_1 S_1 \to \varepsilon_1 S_1(1+i_2)(1+i_3)\dots(1+i_n) \\ \vdots \\ \varepsilon_{n-1} S_{n-1} \to \varepsilon_{n-1} S_{n-1}(1+i_n) \\ \varepsilon_n S_n \to \varepsilon_n S_n \end{cases} \tag{449}$$

Under the circumstances, if P_n is the capitalization polynomial, then its analytical expression is the following:

$$P_n(S_k,\varepsilon_k,k=\overline{1,n}) = \sum_{k=0}^{n} \varepsilon_k S_k \prod_{j=k+1}^{n} (1+i_k) \tag{450}$$

In the case $i_1 = i_2 = \cdots i_n$ we shall note the commune value with i.

If we use the notation $u = 1+i,\, a_k = \dfrac{\varepsilon_k S_k}{\varepsilon_0 S_0}, k=\overline{1,n}$ for the capitalization factor, then the capitalization polynomial can be written as follows:

$$P_n(a_k,\varepsilon_k,k=\overline{1,n};i_k,k=\overline{1,n}) = \varepsilon_0 S_0(u^n + \sum_{k=1}^{n} a_k u^{n-k}) \tag{451}$$

In case $a_1 = a_2 = \dots = a_n = 1$ (which is equivalent to $S_1 = S_2 = \dots = S_n$), the analytical expression of the capitalization polynomial becomes:

$$P(S_0,n) = S_0(u^n + u^{n-1} + \dots + u + 1) = S_0 \frac{u^n - 1}{u-1} \tag{452}$$

Because $u = 1+i$ (u represents the capitalization factor), the capitalization polynomial can be written by taking into consideration the initial invested sum - S_0 and the level of the unitary interest – i, as follows:

$$P(S_0,i)=S_0\frac{(1+i)^{n+1}-1}{i} \tag{453}$$

When studying the properties of compound interest capitalization, the key issues and the context in which they appear are the following:

5.1.1 The Annulment Problem

In this case, i is constant, n and S_k, $k=\overline{0,n}$ are variable, and the problem that has to be solved represents the determination of the solutions of the equation:

$$u^n+\sum_{k=1}^{n}a_k u^{n-k}=0 \tag{454}$$

We shall use the notations $u_1^*,u_2^*,\ldots,u_n^*$ for the solutions of equation (454).

One can easily demonstrate the following properties:

1) $$\left|u_j^*\right|\le 1+max\left\{\left|\frac{a_k}{a_0}\right|\right\},\quad j=\overline{1,n}\ ,\quad k=\overline{1,n}\ ; \tag{455}$$

2) $$\left|u_j^*\right|\le 2\,max\sqrt[k]{\left|\frac{a_k}{a_0}\right|},\quad j=\overline{1,n}\ ,\quad k=\overline{1,n}\ ; \tag{456}$$

3) $$\left|u_j^*\right|\le\left|\frac{a_1}{a_0}\right|+max\sqrt[k-1]{\frac{a_k}{a_1}},\quad j=\overline{1,n}\ ,\quad k=\overline{1,n}\ ; \tag{457}$$

Practically, these increases are not conclusive in many cases. For example, we shall take into consideration the capitalization polynomial corresponding to the following data:

- at times t=0, t=1, we have cash investments $S_0=6$ monetary units and $S_1=1$ monetary unit, respectively;
- at time t=2, we have a financial withdrawal $S_2=12$ monetary units; as a result $P(u)=6u^2+u-12=0$.

The roots of equation $6u^2+u-12=0$ are $u_1=-\frac{3}{2}$, $u_2=\frac{4}{3}$.

It is obvious that $(u_1)=-\frac{3}{2}$, $(u_2)=\frac{4}{3}$, $\frac{S_1}{S_0}=\frac{1}{6}$, $\frac{S_2}{S_0}=-\frac{12}{6}=-2$

Hence it follows:

$$1+max\left\{\left|\frac{1}{6}\right|,\left|-\frac{12}{6}\right|\right\}=3$$

$$2\,max\left\{\sqrt{\frac{1}{6}},\sqrt{\frac{12}{6}}\right\}=2\sqrt{2}$$

$$\frac{1}{6}+\left|-\frac{12}{6}\right|=\frac{13}{6}$$

Remark 5.1

The results below prove the importance of the initial investment.

1) If $S_0=S_1=\ldots=S_n$, the unit interest for which capitalization polynomials are null will be below 1.

2) If $S_0=n,S_1=S_2=\ldots=S_{n-1}=S_n=-1$, then there are no solutions to the problem of the capitalization polynomial annulment. In other words, the equation $nu^n-(u_1^{n-1},u_2^{n-2},\ldots,u+1)=0$ has no solutions.

Economically, we cannot determine an interest for the cases in which we invest a certain amount given by the number of periods, and then, we successively withdraw one monetary unit, every period $t=1,t=2,\ldots,t=n$.

Obviously, the problem can be easily generalized in the case when during each time $t=1,t=2,\ldots,t=n$, there are withdrawals which complies with the property: $S_1+S_2+\ldots+S_n=n$.

Obviously, for the particular case $S_0=1,\ S_1=S_2=...=S_{n-1}=0,\varepsilon_n S_n=-1$, the equation

$u^n - 1 = 0$ will have the following roots: $u_k = cos\frac{2k\pi}{n} + i\,sin\frac{2k\pi}{n}, k = \overline{0,n-1}$. Of all roots, we shall pick out only those roots that have real values.

5.1.2 The Minimum Deviation Problem

For a function of a single variable $f : [a,b] \to \mathbb{R}$, the minimum deviation (in respect with 0), can be defined by size A, given by the following equality:

$$A = \sup_x |f(x)|, x \in [a,b]$$

For example, for the function f defined by the following equality:

$$f = \begin{cases} x - n, x \in \left[n, n + \frac{1}{2}\right] \\ -x + n, x \in \left[n + \frac{1}{2}, n + 1\right] \end{cases} \quad n = 0,1,2,... \tag{458}$$

one can establish the following equalities:

$$\sup_x |f(x)| = \frac{1}{2}, \quad \inf_x |f(x)| = 0, x \in [n,n+1], n = 0,1,2,... \tag{459}$$

Practically, $f(x)$ is defined as the distance between the point x and the nearest point with an integer coordinate on Ox axis. We can immediately observe that f is a periodic function which has the principal period equal to 1 (fig.23).

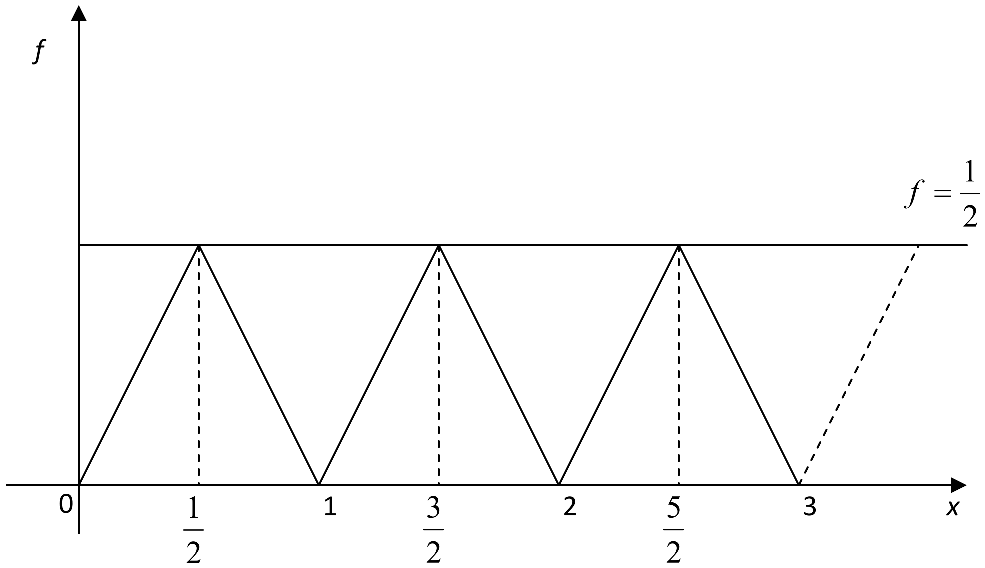

Figure 23

In case of capitalization polynomial, the minimum deviation issue represents an extension of the above enunciated problem.

In this case, n is fixed, i and S_k, $k = \overline{0,n}$ are variable, and the problem that needs to be analyzed is:

$$(P_1) \quad \min_{\tilde{S}_k} \max_i P(i, \tilde{S}_n) \tag{460}$$

where $\tilde{S}_n = (S_0, S_1, \ldots, S_n)$

The main results around the analysis of the problem (P_1) are the following:

1. The minimum deviation of the capitalization polynomial is always greater or equal to the size $\frac{S_0}{2^{n-1}}$;

2. If the capitalization polynomial is of Cebişev type, than the minimum deviation always equals $\frac{S_0}{2^{n-1}}$.

Remark 5.2

Cebişev polynomial of degree *n* is marked with T_n and it can be defined as:

$T_n : [-1,1] \to R, T_n(x) = cos(n \, arccos \, x), n \in N$

The main properties of this polynomial are the following:

- the degree of the polynomial is *n* and the coefficient of x^n is 2^{n-1};
- the extreme points are $x_k = cos\frac{k\pi}{n}, k = \overline{0,n}$;
- the extreme values are ± 1.

In cases $n = 1, n = 2, n = 3$, the analytical expressions of Cebâsev polynomial are as follows:

$$T_1(x) = x, \quad |x| \le 1$$

$$T_2(x) = x^2 - \frac{1}{2}, \quad |x| \le 1$$

$$T_3(x) = 4x^3 - 3x$$

The corresponding graphs are shown in figure 24, figure 25, figure 26.

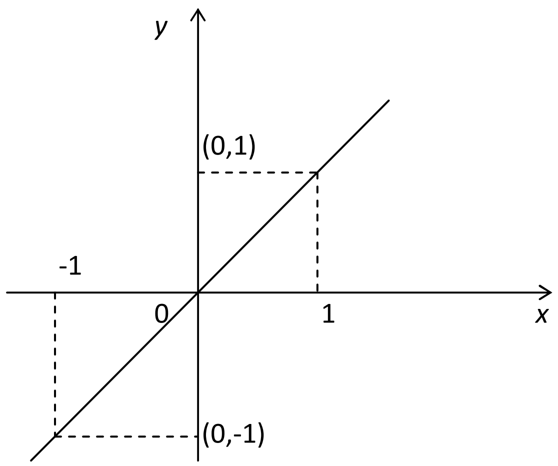

Figure 24

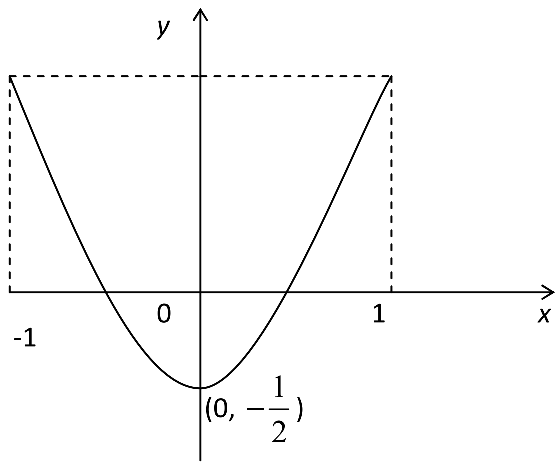

Figure 25

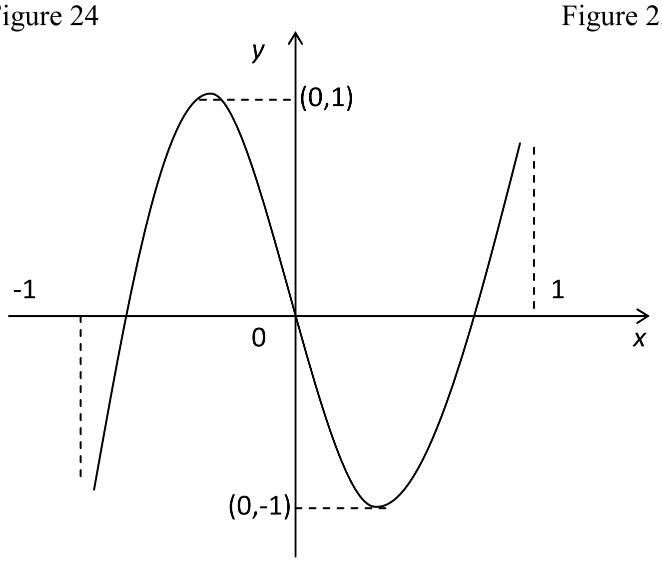

Figure 26

We shall prove that the deviation in relation to zero of any polynomial $P_n(x) = x^n + a_1 x^{n-1} + a_{n-1} x + a_n$ can not be smaller than $\frac{1}{2^{n-1}}$ on the interval [-1,1] and it equals $\frac{1}{2^{n-1}}$ only in the case of Cebâsev polynomial.

Indeed, we shall assume by reduction ad absurdum that there is a polynomial P_n which has the form stated above, so that:

$$\sup_x \left|P_{(x)}\right| < \frac{1}{2^{n-1}} \quad , \quad |x| \le 1 \tag{461}$$

We shall write the polynomial R_n which is defined by the following equality:

$$R_n(x) = P_n(x) - \frac{1}{2^{n-1}} T_n(x) \tag{462}$$

Hence it results that:

$$R_n\left(\cos k\frac{\pi}{n}\right) = P_n\left(\cos k\frac{\pi}{n}\right) - \frac{1}{2^{n-1}} T_n\left(k\frac{\pi}{n}\right), \quad k = \overline{0,n} \tag{463}$$

Since $T_n\left(k\frac{\pi}{n}\right) = (-1)^k \;, \quad k = \overline{0,n}$, we shall have:

$$R(\cos 0) < , R\left(\cos\frac{\pi}{n}\right) > 0 \,, R\left(\cos\frac{2\pi}{n}\right) < 0 \ldots$$

and as a consequence, polynomial R_n is null in each of the following intervals:

$$\left(\cos\frac{\pi}{n}, 0\right), \left(\cos\frac{2\pi}{n}, \cos\frac{\pi}{n}\right), \ldots, \left(\cos n\frac{\pi}{n}, \cos(n-1)\frac{\pi}{n}\right)$$

From this assertion, we are led to a contradiction with our initial hypothesis.
Likewise, we shall demonstrate that neither the equality

$$\sup_x \left|P_{(x)}\right| = \frac{1}{2^{n-1}} \tag{464}$$

can take place unless

$$P_n(x) = \frac{1}{2^{n-1}} T_n(x) \tag{465}$$

5.1.3 The Equilibrium Problem in Simple and Mixed Strategies

In the case of simple strategies, we may consider S_k a known quantity, $k = \overline{0,n}$, i and n variable. The problem that needs to be solved is the following:

$$(P_2) \quad \max_n \min_i P(n,i) = \min_i \max_n P(n,i)$$

This problem has a saddle point in simple strategies; the saddle point is given by the pair $(n,0)$. From economic point of view, this case has no importance because it implies the condition that the unitary interest equals zero: $i = 0$.

De facto, from minmax optimality conditions we can immediately prove that the unitary interest that we are searching for represents the solution of the following equation:

$$(1+i)^n = \frac{1}{1-in} \tag{466}$$

The graphical representations of functions $f_1(i) = (1+i)n$, $f_2(i) = \frac{1}{1-in}$ (fig.27) show that $i = 0$ represent the unique solution of the above equation.

In economic practice, for $\alpha > 0$ and sufficiently close to 0, we can approach the problem of using a positive value of the interest so that the area of the curvilinear triangle (the shaded part from the graphical representation) should be increased by $\varepsilon > 0$ - a fixed value which is sufficiently small. Economically thinking, we are interested to determine a saddle point -ε; from mathematical point of view, we must solve the following equation:

$$\frac{1}{n}\ln\frac{1}{1-n\alpha} - \frac{(1+\alpha)^{n+1} - 1}{n+1} = \varepsilon \tag{467}$$

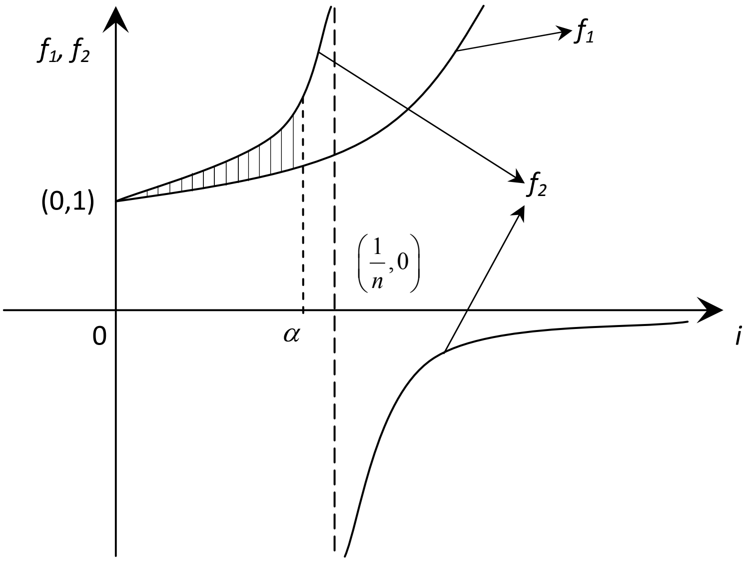

Figure 27

It is extremely difficult to solve the above equation in respect of α, because this is a transcendental equation.

We can approximate this area by using the quantity S, given by the following equality:

$$S = \frac{\left(\frac{1}{1-4\alpha} - 1\right)^{\alpha}}{2} \tag{468}$$

Therefore, equation (467) becomes:

$$\frac{n\alpha^2}{2} = \varepsilon \tag{469}$$

and the interest, approximated by α, results from:

$$i \approx \sqrt{\frac{2\varepsilon}{n}} \tag{470}$$

For the particular case $\varepsilon = \frac{1}{8n}$, we get $i \approx \frac{1}{2n}$; this result is known from [50], [54]. The dependence between i and n is shown by figure 28.

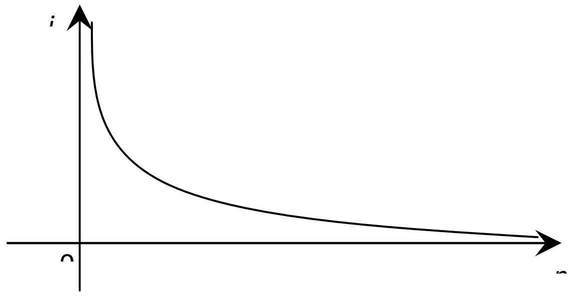

Figure 28

5.2 Maxmin and Minmax Capitalization Problems in the case of Variable Interest

5.2.1 The Formulation of the Problem

We shall start from the following elements:

❖ At the moments *1, 2, ..., k* the following financial operations take place:

- we deposit the sums $S_1, S_2, \dots, S_k$, $\sum_{j=1}^{k} S_j = A$, A being fixed
- we withdraw the sums $s_1, s_2, \dots, s_k$, $\sum_{j=1}^{k} s_j = B$, B being fixed

❖ $i_1, i_2, \dots, i_k$ represent the unit interest values used for the intervals [0,1), [1,2), …, [k-1,k).

The problem is to determine the amounts needed to be deposited and withdrawn in order to obtain the maximum capitalized sum in the moment k. Because the withdrawal of sums $s_1, s_2, \dots, s_k$ implies actually the withdrawal of $\overline{s_1}, \overline{s_2}, \dots, \overline{s_k}$ which represent the potential capitalization using compound interest for $s_1, s_2, \dots, s_k$ we get (table 1):

Moment	The withdrawn sum	The capitalized sum
1	s_1	$\overline{s_1} = s_1(1+i_2)(1+i_3)\dots(1+i_k)$
2	s_2	$\overline{s_2} = s_2(1+i_3)(1+i_4)\dots(1+i_k)$
$\vdots$	$\vdots$	$\vdots$
t	s_t	$\overline{s_t} = s_t(1+i_{t+1})\dots(1+i_k)$
$\vdots$	$\vdots$	$\vdots$
k	s_k	$\overline{s_k} = s_k$

Table 1

Remark 5.3

If we note $\pi = (1+i_1)(1+i_2)\dots(1+i_k)$, then the following equalities take place:

$$\begin{cases} \overline{s_1} = s_1 \dfrac{\pi}{1+i_1} \\ \overline{s_2} = s_2 \dfrac{\pi}{(1+i_1)(1+i_2)} \\ \vdots \\ \overline{s_k} = s_k \dfrac{\pi}{(1+i_1)(1+i_2)\dots(1+i_k)} \end{cases} \tag{471}$$

We mark with f the adopted efficiency function, $f: R^2 \to R$ defined by the following equation:

$$f(S,s) = \sum_{j=1}^{k} max\{S_j - c_j s_j; 0\} \tag{472}$$

where $S = (S_1, S_2, \dots, S_k), s = (s_1, s_2, \dots, s_k)$

$$c_j = \frac{\pi}{(1+i_1)(1+i_2)\dots(1+i_j)}, j = \overline{1,k}$$

Remark 5.4

Since the efficiency function f is not continuous, it is obvious that it is not differentiable either, so the regular operations related to the differential calculation have no effect. The formulated problem may be interpreted as an uncooperative game problem with two decision makers where the efficiency functions W_1 and W_2 admit the following analytical expressions:

$$W_1(S,s) = \sum_{j=1}^{k} max\{S_j - c_j s_j; 0\} \tag{473}$$

$$W_2(S,s) = \sum_{j=1}^{k} min\{S_j - c_j s_j; 0\} \tag{474}$$

We notice that the following relation is verified:

$$W_1(S,s) = -W_2(S,s) \tag{475}$$

therefore, we are dealing with a zero sum game. This game does not present equilibrium points for simple strategies because of the efficiency functions particular forms; consequently the main problem regarding this issue is to determine the guaranteed optimal strategies and values.

5.2.2 Solving the Problem and Economic Interpretation

For the following equations:

$$\max_{s}\min_{S} W_2(S,s) = -\min_{s}\max_{S} W_2(S,s) = \min_{s}\max_{S} W_1(S,s) \tag{476}$$

$$\min_{S}\max_{s} W_2(S,s) = -\max_{S}\min_{s}(-W_2(S,s)) = +\max_{S}\min_{s} W_1(S,s) \tag{477}$$

it is obvious that we need to determine only two guaranteed optimal values (and the corresponding guaranteed optimal strategies). Practically, we need to analyze the following problems:

$$(P_1)\max_{s}\min_{S} W_2(S,s) \tag{478}$$

$$(P_1)\max_{s}\min_{S} W_2(S,s) \tag{479}$$

The problem (P_1) is solved using the equalization principle [29], [47]:

- the optimal solution $s^* = (s_1^*, s_2^*, ..., s_k^*)$ that we are searching for, is determined by solving the following algebraic system:

$$\begin{cases} c_1 s_1 - B = c_j s_j - B, j = \overline{2,k} \\ \sum_{j=1}^{k} s_j = B \end{cases} \tag{480}$$

After an immediate calculation, we get:

$$s_j^* = \frac{1}{c_j}\frac{B}{\sum_{j=1}^{k}\frac{1}{c_j}}, j = \overline{1,k} \tag{481}$$

$$\text{therefore } \begin{cases} s_1^* = \frac{1}{c_1}\frac{B}{D} \\ s_2^* = \frac{1}{c_2}\frac{B}{D} \\ \vdots \\ s_k^* = \frac{1}{c_k}\frac{B}{D} \end{cases}, \tag{482}$$

where $D = \sum_{j=1}^{k}\frac{1}{c_j}$

- the minmax guaranteed optimal value is given by the following relationship:

$$\max_{s}\min_{S} W_2(S,s) = \min\left(\frac{B}{\sum_{j=1}^{k}\frac{1}{c_j}} - A; 0 \right) \tag{483}$$

Obviously, the problem (P_2) $\min_{S}\max_{s} W_2(S,s)$ is equivalent with the problem (P_3) $\max_{S}\min_{s} W_1(S,s)$ and it can be also solved by using the equalization principle [54]. Assuming that $c_1 \geq c_2 \geq c_3 \geq ... \geq c_k$, the optimal solution that we are looking for is the following:

$$s_i^* = \begin{cases} \min\left(B - \sum_{j=1}^{i-1} s_j^*; S_i/c_i \right), s_1^* = \min\left(B; \frac{S_1}{c_1} \right), \text{if } \min\left(B - \sum_{j=1}^{i-1} s_j^*; \frac{S_i}{c_i} \geq 0 \right) \\ 0, \text{contrariwise} \end{cases} \tag{484}$$

The guaranteed optimal solution type minmax is the following:

$$\min_S \max_s W_2(S,s) = \min\left\{\min\left[c_i B - A\right];0\right\} \tag{485}$$

Remark 5.5

It is obvious that $\max_S \min_s W_1(S,s) = \min\left\{\min_i\left[c_i B - A\right];0\right\}$ (486)

Remark 5.6

Since W_1 is convex reported to the *s* variable, it is obvious that the game value is given by the guaranteed optimal value of W_1 in minmax sense.

Taking into consideration the equation $\max_s \min_S W_2(S,s) = \min_s \max_S W_1(S,s)$, the following relation takes place:

$$\min_s \max_S W_1(S,s) = \max\left[A - \frac{B}{\sum_{i=1}^{k} \frac{1}{c_i}};0\right] \tag{487}$$

Because $c_1 = \frac{\pi}{1+i_1}, c_2 = \frac{\pi}{(1+i_1)(1+i_2)}, ..., c_k = \frac{\pi}{(1+i_1)(1+i_2)...(1+i_k)}$, after an immediate calculation we shall get:

$$\sum_{j=1}^{k} \frac{1}{c_j} = \frac{1}{\pi}\left((1+i_1)+(1+i_1)(1+i_2)+...+(1+i_1)(1+i_2)...(1+i_k)\right) \tag{488}$$

and, consequently the optimal strategies that we are searching for are given by the following relations:

$$s_j^* = B\left(\frac{(1+i_1)(1+i_2)...(1+i_k)}{(1+i_1)+(1+i_1)(1+i_2)+...+(1+i_1)(1+i_2)...(1+i_k)}\right), j=\overline{1,k} \tag{489}$$

Particular cases

1) If $i_1 = i_2 = … = i_k$ and we mark with *I* the common value, then the optimal strategies that we are searching for allow the following expression:

$$s_j^* = \frac{B(1+i)^j}{(1+i)+(1+i)^2+...+(1+i)^k}, j=\overline{1,k} \tag{490}$$

hence

$$s_j^* = \frac{B_i(1+i)^{j-1}}{(1+i)^k - 1}, j=\overline{1,k} \tag{491}$$

If *I* is small enough, then $(1+i)^k \approx 1+ik$; consequently

$$s_j^* = \frac{B}{k}(1+i)^{j-1}, j=\overline{1,k} \tag{492}$$

We can immediately observe that the optimal solutions are given by the product between the ratio $\frac{B}{k}$ and the powers of the capitalization factor *(1+i)*.

Furthermore, all the optimal solutions are found on the line:

$$s = \frac{B}{k}\left(1+i(x-1)\right) \tag{493}$$

if we consider that $(1+i)^{j-1} = 1+i(j-1), j=\overline{1,k}$

Consequently, if $a = \frac{B}{k}$, $b = \frac{B}{k}i$, we have (figure 29):

$$
\begin{aligned}
&s_1 = a \\
&s_2 = a + b \\
&s_3 = a + 2b \\
&\vdots \\
&s_k = a + (k-1)b
\end{aligned}
$$

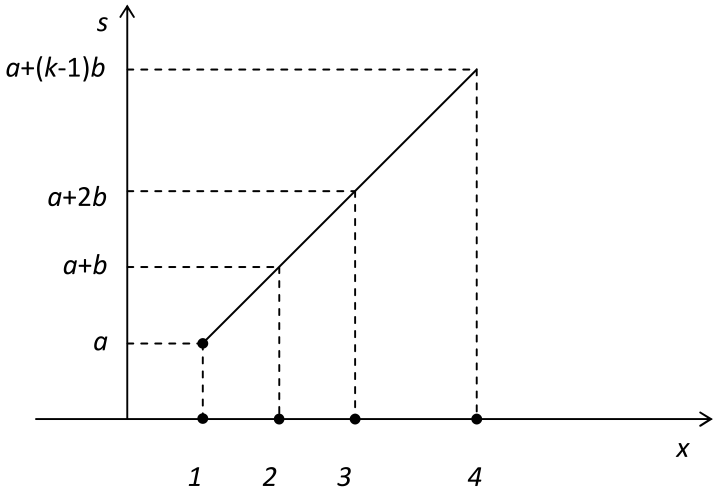

Figure 29

In this case, the optimum guaranteed values are the following [53]:

a) $\min_s \max_S W_1(S,s) = \max\left(A - \frac{B(1+i)}{k}; 0 \right)$ (494)

This size must have a non-zero value; this leads us to the condition that the capitalization factor has to verify the following inequality: $1 + i \prec \frac{A}{B} \prec k$. In other words, the unit interest used must meet the requirement: $i \prec \frac{A}{B} k - 1$.

b) $\max_S \min_s W_1(S,s) = \min\left\{ \min_i [c_i B - A]; 0 \right\}$ (495)

This optimum guaranteed value has to be non-zero; in order to meet this condition, it is necessary that the unit interest should verify the inequality: $i \prec \frac{A-1}{B(k-j)}$.

Let us assume that the unitary interests have the following form:

$$i_1 = a \,;\, i_2 = a^2 \,;\, i_3 = a^4 \,;\ldots;\, i_k = a^{2^k}$$

In this case, we can prove by induction the following equality:

$$(1+i_1)(1+i_2)\ldots(1+i_k) = (1+a)(1+a^2)\ldots\left(1+a^{2^k}\right) = 1 + a + a^2 + \ldots a^{2^{k+1}-1} = \frac{1-a^{2^{k+1}}}{1-a} \quad (496)$$

The following equalities will consequently take place:

$$
\begin{cases}
1+a=\dfrac{1-a^{2^{0+1}}}{1-a} \\
\left(1+a\right)\left(1+a^2\right)=\dfrac{1-a^{2^{1+1}}}{1-a} \\
\left(1+a\right)\left(1+a^2\right)\left(1+a^{2^2}\right)=\dfrac{1-a^{2^{2+1}}}{1-a} \\
\vdots \\
\left(1+a\right)\left(1+a^2\right)\dots\left(1+a^{2^k}\right)=\dfrac{1-a^{2^{k+1}}}{1-a}
\end{cases} \tag{497}
$$

Therefore, the optimum strategies that we are searching for have the form below:

$$
s_j^* = B\frac{1-a^{2^{k+1}}}{\left(1-a^{2^{0+1}}\right)+\left(1-a^{2^{1+1}}\right)+\left(1-a^{2^{2+1}}\right)+\dots+\left(1-a^{2^{j+1}}\right)} \tag{498}
$$

So

$$
s_j^* = B\frac{1-a^{2^{j+1}}}{\left(j+1\right)-\left(a^{2^1}+a^{2^2}+\dots a^{2^{j+1}}\right)}\ , \quad j=\overline{1,k} \tag{499}
$$

Because the values of interests are smaller than 1, it is obvious that the following sum: $a^2+a^4+a^8+a^{16}+a^{32}+\dots+a^{2^{k+1}}$ can be written as $a^2+\left(a^4\right)^1+\left(a^4\right)^2+\left(a^4\right)^4+a^{32}+\dots a^{2^{k+1}}$, from which, we can keep the first three terms maximum.

By performing calculations, we shall get:

$$
\begin{cases}
s_1^* = B\dfrac{1-a^4}{2-\left(a^2+a^4\right)}=\dfrac{B\left(1+a^2\right)}{2+a^2} \\
s_2^* = B\dfrac{1-a^8}{3-\left(a^2+a^4+a^8\right)}=\dfrac{B\left(1+a^2+a^4\right)}{3+2a^2+a^4} \\
s_4^* \approx B\dfrac{1}{5-\left(a^2+a^4\right)}=\dfrac{B}{3+\left(1-a^2\right)\left(2+a^2\right)} \\
\vdots \\
s_k^* \approx B\dfrac{1}{k+1-\left(a^2+a^4\right)}=\dfrac{B}{k-1+\left(1-a^2\right)\left(2+a^2\right)}
\end{cases}
$$

Obviously, for a sufficiently small, we can consider the following approximation:

$$
\left(1-a^2\right)\left(2+a^2\right)\approx 2-a^2
$$

5.2.3 The Ideal Interest

We assume that at the time $t=0$ we invest the sum S_0 and at times $1,2,...,n$ no other financial operations take place; therefore $S_1=S_2=...=S_n=0$, $s_1=s_2=...=s_n=0$. For each moment $t\succ p$ only fund withdrawals are made. The problem which arises aims to determine the unit interest *i* needed to be used in order to withdraw the sum S_0 at times $t=p+1, t=p+2,...$ (in other words, $s_{p+1}=s_{p+2}=...=S_0$).

In view of the specific form of this problem, we can find its solution by solving the following equation:

$$
S_0(u^k-(u^{n-p-1}+u^{n-p-2}+...+u+1))=0 \tag{500}
$$

where $u=1+i$ represents the capitalization factor.

Practically, this equation can be written as follows:

$$u^k - \frac{u^{k-p} - 1}{u-1} = 0 \tag{501}$$

and, accordingly, we are led to solve the following equation:

$$u^{k-p}(u^{p+1} - u^p - 1 + \frac{1}{u^{k-p}}) = 0 \tag{502}$$

Because the capitalization factor u is always non-zero and $\frac{1}{u^{k-p}}$ is sufficiently closed to 0 for *k* great enough, we are led to solve the following higher degree equation:

$$(1+i)^{p+1} - (1+i)^p - 1 = 0 \tag{503}$$

where the variable *i* represents the unit interest that we are searching for.

It is obvious that this equation can be written in the following equivalent form: $i(1+i)^p - 1 = 0$ and admits a single solution $i \in (0,1)$.

Remark 5.7

If $p = 1$, then the unit interest that we are searching for is the positive solution of the equation $i^2 + i - 1 = 0$; therefore $i = \frac{-1+\sqrt{5}}{2} \approx 0.6$.

Generally, the equation of "the optimum interest" $i(1+i)^p - 1 = 0$ is a higher degree equation and it can only be solved by means of approximate methods.

The most convenient method implies, practically, the linearization of the expression $(1+i)^p$. We can immediately observe that $(1+i)^p \approx 1 + ip$ when *i* is small enough.

In this case, the equation $i(1+i)^p - 1 = 0$ turns into $i^2 p + i - 1 = 0$ and the positive solution of this equation is the following:

$$i = \frac{-1+\sqrt{1+4p}}{2p} \tag{504}$$

If we linearize the factor $(1+i)^p$ in an arbitrary chosen point $i_0 \in (0,1)$, then the "optimum interest equation" becomes:

$$(1+i_0)(1+pi+i_0-pi_0)i = 1 \tag{505}$$

Because the waiting period *p* is in inverse ratio to the optimum interest, it is very convenient form calculation point of view to consider $i_0 = \frac{1}{p-1}$. In this case, the previous equation becomes $\frac{p^2 i^2}{p-1} = 1$ and, therefore, the unit interest that we are searching for is:

$$i = \frac{\sqrt{p-1}}{p} \tag{506}$$

We can immediately observe that for $p = 2$, the optimum interests $i = \frac{\sqrt{p-1}}{p}$ and $i = \frac{-1+\sqrt{1+4p}}{2p}$ coincide and this result is in concordance with the graphical images of this interests (fig.30).

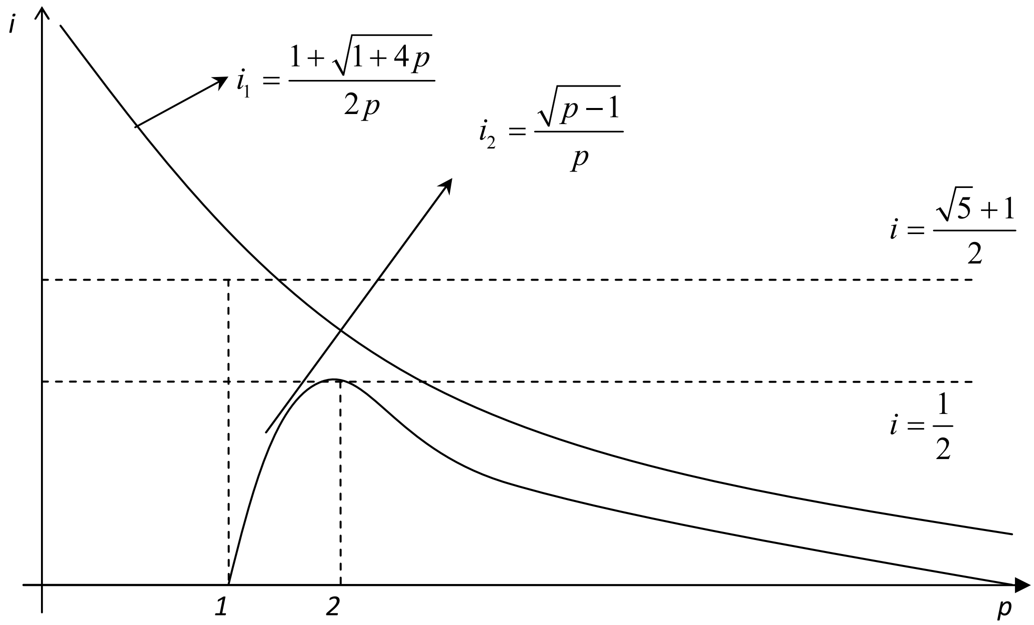

Figure 30

5.3 The Determination of Market Equilibrium Interest

5.3.1 The Case When the Elasticity Coefficients of the Funds Demand and Supply are Linear

The problem of determining the market equilibrium interest may be approached as an optimum problem type maxmin for the capitalization polynomial. In a simplified form, for the moments *t=0, t=1, ..., t=n* we place the amount S_0 with the unit interest *i* as composite interest, the capitalized amount is noted with $S = S(n,i)$ and is determined as (the capitalization polynomial):

$$S(n,i) = \frac{(1+i)^n - 1}{i} \tag{507}$$

The equilibrium interest i^* is determined as solution of the problem

$$(P)\max_n \min_i S(n,i) = \min_i \max_n S(n,i) \tag{508}$$

and after calculation, practically, the searched optimal interest is the solution of the following equation.

$$(1+i)^n = \frac{1}{1-ni} \tag{509}$$

On the monetary market, the problem of equilibrium interest may be approached in differential form, starting from the following elements:

❖ The capitalized sum is denoted by *f* and depends on the unitary interest *i*; hence $f = f(i)$ or, if we pertain to the capitalization factor $1+i$, we shall have: $f = f(1+i)$;

❖ The problem that faces us is to approximate the capitalized sum *f* by the polynomial P_n with the fixed degree *n*. This polynomial will have the following form: $P_n(i) = \sum_{k=0}^{n} a_k i^k$ and its coefficients $a_0, a_1, ..., a_n$ will be undetermined;

❖ The efficiency function *F* is defined by the following equality:

$$F(i,a) = -\left(f(i) - \sum_{k=0}^{n} a_k i^k \right)^2, \ a = (a_0, a_1, ..., a_n) \tag{510}$$

The main results can be synthesized as follows:

❖ The following equality will always take place:

$$\max_{a} \min_{i} F(i,a) = -\min_{a} \max_{i}(-F(i,a)) \tag{511}$$

❖ The function F is a convex function with respect to the variable *a*, and consequently, we can ascertain the following results for the two-person game for which *F* represents the payoff function:

- o The game has a saddle point;
- o The decision maker which will choose *a* (as a decisional alternative), will have an optimum simple strategy;
- o For the second decision maker, there is an optimum mixed strategy which corresponds to at most *n+2* simple strategies $i_j, j = \overline{1,n+2}$.

❖ Let us consider $P = (p_1, p_2, ..., p_{n+2})$ a certain mixed strategy for which p_j represents the probability of using the simple strategy i_j; obviously, $p_j \geq 0, j = \overline{1,n+2}, \sum_{j=1}^{n+2} p_j = 1$.

If $I = (i_1, i_2, ..., i_{n+2})$ represents a vector which consists of various levels of interest, we can consider the game with the payoff function F_1, defined as follows:

$$F_1(P,I,a) = \sum_{j=1}^{n+2} p_j \left(f(i_j) - \sum_{k=0}^{n} a_k i_j^k \right)^2$$

The problem of determining the saddle point for the game with the payoff function F_1 consists in solving two optimum problems in the following way: if P_0, I_0, a_0 are the components of the saddle point, then:

$$F_1(P_0, I_0, a_0) = \min_{a} F_1(P_0, T_0, a) = \max_{P,T} F_1(P, T, a_0)$$

Practically, Cebâşev's approximation issue was written in an equivalent form as a game problem and this problem is equivalent with two extremal problems. These problems can be relatively easy solved.

❖ A few important conclusions can be drawn:

- o If $\overline{a} = (\overline{a}_0, \overline{a}_1, ..., \overline{a}_n)$ is the optimum vector that we were searching for, then for the efficiency function *F*, there are $n+2$ values of unitary interest for which *F* attains, alternatively, the minimum and the maximum value;
- o The modules of the minimum and maximum values coincide.

1) r_1 is the minimum interest at which are constituted the bank deposits and r_2 is the minimum interest at which the credit is given;
2) If we apply this interests, we shall note with O_0 the value of deposits and with C_0 the value of credit demand (obviously, O_0 and C_0 are known values);
3) e_1 is the function of elasticity of saving through deposits, and e_2 is the function of elasticity of credit demand. We shall consider e_1 and e_2 as linear functions in respect with interest rate *r*, where:

$$e_i = a_i r + b_i, i = \overline{1,2} \quad (a_1, a_2, b_1, b_2 \text{ values are known}) \tag{512}$$

4) Both the credit supply *O* and the credit demand *C* depend on interest rate *r*, that is $O = O(r)$, $C = C(r)$. The analytical expressions of functions *O* and *C* are known.

The problem can be solved starting from the definition of elasticity functions e_1 and e_2.

The elasticity of credit supply (savings through deposits) e_1 is determined based on the following relation:

$$e_1 = \frac{\frac{\Delta O}{O}}{\frac{\Delta r}{r}} \tag{513}$$

Otherwise, e_1 is the relation between the absolute variations rapports (ΔO, respectively Δr) and the reference levels (*O*, respectively *r*).

If the absolute variations are sufficiently small, the previous equality can write in differential form, like this:

$$e_1 = \frac{\frac{dO}{O}}{\frac{dr}{r}} \tag{514}$$

d being the differential operator.

Therefore, we are lead to the first order differential equation [50]:

$$\frac{dO(r)}{O(r)} = (a_1 r + b_1)\frac{dr}{r} \tag{515}$$

Actually, this is an equation with separable variables.

If r^* is the equilibrium interest level of the market (r^* is that interest for which the equality $O(r^*)=C(r^*)$ takes place) and r_1 is the minimum interest level at which the bank deposits are constituted, by integrating the last equation, we obtain:

$$\int_{r_1}^{r^*} \frac{dO(r)}{O(r)} = \int_{r_1}^{r^*} (a_1 r + b_1)\frac{dr}{r} \tag{516}$$

from where immediately results:

$$\ln O(r)\Big|_{r_1}^{r^*} = a_1(r^* - r_1) + b_1 \ln r\Big|_{r_1}^{r^*} \tag{517}$$

and the following relation is verified:

$$\ln O(r^*) - \ln O(r_1) = a_1(r^* - r_1) + b_1(\ln r^* - \ln r_1) \tag{518}$$

Taking account of $O(r_1)=O_0$, from the equality (518), we get:

$$\ln O(r^*) = a_1 r^* + b_1 \ln r^* + \ln O_0 - a_1 r_1 - b_1 \ln r_1 \tag{519}$$

Applying the same technique for the credit demand, we get the following equality:

$$\ln C(r^*) = a_2 r^* + b_2 \ln r^* + \ln C_0 - a_2 r_2 - b_2 \ln r_2 \tag{520}$$

Taking into consideration that $O(r^*)=C(r^*)$, from equalities (519) and (520), it immediately results the following equality:

$$a_1 r^* + b_1 \ln r^* + \ln O_0 - a_1 r_1 - b_1 \ln r_1 = a_2 r^* + b_2 \ln r^* + \ln C_0 - a_2 r_2 - b_2 \ln r_2 \tag{521}$$

If we note: $A = \ln C_0 - \ln O_0 + a_1 r_1 - a_2 r_2 + b_1 \ln r_1 - b_2 \ln r_2$, the previous equality becomes:

$$r^*(a_1 - a_2) + (b_1 - b_2)\ln r^* = A \tag{522}$$

Practically, the equality (521) is an equation that leads us to the values of the searched equilibrium interest r^*. Because the equation (521) is a transcedentary equation, its solutions can only be approximated, by using special techniques of solving these kind of equations.

Equation (522) can be written in the following form:

$$\ln r^* = \frac{A}{b_1 - b_2} - \frac{a_1 - a_2}{b_1 - b_2} r^* \tag{523}$$

From the diagrams of functions $y = \ln r^*$, $y = \frac{A}{b_1 - b_2} - \frac{a_1 - a_2}{b_1 - b_2} r^*$, it is immediately noticeable that there exist a single root of equation placed in the (0,1) interval, if $\frac{A}{b_1 - b_2} \le 0$, $\frac{a_1 - a_2}{b_1 - b_2} > 0$ (figure 31), or placed in the $\left(1, \frac{A}{a_1 - a_2}\right)$ interval, if $\frac{A}{b_1 - b_2} > 0$, $\frac{a_1 - a_2}{b_1 - b_2} > 0$ (figure 32).

After the determination of the interval that contains the solution of equation (521), this solution can be found immediately by using one of the forenamed methods.

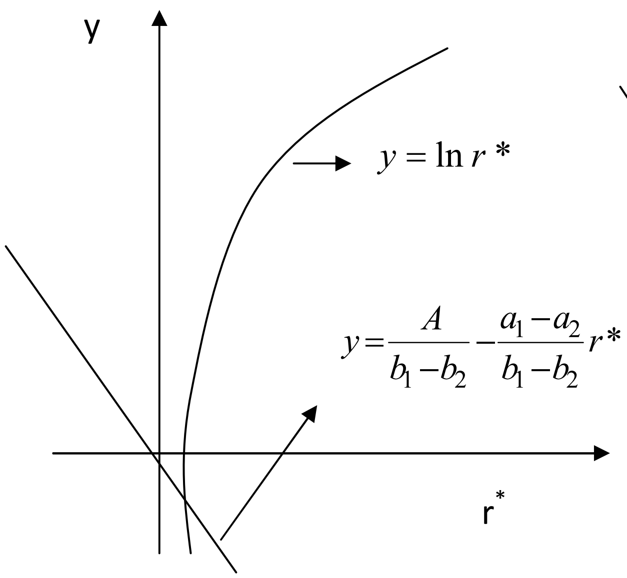

Figure 31

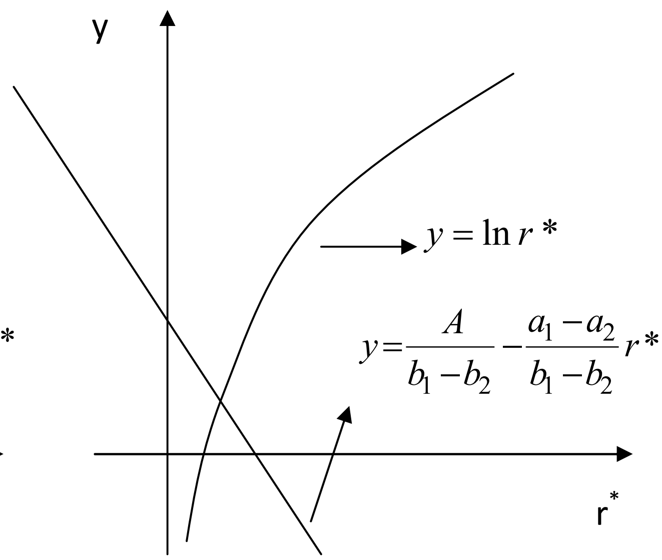

Figure 32

Remark 5.8

By linearizing the logarithmic function through Taylor's method, in point $r_0=1$, we obtain $ln\, r = r-1$ and the equation (522) becomes:

$$r^* - 1 = \frac{A}{b_1-b_2} - \frac{a_1-a_2}{b_1-b_2} r^*$$

This solution of this equation is:

$$r^* = \frac{A+b_1-b_2}{a_1-a_2+b_1-b_2} \tag{524}$$

It is obvious that the solution (524) represents an approximate solution of the equation (523). If we apply the method of successive approximation, then the solution that we are searching for - r^* - can be determined as the limit of the sequence $(r_n)n$, where

$$r_n = \frac{1}{B^n} - \left(\overline{A}+1\right)\left(\frac{1}{B^n}+\frac{1}{B^{n-1}}+\cdots+\frac{1}{B}\right) \tag{525}$$

and

$$\overline{A} = \frac{A}{b_1-b_2} \quad , \quad B = \frac{a_2-a_1}{b_1-b_2} \tag{526}$$

Because $r^* = \lim_n r_n$, after performing calculations, we shall get:

$$r^* = \frac{A+b_1-b_2}{a_1-a_2} + \frac{b_1-b_2}{b_1-b_2+a_1-a_2} \tag{527}$$

and, consequently:

$$r^* = \frac{ln\frac{C_0}{O_0} + ln\frac{b_1^{r_1}}{b_2^{r_2}} + a_1 r_1 - a_2 r_2}{a_1-a_2} + \frac{b_1-b_2}{b_1-b_2+a_1-a_2} \tag{528}$$

Particular cases

The equilibrium interest can be determined immediately in some particular cases of supply elasticity, respectively of demand elasticity.

Case I: We analyze the following situation: $b_1 = b_2 = 0$

In this case, we have $e_1 = a_1 r,\ \ e_2 = a_2 r$, and the equilibrium interest is solution of the following equation:

$$r^*(a_1-a_2) = A\,. \tag{529}$$

Accordingly, it results:

$$r^{*}=\frac{A}{a_1-a_2}=\frac{ln\frac{C_0}{O_0}+a_1r_1-a_2r_2}{a_1-a_2} \tag{530}$$

Case II: In the second situation, we assume that: $a_1=a_2=0$

In this case, the elasticity functions admit the analytical expressions $e_1=b_1, e_2=b_2$ (therefore, the elasticity functions are constant).

The equation (523) becomes:

$$ln\, r^{*}=\frac{A}{b_1-b_2} \tag{531}$$

Taking into account that: $A=ln\frac{C_0}{O_0}+ln\frac{r_1^{b_1}}{r_2^{b_2}}=ln\frac{C_0}{O_0}\frac{r_1^{b_1}}{r_2^{b_2}}$, the equation (531) can be written as follows:

$$ln\, r^{*}=ln\left(\frac{C_0}{O_0}\frac{r_1^{b_1}}{r_2^{b_2}}\right)^{\frac{1}{b_1-b_2}} \tag{532}$$

and, therefore, the equilibrium interest is: $r^{*}=\left(\frac{C_0}{O_0}\frac{r_1^{b_1}}{r_2^{b_2}}\right)^{\frac{1}{b_1-b_2}}$, i.e:

$$r^{*}=\sqrt[b_1-b_2]{\frac{C_0}{O_0}\frac{r_1^{b_1}}{r_2^{b_2}}} \tag{533}$$

Obvious, if the minimum interests are equal, $r_1=r_2$, we obtain: $r^{*}=\sqrt[b_1-b_2]{\frac{C_0}{O_0}}$. Thus, the equilibrium interest depends only on the minimal values of the credit supply and demand, as well as on the elasticity of savings through deposits, and of elasticity of credit demand respectively.

5.3.2 The General Case

Unlike the previous case, we suppose that the analytical expressions of elasticity functions e_1 (savings through deposits) and e_2 (credit demand) are known. Obvious, if r is the interest rate, we have $e_1 = e_1(r)$, $e_2 = e_2(r)$. We shall use the same notations as in the previous paragraph.

We shall start from the economical signification of elasticity functions e_1 and e_2, and we will be lead to the following differential equation [50]:

- for the supply: $$\frac{dO}{O}=\frac{e_1(r)}{r}dr \tag{534}$$
- for the demand: $$\frac{dC}{C}=\frac{e_2(r)}{r}dr \tag{535}$$

By integrating both members of equation (534), it immediately results:

$$\int_{O(r_1)}^{O(r)}\frac{dO(r)}{dr}=\int_{r_1}^{r_2}\frac{e_1(r)}{r}dr \tag{536}$$

If G_1 is a primitive of $\frac{e_1(r)}{r}$ function, from (536) we obtain:

$$ln\, O(r)-ln\, O(r_1)=G_1(r)-G_1(r_1) \tag{537}$$

Similarly, by integrating both members of equation (535), we obtain:

$$\int_{C(r_1)}^{C(r)}\frac{dC(r)}{dr}=\int_{r_1}^{r_2}\frac{e_2(r)}{r}dr \tag{538}$$

If G_2 is a primitive of function $\frac{e_2(r)}{r}$, from (538) we obtain:

$$ln\,C(r)-ln\,C(r_1)=G_2(r)-G_2(r_2) \tag{539}$$

By denoting the equilibrium interest with r^*, is obvious that the market equilibrium is given by the following condition:

$$O(r^*)=C\left(r^*\right) \tag{540}$$

Accordingly, the equilibrium interest r^* is determined as a solution of the following equation:

$$G_1\left(r\right)-G_2\left(r\right)=-ln\,O\left(r_1\right)+ln\,C\left(r_2\right)+G_1\left(r_1\right)-G_2\left(r_2\right) \tag{541}$$

We note:

$$A=-ln\,O\left(r_1\right)+ln\,C\left(r_2\right)+G_1\left(r_1\right)-G_2\left(r_2\right) \tag{542}$$

$\varepsilon_1,\varepsilon_2$ the assumed errors for the deviation from the equilibrium interest in respect with the savings through deposits, and credit demand respectively.

We develop in Taylor series the primitive G_1 and G_2 in $r_1+\varepsilon_1, r_2+\varepsilon_2$ points and we keep just the first two terms. Taking into account that $G_1'\left(r\right)=\dfrac{e_1\left(r\right)}{r}, G_2'\left(r\right)=\dfrac{e_2\left(r\right)}{r}$, after an immediately calculation, it result that:

$$G_1\left(r\right)=G_1\left(r_1+\varepsilon_1\right)+r\frac{e_1\left(r_1+\varepsilon_1\right)}{r_1+\varepsilon_1}-e_1\left(r_1+\varepsilon_1\right) \tag{543}$$

$$G_2\left(r\right)=G_2\left(r_2+\varepsilon_2\right)+r\frac{e_2\left(r_2+\varepsilon_2\right)}{r_2+\varepsilon_2}-e_2\left(r_2+\varepsilon_2\right) \tag{544}$$

Accordingly, we are lead to the following equation:

$$\begin{gathered}G_1\left(r_1+\varepsilon_1\right)+r\frac{e_1\left(r_1+\varepsilon_1\right)}{r_1+\varepsilon_1}-e_1\left(r_1+\varepsilon_1\right)-G_2\left(r_2+\varepsilon_2\right)-\\-r\frac{e_2\left(r_2+\varepsilon_2\right)}{r_2+\varepsilon_2}+e_2\left(r_2+\varepsilon_2\right)=A\end{gathered} \tag{545}$$

from where:

$$\begin{gathered}r\left[\frac{e_1\left(r_1+\varepsilon_1\right)}{r_1+\varepsilon_1}-\frac{e_2\left(r_2+\varepsilon_2\right)}{r_2+\varepsilon_2}\right]=A-G_1\left(r_1+\varepsilon_1\right)+G_2\left(r_2+\varepsilon_2\right)+\\+e_1\left(r_1+\varepsilon_1\right)-e_2\left(r_2+\varepsilon_2\right)\end{gathered} \tag{546}$$

After an immediately calculation, from equation (546), it results that:

$$\begin{gathered}r\left[\frac{e_1\left(r_1+\varepsilon_1\right)}{r_1+\varepsilon_1}-\frac{e_2\left(r_2+\varepsilon_2\right)}{r_2+\varepsilon_2}\right]=ln\frac{C\left(r_2\right)}{O\left(r_1\right)}+\varepsilon_1\frac{e_1\left(a\right)}{a}-\varepsilon_2\frac{e_2\left(b\right)}{b}+\\+e_1\left(r_1+\varepsilon_1\right)-e_2\left(r_2+\varepsilon_2\right)\end{gathered} \tag{547}$$

where: a is a value from the $\left(r_1,r_1+\varepsilon_1\right)$ interval;

b is a value from the $\left(r_2,r_2+\varepsilon_2\right)$ interval.

After performing calculations, the approximate solution that we were looking for is the following:

$$r^*=\frac{ln\dfrac{C\left(r_2\right)}{O\left(r_1\right)}+e_1\left(r_1\right)-e_2\left(r_2\right)+\varepsilon_1\dfrac{e_1\left(r_1\right)}{r_1}-\varepsilon_2\dfrac{e_2\left(r_2\right)}{r_2}}{\dfrac{e_1\left(r_1\right)}{r_1}-\dfrac{e_2\left(r_2\right)}{r_2}} \tag{548}$$

For $e_1\left(r\right)=a\,,e_2\left(r\right)=b$ and $\varepsilon_1=\varepsilon_2=0$, this solution is in concordance with the solution marked out within the previous paragraph.

Remark 5.9

The solution r^* given by the previous equality is an approximate one, because in order to solve the equation *O(r*) = C(r*),* we took into account the following elements:

1) the development in Taylor series keeps only the first two terms;

2) the values $a \in (r_1, r_1 + \varepsilon_1), b \in (r_2, r_2 + \varepsilon_2)$ were approximated through r_1, respectively r_2.

5.4 Bibliographical Notes and Comments

The content of this chapter approaches an optimum problem regarding the capitalization of compound interest which is different from other known problems (the problem of annulment, the minimum deviation problem, the equilibrium problem) [54]. Even these known problems are analyzed insufficiently at present and, therefore, few theoretical results are marked out.

By virtue of the form of the efficiency function, the approached optimum problem requires a special mathematical apparatus which calls into requisition to basic results from the game theory (especially the results related to minmax optimization).

The application performed, extremely significant from practical point of view, represents in fact, a particular case of the optimum problem approached. Actually, the solving of the problem can be considered an annulment problem for the capitalization polynomial, which has a particular form. Practically, the unit interest that we are searching for can be determined as a solution of a higher degree algebraic equation; hence the process of solving this equation is performed by means of an approximate method. Therefore, we appealed to linearization method (in fact, a variant of the method, which implies the development into Taylor series) because Newton method and successive approximation method require extremely difficult calculations.

The results presented in this chapter put forward the following basic problems which imply a suitable mathematical device:

1. The problems of the capitalization polynomial problems and their solutions: the annulment problem, the minimum deviation problem and the equilibrium problem;
2. We have also analyzed the capitalization problem with variable interest, by using special efficiency functions; because these functions are not continuous, the optimality problem was approached by means of maxmin and minmax optimum conditions;
3. As on the monetary market the determination of equilibrium interest is a core problem, we have presented a general method of computing this level of interest, starting from an equilibrium type equation, written in a differential form, between the credit supply and the credit demand.

MAXMIN OPTIMAL METHOD FOR ANALYZING THE STABILITY OF WORKS IN OPEN PIT MINING

6.1 Fundamental Results

When analyzing the stability of mining activities within the open pits, one can use several methods which involve numerous calculations and low precision of results. Another weakness of these techniques is that the break curve (in plane section) is considered to be well approximated through an arc, which contradicts practice.
In an open pit mining, the stability of works is specified with the help of the so-called stability factor, practically with an angle determined by a specific rule.
The purpose of this chapter is to present a combined method of establishing an approximate slope breaking curve considering the fact that not one approximation through an arc is correct and not one stability angle is enough as an indicator.
This method is based on some experimental results obtained by Felenius (in his opinion, the breaking of the slope is possible by an arc) as well as on a methodology proposed by Frölilich and Förster [26] according to which the breaking curve follows a normal distribution N (not σ) for which the rule of calculating the average value m and the average square deviation are specified.
Since in reality the breaking curve there is neither an arc nor a normal distribution, the problem was solved based on a game structure. Basically it appears that for the chosen efficiency function the game has no equilibrium points (therefore one can never determine exactly breaking curve). As a consequence, the curve will be in a band limited by two extreme curves determined as optimal solutions of maxmin and minmax type. The proposed method will approximate, in fact, the real breaking curve through a precise curve that meets the conditions required by the maxmin and minmax optimal curves.

6.1.1 The Determination of the Optimal Stability Angle for Plane Sliding Surfaces

In this field of study, in the case of the earth slide of the high walls according to the flat slide, we use the quantity:

$$s = \frac{tg\,\varphi}{tg\,\beta} + \frac{2c\sin\alpha}{\gamma h \sin(\alpha-\beta)\sin\beta} \tag{549}$$

where:

tgφ	is the coefficient of friction;
c	is the coefficient of cohesion;
α	is the angle of the high wall;
β	is the angle between the slide plan and the horizontal ones;
h	is the height of the high wall;
γ	is the volumetric weight.

In order to determine the optimum β (marked β_0), we assume the conditions: $\frac{ds}{d\beta} = 0, \frac{d^2 s}{d\beta^2} < 0$

The equation $\frac{ds}{d\beta} = 0$ leads to the equation:

$$\frac{\sin(2\beta-\alpha)}{\sin^2(\alpha-\beta)} = \frac{h\gamma\, tg\,\varphi}{2c\sin\alpha} \tag{550}$$

The equation (550) becomes (after an immediate calculus):

$$\sin 2\beta(\cos\alpha + A\sin\alpha\cos\alpha) + \cos 2\beta\left(-\sin\alpha - \frac{A}{2}\sin^2\alpha + \frac{A}{2}\cos^2\alpha\right) - \frac{A}{2} = 0$$
$$a = \frac{h\gamma\, tg\,\varphi}{2c},\ A = \frac{a}{\sin\alpha} \tag{551}$$

According to the substitution $tg\varphi = A$ there occurs $\sin 2\beta = \frac{2t}{1+t^2}$, $\cos 2\beta = \frac{1-t^2}{1+t^2}$.

Thus, the equation (551) becomes:

$$t^2\left(A\sin\alpha - A + \sin\alpha\right) + 2t\cos\alpha\left(1 + A\sin\alpha\right) - \sin\alpha - A\sin^2\alpha = 0 \tag{552}$$

which has the following solutions:

$$t_{1,2} = \frac{-\cos\alpha\left(1+a\right) \pm \sqrt{1+a}}{\sin\alpha - A\cos^2\alpha} = \frac{-\sin\alpha\left[-\left(1+a\right)\cos\alpha \pm \sqrt{1+a}\right]}{\sin^2\alpha - a\cos^2\alpha} \tag{553}$$

Taking into account the condition $\frac{d^2 s}{d\beta^2} < 0$ it results at once that the only accepted solution is:

$$t_1 = \frac{-\sin\alpha\left[-\left(1+a\right)\cos\alpha - \sqrt{1+a}\right]}{\sin^2\alpha - a\cos^2\alpha} \tag{554}$$

Thus the optimum angle β_0 is given by:

$$tg\,\beta_0 = \frac{\sin\alpha\left[\left(1+a\right)\cos\alpha + \sqrt{1+a}\right]}{\sin^2\alpha - a\cos^2\alpha} \tag{555}$$

This solution is more comfortable from the point of view of calculations than the usual one used in this field of study:

$$\sin\beta_0 = \sqrt{\frac{\left(a+1\right)\left(a+2\right)\sin^2\alpha}{a^2 + 4\left(a+1\right)\sin^2\alpha} - \sqrt{\frac{4\left(a+1\right)^2\left(a+2\right)^2\sin^4\alpha}{4\left[a^2 + 4\left(a+1\right)\sin^2\alpha\right]} - \frac{\left(a+1\right)\sin^4\alpha}{a^2 + 4\left(a+1\right)\sin^2\alpha}}} \tag{556}$$

For the optimum determined β_0 we can calculate, at once, the coefficient of stability of the high wall:

$$s_0 = \frac{tg\,\varphi}{tg\,\beta_0} + \frac{2c\sin\alpha}{h\sin\left(\alpha - \beta_0\right)\sin\beta_0} \tag{557}$$

6.1.2 Frölilich-Förster Method and its Approximation

Based on initial data (α, h) and according to Felenius method one can draw Table 2 and figure 33. Obviously, h is the height of the slope and α is the slope angle.

Table 2

Gradient		Angles	
1: m	α	β_1	β_2
1:0,58	60^0	29^0	40^0
1:1	45^0	28^0	38^0
1:1,5	$33^0,41'$	26^0	35^0
1:2	$26^0,30'$	25^0	35^0
1:3	$18^0,26'$	25^0	35^0
1:5	$11^0,19'$	25^0	37^0

The Cartesian coordinates of point A follow immediately:

$$\begin{cases} x = htg\alpha \\ y = h \end{cases} \tag{558}$$

As a consequence, the equations of lines (D_1) and (D_2) are as follows:

$$(D_1) \quad y = xtg(\alpha + \beta_1)$$
$$(D_2) \quad y = h - (x - htg\alpha)tg\beta_2 \tag{559}$$

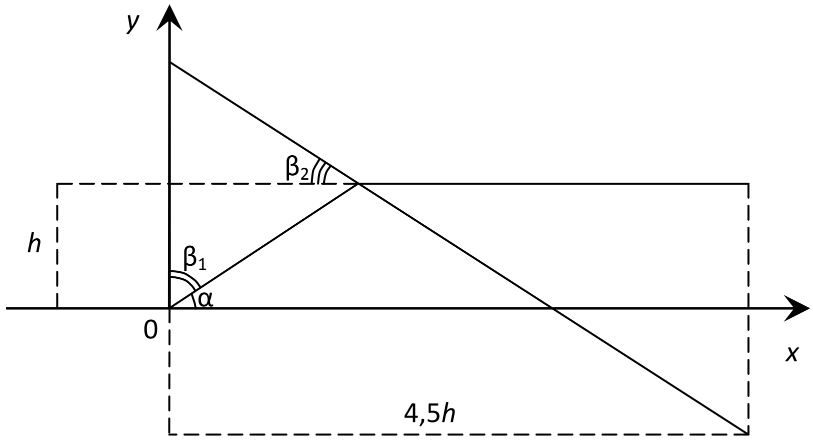

Figure 33

Since the center of the circle $\bar{0}$ is determined as an intersection of lines (D_1) and (D_2), the Cartesian coordinates $\bar{x}$, $\bar{y}$ of point $\bar{0}$ result immediately after calculations.

$$\begin{cases} \bar{x} = \dfrac{h\sin(\alpha+\beta_2)\cos(\alpha+\beta_1)}{\sin\alpha\sin(\alpha+\beta_1+\beta_2)} \\ \bar{y} = \dfrac{h\sin(\alpha+\beta_1)\sin(\alpha+\beta_2)}{\sin\alpha\sin(\alpha+\beta_1+\beta_2)} \end{cases} \tag{560}$$

Therefore, the radius that approximates the breaking curve will be denoted by r and can be determined immediately:

$$r = h\frac{\sin(\alpha+\beta_2)}{\sin(\alpha+\beta_1+\beta_2)\sin\alpha} \tag{561}$$

According *to* Frölilich – Förster methodology, the breaking curve is an arc of normal distribution N (not σ) (fig.34) where statistical indicators, not σ, are determined precisely.
It is clear that actual deployment will be approximated by the so-called theoretical dislocation made on the approximation of a breaking curve through an arc.

From Forster, the slope breaking curve presents a normal type distribution $N(m,\sigma)$, result proven only experimentally. For determining the average value m and mean square deviation σ we have a precise calculation method:

$$m = A_c + Btg\varphi,\ \sigma = \sqrt{D},\ D = A^2 D_c + B^2 D_\varphi + 2ABD_c D_\varphi \rho_{c\varphi} \tag{562}$$

where:

$$A = \frac{2r^2\sin\dfrac{\alpha}{2}}{Qa},\quad B = \frac{r\cos\sigma}{2a}\left(1+\frac{\alpha}{\sin\dfrac{\alpha}{2}}\right) \tag{563}$$

c, φ represents the cohesion coefficient, respectively the friction coefficient;
Q represents the weight of the dislocated volume;
D_c, D_φ represent the distribution of measurement errors for c and φ;
$r_{c\varphi}$ represents the correlation coefficient between cohesion and friction;

$$\sin\frac{\alpha}{2} = \sqrt{\frac{h}{2\cdot r}},$$

$$a = h \cdot \sqrt{\left(ctg\alpha - \frac{x_0^2 ctg\alpha + x_0 y_0}{x_0^2 + y_0^2} \right)^2 + \left(1 - \frac{y_0^2 \left(\frac{x_0}{y_0} ctg\alpha + 1 \right)}{x_0^2 + y_0^2} \right)^2} \tag{564}$$

r, x_0, y_0 represent the radius, respectively the Cartesian coordinates of the circle from figure 34.

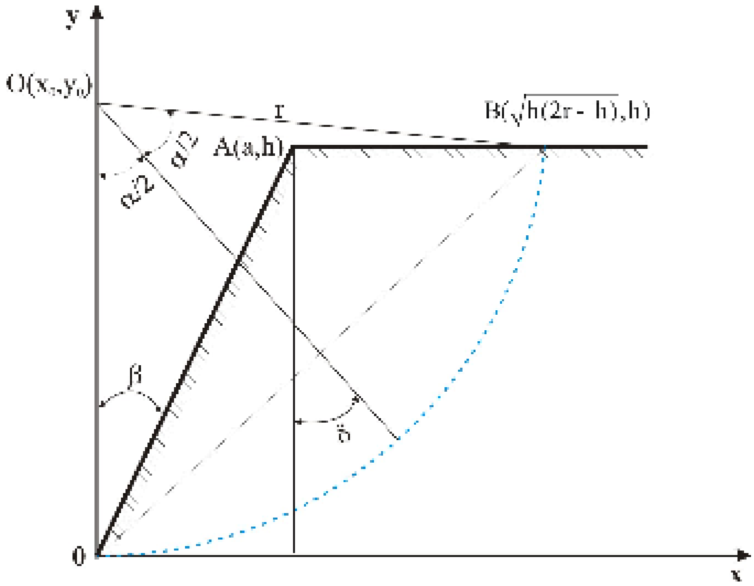

Figure 34

The breaking curve may be approximated through a circle arc which passes through the points C_1 and C_2 (where the normal distribution $N(m,\sigma)$ intersects Oy and the right $y = h$ - figure 35) and is tangent to the optimum slope right.

The coordinates $\bar{x}, \bar{y}$ of the tangent point M($\bar{x}, \bar{y}$) are determined starting form the curve f_1 given by:

$$f_1(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \tag{565}$$

and the right f_2, $f_2(x) = \lambda x, \lambda$ real parameter arbitrary chosen.

Applying the tangential conditions of the right to the curve in the point M($\bar{x}, \bar{y}$), $f_1(\bar{x}) = f_2(\bar{x})$, $f_1'(\bar{x}) = f_2'(\bar{x})$, after an immediate calculation it results:

$$\bar{x} = \frac{m + \sqrt{m^2 - 4\sigma^2}}{2} \qquad \bar{y} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{m - \sqrt{m^2 - 4\sigma^2}}{8\sigma^2}}, \qquad m > 2\sigma \tag{566}$$

The right determined by the origin and the point M($\bar{x}, \bar{y}$) represents the slope given by the following relation:

$$m_0 = \frac{2e^{-\frac{m - \sqrt{m^2 - 4\sigma^2}}{8\sigma^2}}}{\sqrt{2\pi}\sigma\left(m + \sqrt{m^2 - 4\sigma^2} \right)} = tg\beta_0 \tag{567}$$

The circle $(O(x_0,y_0),r)$ determined by the points C_1, C_2 and M (figure 36) may be found easily, as the Cartesian coordinates x_0, y_0 and the radius r are given by the following relations (568). C_1 and C_2 are the points in which the normal distribution intersects Oy and $y = h$, and M is the tangential point to the optimal slope (figure 35).

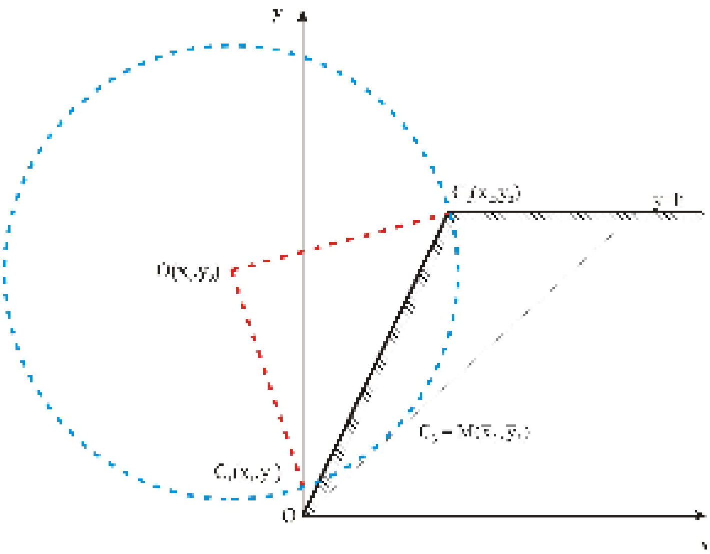

Figure 35

The center $O(x_0, y_0)$ and the radius r of this circle are determined by solving the following nonlinear system:

$$\begin{cases} x_0 = \dfrac{\left(\tau^2 + h^2 - p^2\right)\left(\tilde{y} - p\right) + \left(p - h\right)\left(\tilde{x}^2 + \tilde{y}^2 - p^2\right)}{2\left[\tau\left(\tilde{y} - p\right) + \tilde{x}\left(p - h\right)\right]} \\ y_0 = \dfrac{\tilde{x}^2 + \tilde{y}^2 - p^2}{2} - \tilde{x}\dfrac{\left(\tau^2 + h^2 - p^2\right)\left(\tilde{y} - p\right) + \left(p - h\right)\left(\tilde{x}^2 + \tilde{y}^2 - p^2\right)}{2\left[\tau\left(\tilde{y} - p\right) + \tilde{x}\left(p - h\right)\right]} \\ r = \sqrt{p^2 + y_0^2 - 2py_0 + x_0^2} \end{cases} \tag{568}$$

where

$$\tau = m - \sqrt{\frac{\sigma}{2}} \qquad p = \frac{e^{-\frac{m^2}{\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$\tilde{x} = \frac{x_0^2 - r^2}{x_0} \quad \tilde{y} = \frac{t}{x_0}(x_0^2 - r^2) \quad t = \frac{2}{\sigma\sqrt{2\pi}} \frac{e^{-\left(\frac{m - \sqrt{m^2 - 4\sigma^2}}{64\sigma^4}\right)^2}}{m + \sqrt{m^2 - 4\sigma^2}} \tag{569}$$

Besides,

$$m = \frac{2r^2 \frac{\gamma}{2} c}{Qa} + \frac{r\cos\delta}{2a}\left(1 + \frac{\frac{\gamma}{2}}{\sin\frac{\gamma}{2}}\right) tg\varphi \ , \quad \sin\frac{\gamma}{2} = \frac{\sqrt{r}}{h\sqrt{2}} \tag{570}$$

$$s = \frac{2r^2 \frac{\alpha}{2} C}{Qa} + \frac{r\cos\delta}{2a}\left(1 + \frac{\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\right) tg\varphi \tag{571}$$

If s is the stability angle of the slope, its caving probability ca be immediately determined:

$$P(s \prec 1) = \phi\left(\frac{1 - m}{\sigma}\right) \tag{572}$$

where ϕ is the Laplace function.

In this situation, the coefficient s and the stability reserve D_s are the following:

$$s = \frac{Qtg\gamma\cos\beta_0 + Ch}{Q\sin\beta_0}$$

$$D_s - 1 = \frac{\sin(\gamma - \beta_0)}{\cos\gamma\sin\beta_0} + \frac{2C\sin\alpha}{\gamma h\sin(\alpha - \beta)\sin\beta_0} \tag{573}$$

where γ is the specific weight.

The additional dislocated surface S may be calculated approximately

$$S \approx \frac{\overline{\overline{x}} - \overline{x}}{2}\left(1 - \overline{y}\right) \quad \text{(figure 36)} \tag{574}$$

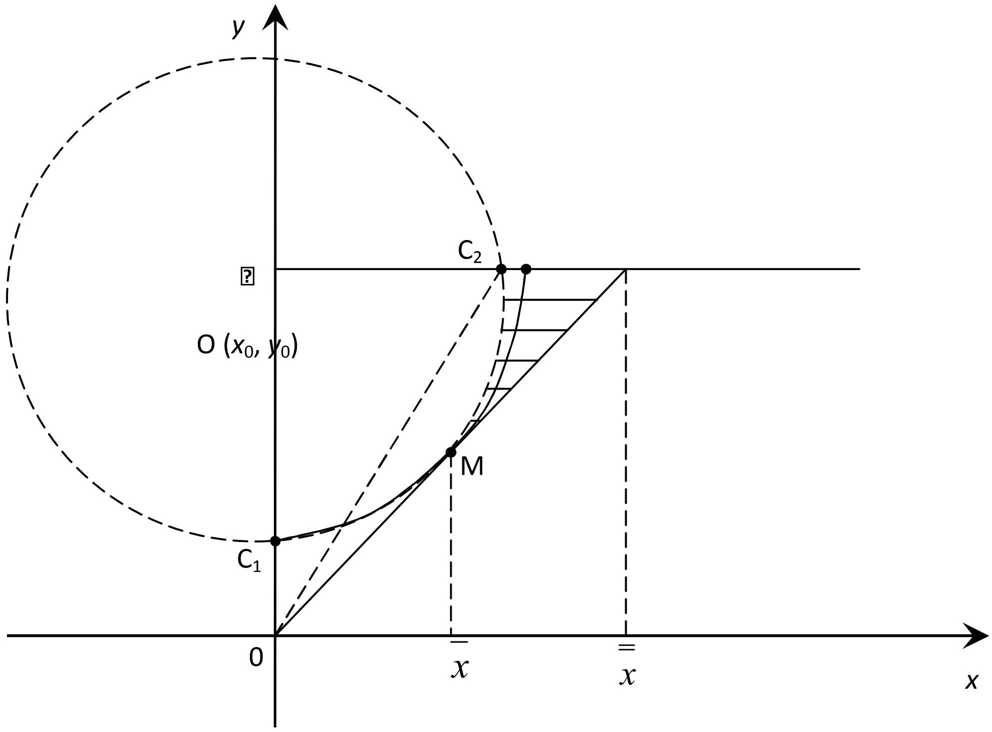

Figure 36

Remark 6.1

The accuracy of this method is proven only experimentally.

Performing calculations, we get:

$$\begin{cases} \overline{x} = \dfrac{m + \sqrt{m^2 - 4\sigma}}{2} \\ \overline{y} = 2e^{-\Delta} \qquad \Delta = \dfrac{\left(m - \sqrt{m^2 - 4\sigma^2}\right)^2}{8\sigma^2} \\ \overline{\overline{x}} = \dfrac{\sqrt{2\pi}\sigma\left(m + \sqrt{m^2 - 4\sigma 2}\right)}{4e^{-\Delta}} \end{cases} \tag{575}$$

Given data from table 2, we can immediately show that:

$$\beta_0 = \alpha_{opt} = 41^0$$

Therefore, for $h = 20m$ and $\alpha = 41^0$, we can determine the stability coefficient $s = 1.26$ (we assumed that the sliding curve is not a plane one). Besides, $\Delta_{s_n} = 0.26$.

6.2 Approaching the Problem from Maxmin Optimality Point of View

We start from Forster's idea which states that the slope breaking curve is an arc correspondent to a distribution *N (m,σ)*, and the statistical indicators *m* (average value) and *σ* (mean square deviation) are calculated precisely depending on the geo-mechanic characteristics of the rocks.

Practically, since there are infinity of probabilistic distributions with the same average value *m* and mean square deviation *σ* and we need to select them with an efficiency function that imposes a certain optimum criteria.

If we note with *p* the distribution function and with *x* the slope breaking point, the efficiency function is considered the function *f* defined as [51]:

$$f\left(x,p\left(t\right)\right)=\int_{0}^{x}p\left(t\right)dt+mp\left(x\right) \tag{576}$$

with the following conditions:

$$\int_{0}^{\infty}p\left(t\right)dt=m \qquad 2\int_{0}^{\infty}tp\left(t\right)dt=m^{2}+\sigma \qquad p\left(0\right)=1 \tag{577}$$

This efficiency function is alike the one introduced by Ghermeier in the system reliability study and it practically means that the area of the curved triangle ABC is equal to the area between *Ox* and *p* graphic, $x \geq x_{*}+m$, x_{*} representing the slope breaking point (figure 37).

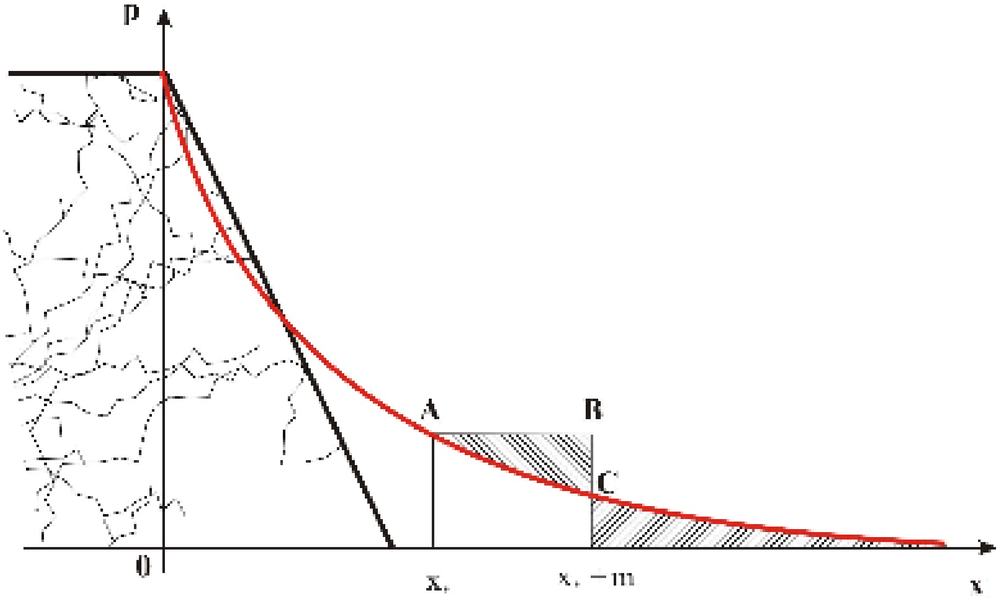

Figure 37

Analytically, this property is expressed by the following relation:

$$mp\left(x\right)=\int_{x}^{\infty}p\left(t\right)dt \tag{578}$$

If we consider a game against nature where the efficiency function is F, we can show easily that, regarding the games theory, this game doesn't present an equilibrium point, so the following problem:

$$\max_{x}\min_{p(t)}F\left(x,p\left(t\right)\right)=\min_{p(t)}\max_{x}F\left(x,p\left(t\right)\right) \tag{579}$$

doesn't have any solutions.

Practically, this means that the real breaking curve can never be determined precisely, it can be mostly approximated.

Therefore we need to solve the following problems:

$$\left(P_{1}\right)\max_{x}\min_{p(t)}F\left(x,p\left(t\right)\right),\ \left(P_{2}\right)=\min_{p(t)}\max_{x}F\left(x,p\left(t\right)\right) \tag{580}$$

and to determine only one belt where the real breaking curve is found (figure 38).

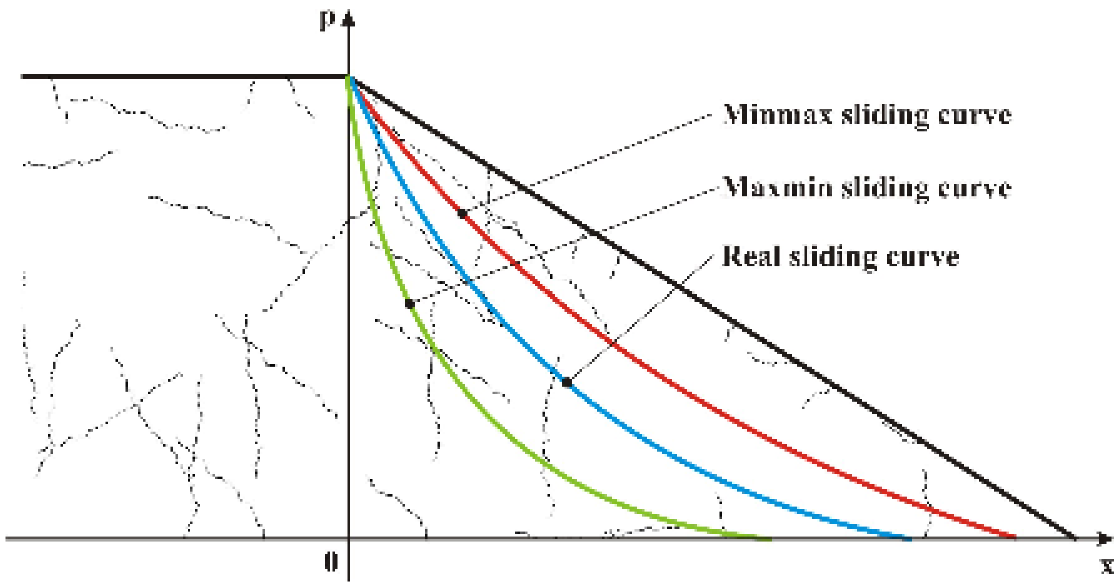

Figure 38

The problem $(P_1)\max_x \min_{p(t)}\left(\int_0^x p(t)dt + mp(x)\right)$ is solved by using method from the games theory; the breaking point x^* is the solution of the following equation:

$$(m-x)^4 - \sigma x(2m-x) + \sigma^2 = 0 \tag{581}$$

This equation may only approximately be solved, the accepted solution is $x^* \approx m - \sqrt{\frac{\sigma}{2}}$, and the correspondent distribution function being exponential type.

Remark 6.2

The equation below has four real roots x_1, x_2, x_3, x_4 located in the following intervals:

$$x_1 \in \left(-\infty, m-\sqrt{\frac{\sigma}{2}}\right),\quad x_2 \in \left(m-\sqrt{\frac{\sigma}{2}}, m\right),\ x_3 \in \left(m, m+\sqrt{\frac{\sigma}{2}}\right),\ x_4 \in \left(m+\sqrt{\frac{\sigma}{2}}, \infty\right)$$

After approximating the solutions of equations using the values $m-\sqrt{\frac{\sigma}{2}}$, not $m+\sqrt{\frac{\sigma}{2}}$, the errors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ that are obtained are as follows:

$$\begin{cases} \overline{x_1} = m-\sqrt{\frac{\sigma}{2}} \Rightarrow \quad \varepsilon_1 = \sigma\left(m^2 - \frac{3}{4}\sigma\right) \\ \overline{x_2} = m \Rightarrow \quad \varepsilon_2 = \sigma\left(\sigma - m^2\right) \\ \overline{x_3} = m+\sqrt{\frac{\sigma}{2}} \Rightarrow \quad \varepsilon_3 = \sigma\left(\frac{7}{4}\sigma - m^2\right) \end{cases} \tag{582}$$

From the experimental data presented in [26], for the height $h = 20m$ of the slope, one can get the smallest errors in the approximate case $x = m-\sqrt{\frac{\sigma}{2}}$, according to the following table (Table 3).

Table 3

h	α	m	σ	$m-\sqrt{\dfrac{\sigma}{2}}$	ε
20	40^0	1.51	0.378	1.08	-0.615
20	50^0	1.26	0.342	0.849	-0.337
20	55^0	1.12	0.325	0.753	-0.195
20	60^0	1.17	0.306	0.676	-0.183
20	65^0	0.98	0.301	0.594	-0.118

Approximation $x \approx m-\sqrt{\dfrac{\sigma}{2}}$ was adopted on the consideration that it is only rational to choose the root that is the nearest to point where the curve proposed by Förster meets top edge of the slope.
Based on the condition that the graph of the function:

$$f_{(x)} = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-m)^2}{2\sigma^2}} \tag{583}$$

should cross the line corresponding to the height of the slope, the point $\bar{x}$ that we were searching for results immediately:

$$\bar{x} = m - \sqrt{-2\sigma^2 \ln\left(\sqrt{2\pi\sigma}\right)} \tag{584}$$

The problem $(P_2) = \min_{p(t)} \max_{x} \left(\int_0^x p(t)dt + mp(x) \right)$ admits also an exponential distribution function and the falling point x^{**} is found in the interval $[x_1, x_2]$, where $x_1 = me^{-z}$, $x_2 = me^{-z} + mz$, z being the solution of the following equation:

$$e^{-2z} + 2e^{-z} + \frac{\sigma}{m^2} - 1 = 0 \tag{585}$$

Obviously, the most disadvantageous situation corresponds to the solution of the problem (P_1); basically it corresponds to the case where the falling curve is least favorable.

Remark 6.3

Determining the equation of the circle defined by the three points M_1, M_2, M_3 allows the immediate calculation of the following elements:

- The breakage curve (represented by M_1, M_2 circle arch)
- The optimum slope angle (represented by the tangent angle from the origin to the circle determined by the three points M_1, M_2, M_3)
- The breakage probability (calculated based on the equality $p = \phi\left(\dfrac{1-m}{\sigma}\right)$, where ϕ is Laplace function).

For a height of the slope $h = 20m$ and different variants α of the slope's angle, the resulted presented so far can be grouped in the following table 4.

Table 4

h	α	m	σ	Breakage probability	Optimum angle
20	40^0	1.5143	0.378	0.08	39^0
20	50^0	1.2585	0.342	0.22	46^0
20	55^0	1.1167	0.325	0.31	50^0
20	60^0	1.0676	0.360	0.41	57^0
20	65^0	0.9825	0.301	0.52	58^0

6.3 The Determination of Optimal Stability Angle by Using a Combined Maxmin Method

The aim of this paragraph is to present a combined method for determining the optimum slope angle. The theoretic support is representing by using the maxim probability criteria, which is equivalent, for the analyzed problem, with the maxmin and minmax criteria.

Starting from the initial data (α,h) (α represents the gradient angle of the slope and h represents the height of the slops), the angles β_1 and β_2 are determined according to the following table 5:

Table 5

Gradient		Angles	
$1:m$	α	β_1	β_2
1:0,58	$60°$	$29°$	$40°$
1:1	$45°$	$28°$	38^0
1:1,5	$33° 41'$	$26°$	$35°$
1:2	$26° 30'$	$25°$	$35°$
1:3	$18° 26'$	$25°$	$35°$
1:5	$11° 19'$	$25°$	$37°$

The combined method suggested above implies the following stages [51].

Stage 1

Felenius: $(\alpha,h)\rightarrow(\beta_1,\beta_2)\rightarrow C(O_1,r_1)$

The center O_1 of the first circle is determined as an intersection of two lines (D_1), (D_2) of gradients $m_1 = tg(\alpha+\beta_1)$, $m_2 = -tg\,\beta_2$ which pass through the origin and the point A respectively (figure 39).

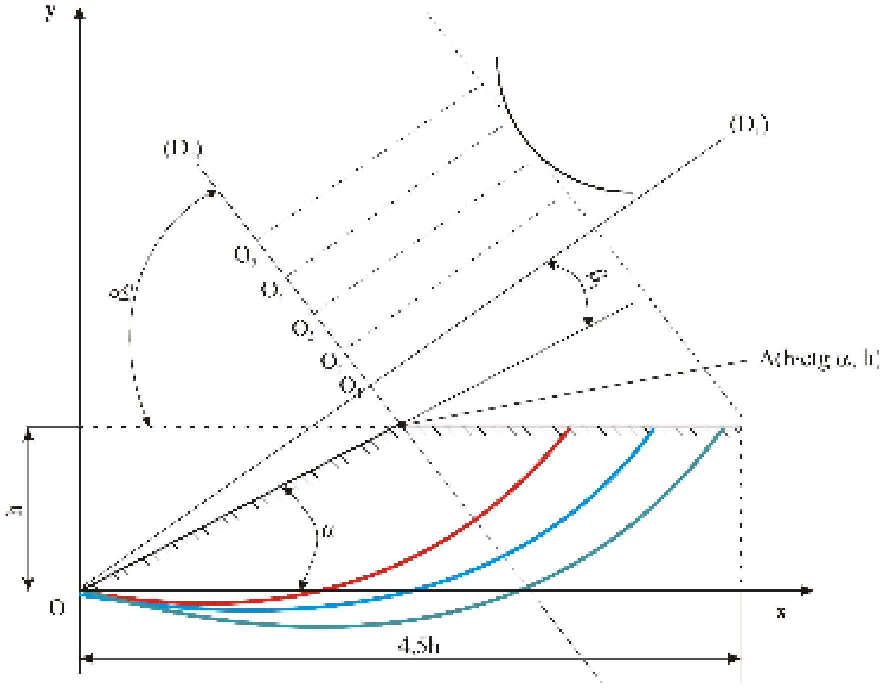

Figure 39

For a height h of the slope and a gradient α, the Cartesian coordinates of the point A result directly:

$$\begin{aligned} x &= h\,ctg\,\alpha \\ y &= h \end{aligned} \tag{586}$$

Therefore, we have:

$$
\begin{aligned}
&(D_1)\ y = tg(\alpha+\beta_1)x \\
&(D_2)\ y = h - tg\,\beta_2(x - h\,ctg\,\alpha)
\end{aligned}
\tag{587}
$$

As $O=(D_1)\cap(D_2)$, the Cartesian coordinates x_1, y_1 of the point O_1 are determined directly:

$$
\begin{aligned}
x_1 &= h\frac{sin(\alpha+\beta_2)cos(\alpha+\beta_1)}{sin\,\alpha\,sin(\alpha+\beta_1+\beta_2)} \\
y_1 &= h\frac{sin(\alpha+\beta_1)sin(\alpha+\beta_2)}{sin\,\alpha\,sin(\alpha+\beta_1+\beta_2)}
\end{aligned}
\tag{588}
$$

The radius of the circle with the centre O_1 and which passes through the origin is also determined through direct calculation:

$$
r_1 = h\frac{sin(\alpha+\beta_2)}{sin(\alpha+\beta_1+\beta_2)sin\,\alpha}
\tag{589}
$$

Fröhlich-Förster: $r_1 \to m_1$

The mean value m_1 is calculated on the basis of the equality:

$$
m_1 = r_1^2\frac{C\alpha}{aQ} + r_1\frac{cos\,\delta\,tg\,\varphi}{a}
\tag{590}
$$

where:

$$
a = h\cdot\sqrt{\left(ctg\,\alpha - \frac{x_1^2\,ctg\,\alpha + x_1y_1}{x_1^2+y_1^2}\right)^2 + \left[1 - \frac{y_1^2\left(\frac{x_1}{y_1}ctg\,\alpha + 1\right)}{x_1^2+y_1^1}\right]^2}
\tag{591}
$$

- C is cohesion coefficient of the rocks;
- $tg\,\varphi$ is the friction coefficient;
- Q is the weight of the sliding body.

Maxmin – minmax: $m_1 \to (\sigma_1, P_1)$

The following equation is considered:

$$
(m-x)^4 - \sigma_1 x(2m-x) + \sigma_1^2 = 0
\tag{592}
$$

For $m = m_1$ and $x = m - \sqrt{\frac{\sigma_1}{2}}$, from the equality (592), dispersion σ_1 results directly.

Considering the normal distribution $N(m_1,\sigma_1)$, $P_1 = \phi\left(\frac{1-m_1}{\sigma_1}\right)$ can be computed directly, where ϕ is Laplace's function.

For a fixed error $\varepsilon > 0$ (and sufficiently small), P_1 is compared with ε.

If $P_1 \le \varepsilon$ one can immediately calculate the optimal angle of the slope from:

$$
tg\,\alpha = \frac{4e^{\frac{-\left(m_1-\sqrt{m_1^2-4\sigma_1^2}\right)^2}{8\sigma_{n+1}^2}}}{\sqrt{2\pi}\sigma_1\left(m_1+\sqrt{m_1^2-4\sigma_1^2}\right)}
\tag{593}
$$

If $P_1 > \varepsilon$, the next stage follows.

Stage 2

Felenius: $C(O_1,r_1) \to C(O_2,r_2)$

The Cartesian coordinates of the new centre $O_2(x_2,y_2)$ are determined; the centre is situated at the distance $0.3h$ from O_1 (figure 39).

The point O_2 being on the line (D_2) we shall have:

$$\begin{cases} y_2 = -x_2\, tg\, \beta_2 + h + tg\, \beta_2\, ctg\, \alpha \\ 0,3h = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{cases} \quad (594)$$

Having made the calculations it results immediately that x_2 is greatest solution of the equation:

$$\begin{aligned} &x_2^2\left(1 - tg^2\, \beta_1\right) - 2x_2\left[x_1 + tg\, \beta_2\left(y_1 + h + tg\, \beta_2\, ctg\, \alpha + x_1\right)\right] + \\ &+ x_1^2 + \left(y_1 + h + tg\, \beta_2\, ctg\, \alpha + x_1\right)^2 - 0,09h = 0 \end{aligned} \quad (595)$$

From (594) and (595) the Cartesian coordinates (x_2, y_2) of the point O_2 as well as the radius r_2 result directly:

$$r_2 = \sqrt{x_2^2 + \left(h\, tg\, \beta_2\, ctg\, \alpha - x_2\, tg\, \beta_2\right)^2} \quad (596)$$

Fröhlich-Förster: $r_2 \to m_2$

The mean value m_2 is calculated similarly to stage 1:

$$m_2 = r_2^2 \frac{C\alpha}{aQ} + r_2 \frac{\cos\delta\, tg\, \varphi}{a} \quad (597)$$

and the significance of the parameters is preserved.

Maxmin – minmax: $m_2 \to (\sigma_2, P_2)$

With m_2 known from the equation $(m_2 - x)^4 - \sigma_2 x(2m - x) + \sigma_2^2 = 0$, for $x = m_2 - \sqrt{\frac{\sigma_2}{2}}$, σ_2 is determined.

$P_2 = \phi\left(\frac{1 - m_2}{\sigma_2}\right)$ is calculated; if $P_2 \le \varepsilon$, then the optimal angle α of the slope will be given by:

$$tg\, \alpha = \frac{4e^{-\frac{\left(m_2 - \sqrt{m_2^2 - 4\sigma_2^2}\right)^2}{8\sigma_2^2}}}{\sqrt{2\pi}\sigma_2\left(m_2 + \sqrt{m_2^2 - 4\sigma_2^2}\right)} \quad (598)$$

If $P_2 > \varepsilon$, the next stage follows.

Generally, if in the stage n we have $O_n(x_n, y_n)$, then we shall have:

Felenius: $C_n(O_n, r_n) \to C_{n+1}(O_{n+1}, r_{n+1})$ where the Cartesian coordinates of the new centre $C_{n+1}(O_{n+1}, r_{n+1})$ are determined as follows: x_{n+1} is the greatest solution of the equation:

$$\begin{aligned} &x_{n+1}^2\left(1 + tg^2\, \beta_1\right) - 2x_{n+1}\left[x_n + tg\, \beta_2\left(y_n + h + tg\, \beta_2\, ctg\, \alpha + x_n\right)\right] + \\ &+ x_n^2 + \left(y_n + h + tg\, \beta_2\, ctg\, \alpha + x_n\right)^2 - 0,09h = 0 \end{aligned} \quad (599)$$

y_{n+1} is given by the equality:

$$y_{n+1} = -x_2\, tg\, \beta_2 + h + tg\, \beta_2\, ctg\, \alpha \quad (600)$$

x_{n+1}, y_{n+1} being determined, they allow the immediate calculation of the radius of the circle passing through the origin and through O_{n+1}:

$$r_{n+1} = \sqrt{x_n^2 + \left(h\, tg\, \beta_2\, ctg\, \alpha - x_n\, tg\, \beta_2\right)^2} \quad (601)$$

Fröhlich-Förster: $r_{n+1} \to m_{n+1}$

$$m_{n+1} = r_n^2 \frac{C\alpha}{aQ} + r_n \frac{\cos\delta\, tg\, \varphi}{a} \quad (602)$$

where:

$$a = h \cdot \sqrt{\left(ctg\,\alpha - \frac{x_{n+1}^2\, ctg\,\alpha + x_{n+1} y_{n+1}}{x_{n+1}^2 + y_{n+1}^2} \right)^2 + \left[1 - \frac{y_{n+1}^2 \left(\frac{x_{n+1}}{y_{n+1}} ctg\,\alpha + 1 \right)}{x_{n+1}^2 + y_{n+1}^2} \right]^2} \tag{603}$$

Maxmin – minmax: $m_{n+1} \to \left(\sigma_{n+1}, P_{n+1} \right)$

In a similar way to the previous stages, σ_{n+1} is calculated which allows the calculation of the value $P_{n+1} = \phi\left(\frac{1 - m_{n+1}}{\sigma_{n+1}} \right)$ which will be compared with the error ε.

If $P_{n+1} \le \varepsilon$, then the optimal angle α of the slope will be given by:

$$tg\,\alpha = \frac{4e^{\frac{-\left(m_{n+1} - \sqrt{m_{n+1}^2 - 4\sigma_{n+1}^2} \right)^2}{8\sigma_{n+1}^2}}}{\sqrt{2\pi}\sigma_{n+1}\left(m_{n+1} + \sqrt{m_{n+1}^2 - 4\sigma_{n+1}^2} \right)} \tag{604}$$

If $P_{n+1} > \varepsilon$, the next stage follows.

On the basis of the equalization principle, there will be $m \in N$ so that $P_m \le \varepsilon$ and so the algorithm stops.

Remark 6.4

This combined method has two great advantages:

1) it is more rigorous than each of the three methods taken separately;
2) through calculations are numerous by making up the algorithm for the calculation of the necessary elements, calculation it self raises no problems.

6.4 Bibliographical Notes and Comments

In the technical literature, there are known some important techniques of analyzing the stability of works for open pit mining. Some interesting comments must be brought into focus:

❖ As a matter of fact, the breakage curve (in sectional view) is not a circular arc;

❖ For all that, if we assume that the above mentioned curve were a circular arc, then:

- the methodologies used in order to determine the centers are debatable;
- if the center *O* is not the optimum chosen center (by the same token, this center is roughly estimated), then the signs "+" and "–" based on the positions of active and passive prisms will lead us to erroneous results;
- Frölilich's approximations:

$$\sum_i G_i \sin\theta_i = \frac{aQ}{r}$$

$$\sum_i G_i \cos\theta_i = Q\cos\delta$$

from Goldstein's method does not specify the error that has been made.

❖ For this reason, $s > 1$ does not necessarily imply the condition that the slope breaks;

❖ Frölilich- Förster's idea related to the fact that the breakage curve (in sectional view) complies with the normal distribution $N(m,\sigma)$ is not rigorously demonstrated, but only experimentally proven. Yet, statistical parameters *m* and σ are rigorously determined.

❖ The slope breakage phenomenon can be analyzed through the following condition:

$$\theta\left(\frac{1-m}{\sigma} \right) = 0$$

This requirement is more rigorous than the above condition - $s > 1$, because the calculus technique which was used in order to determine *s* is an approximate one (therefore, it is possible that *s* exceed 1 and the slope still breaks).

❖ We have taken into consideration just a few cases $\left(h = 20m, \alpha \in \left\{ 40^0, 50^0, 55^0, 60^0, 65^0 \right\} \right)$. The probabilities of slope breakage have values between 0.08 and 0.52, while no risk limit is being introduced (this means that we cannot indicate a limit value which allows us to use the breakage probability as a decisional instrument).

❖ The determination of the stability coefficient for the case of plane curve slipping is very convenient in terms of calculation:

$$tg\beta_0 = \frac{\sin\alpha\left[\left(1+a\right)\cos\alpha + \sqrt{1+a}\right]}{a\cos^2\alpha - \sin^2\alpha}$$

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SUBJECT INDEX

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