On order of convergence for one regularizing method

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Abstract: - Modelling many problems of heat transfer and thermal engineering as well as mathematical physics, economy, statistics, actuary mathematics and etc., frequently we obtain operational equations of the first kind. As a rule, these equations concern to ill-posed problems. There are some iterative methods for solution of such problems. In the present work we consider the concrete iterative method and we estimate its degree of convergence without any additional conditions.

Key-Words: Ill posed problem, First kind operator equation, Iterative method, Inverse problem regularization

1 Introduction
Consider an operator equation of the first kind

\[ Az = u, \]
\[ z \in H, \; u \in AH, \]  \hspace{1cm} (1)

where \( H \) is a Hilbert space, the operator \( A : H \to H \) is linear, self-conjugate, positive and completely continuous, \( u \in AH \) is given element and \( z \in H \) unknown element.

It is assumed in this paper that the point zero does not belong to the spectrum of \( A \) and equation (1) is solvable for all \( u \in H \), i.e. \( AH \subset H \).

The following two theorems are known (see [1]):

THEOREM 1. Let \( z_p \) be an exact solution of (1) for \( u = u_p \in H \), i.e. \( Az_p = u_p \). Then the iterative process

\[ z_0 = 0, \]
\[ z_{n+1} = z_n + \theta \cdot (u_p - Az_p), \; n \geq 0 \] \hspace{1cm} (2)

converges to the solution of equation (1) in the norm of the Hilbert space \( H \) under the condition

\[ 0 < \theta < \frac{2}{\|A\|}. \]

THEOREM 2. Suppose that instead of the exactly given right-hand side \( u = u_p \in H \) there is some \( u = u_\delta \) in (1) such that

\[ \|u_p - u_\delta\| \leq \delta. \]

Then the iterative process

\[ z_0^\delta = 0, \]
\[ z_{n+1}^\delta = z_n^\delta + \theta \cdot (u_\delta - Az_n^\delta), \; n \geq 0 \] \hspace{1cm} (3)

converges to the exact solution \( z_p \) of equation (1) in the norm of the Hilbert space \( H \) and the following inequality holds

\[ \|z_n^\delta - z_p\| \leq \|z_n - z_p\| + \varepsilon(n) \cdot \delta, \]

where

\[ \varepsilon(n) = \theta \cdot \sum_{k=0}^{n-1} \|E - \theta \cdot A\|^k, \]

\( E \) is the unit operator and \( z_n \) is given by (2).

Using Theorem 1 and choosing \( n = n(\delta) \) so that \( \varepsilon(n(\delta)) \cdot \delta \to 0 \) as \( \delta \to 0 \) we obtain from (3) that \( z_n^\delta \) converges to \( z_p \).

Thus, \( z_n^\delta \to z_p \) in the norm of the space \( H \) as \( n(\delta) \to \infty \), but the degree of convergence can be arbitrary small. In order to estimate the degree of convergence one needs to assume that the exact solution (which is not known) is sourcewise representable.

The following theorem is proved in [2]:

THEOREM 3. Suppose that the exact solution \( z_p \) of equation (1) is sourcewise representable, i.e.

\[ z_p = A^\sigma s, \; \sigma > 0. \]

Then the following inequality holds

\[ \|z_p - z_n^\delta\| \leq \sigma^\gamma \cdot (n \cdot \theta \cdot e)^{-\gamma} \cdot \|s\| + n \cdot \theta \cdot \delta, \] \hspace{1cm} (4)

where \( 0 < \theta < \frac{2}{\|A\|}. \)

Here \( z_n^\delta \) is calculated from (3).

2 Failure on the Sourcewise Representability Condition of the Exact Solution and Use Energy Norm

It is proved in the present paper that in order
to estimate the degree of convergence of iterative process (3) the sufficiently strong additional condition on sourcewiserepresentability of the exact solution is not required, if energy norm is used, and the following theorem holds for iterative process (3) under the condition

\[ A \cdot \leq < \frac{3}{40} \theta \]

\( \text{THEOREM 4. Iterative process} \)

\[ z_0^0 = 0 , \]

\[ z_{n+1}^0 = z_n^0 + \frac{\theta}{3} (u_0 - A z_n^0) , \quad n \geq 0 \]  \( (3) \)

is convergent in the energy norm of space if the number of iterations \( n \) is chosen from the condition

\[ \sqrt{n} \cdot \delta \rightarrow n \rightarrow \infty \rightarrow \delta \rightarrow 0 . \]

In addition, under the condition

\[ 0 < \theta \leq \frac{4}{3} \| A \| (5) \]

the following error estimates are valid for iterative process (3)

\[ \| z_p - z_n^p \|_A \leq (2 \cdot n \cdot \theta \cdot e)^{\frac{1}{2}} \| z_p \|_A + \]

\[ + \left( \frac{4}{3} \cdot n \cdot \theta \right)^{\frac{1}{2}} \cdot \delta , \quad n \geq 1 , \]

\[ \| z_p - z_n^p \|_A \leq (2 \cdot n \cdot \theta \cdot e)^{\frac{1}{2}} \cdot \| z_p \|_A + \]

\[ + \left( \frac{13}{20} \cdot n \cdot \theta \right)^{\frac{1}{2}} \cdot \delta , \quad n \geq 2 . \]

\( (6) \)

\( \text{PROOF. Let us transform (2) to the form} \)

\[ z_n = z_{n-1} + \theta \cdot (u_p - A z_{n-1}) = \]

\[ = (E - (E - \theta \cdot A))^n z_p , \quad n \geq 1 . \]

Similarly, from (3) we get

\[ z_n^0 = 0 \cdot \sum_{k=0}^{n-1} (E - \theta \cdot A)^k u_0 , \quad n \geq 1 , \]

\[ z_p - z_n^0 = (z_p - z_n) + (z_n - z_n^0) , \]

\[ z_p - z_n = (E - \theta \cdot A)^n z_p . \]

Since it is assumed in the present paper that the operator \( A \) is self-conjugate, one has

\[ A = \int_0^M \lambda d E_{\lambda} , \]

where

\[ m = \inf \{ A x, x \} \geq 0 , \]

\[ M = \sup \{ A x, x \} > 0 , \]

and \( E_{\lambda} \) is a projective operator. Using the formula

\[ \| f(A) \|_A = (A f(A)x, f(A)x) = \]

\[ = \int_m^M \lambda \cdot f^2(\lambda) d (E_{\lambda} x, x) , \]

we obtain that

\[ \| z_p - z_n^p \|_A = (A (E - \theta \cdot A)^n z_p, (E - \theta \cdot A)^n z_p) = \]

\[ = \int_m^M \lambda \cdot (1 - \theta \cdot \lambda)^{2n} d (E_{\lambda} z_p, z_p) . \]

We define the function \( f(\lambda) \) by the formula

\[ f(\lambda) \equiv \lambda \cdot (1 - \theta \cdot \lambda)^{2n} . \]

In order to find the upper bound of \( \| z_p - z_n^p \|_A \)

one needs to maximize the function \( f(\lambda) \) in the closed interval \([m,M]\), i.e.

\[ f(\lambda) \rightarrow \max_{\lambda \in [m,M]} \]

Using the necessary condition for the existence of an extremum \( f'(\lambda) = 0 \) we obtain a stationary point

\[ \lambda = \lambda^* = \frac{1}{\theta \cdot (2 \cdot n + 1)} . \]

It is known that if \( f''(\lambda) < 0 \) then \( \lambda = \lambda^* \) is a point of local maximum. It can easily be verified that \( f''(\lambda^*) < 0 \).

Thus, the function \( f(\lambda) \) has a local maximum at \( \lambda = \lambda^* \). The upper bound of \( f(\lambda^*) \) is

\[ f(\lambda^*) = \frac{1}{2 \cdot n \cdot \theta} \left( \frac{2 \cdot n + 1}{2 \cdot n + 1} \right)^{2n} = \]

\[ = \frac{1}{2 \cdot n \cdot \theta} \left( 1 + \frac{1}{2 \cdot n} \right)^{2n} \cdot \frac{1}{2 \cdot n \cdot \theta} < \frac{1}{2 \cdot n \cdot \theta} . \]

It can be shown that

\[ \max_{\lambda \in [m,M]} f(\lambda) < \frac{1}{2 \cdot n \cdot \theta} . \]

If \( 0 < \theta \leq \frac{4}{3} \| A \| \),

Indeed,

\[ f(\| A \|) = \| A \| \cdot (1 - \theta \cdot \| A \|)^{2n} \]

and the function \( f(\lambda) \) has a maximum at

\[ \lambda^* = \frac{1}{\theta \cdot (2 \cdot n + 1)} \]

under the condition \( \theta \cdot \lambda < 1 \).

If \( \theta \cdot \lambda > 1 \) then the larger is \( \theta \), the greater is
Therefore it is sufficient to calculate $f(A)$ at $\theta = \frac{4}{3 \cdot \|A\|}$.

Thus,

$$f(A) = \frac{\|A\|}{9^n}.$$ 

Let us verify whether the inequality

$$f(A) < \frac{1}{2 \cdot n \cdot \theta \cdot e}$$

is satisfied. This inequality is equivalent to the inequality

$$8 \cdot n \cdot e < 3 \cdot 9^n, \quad n \geq 1.$$ 

Therefore, for $\theta \in \left(0, \frac{4}{3 \cdot \|A\|}\right)$ one has

$$\max_{\lambda \in [m, M]} f(\lambda) < \frac{1}{2 \cdot n \cdot \theta \cdot e}.$$ 

Thus

$$\|z_p - z_n\| \leq (2 \cdot n \cdot \theta \cdot e)^{-\frac{1}{2}} \cdot \|z_p\|_A,$$

so that

$$\|z_p - z_n\| \leq (2 \cdot n \cdot \theta \cdot e)^{-\frac{1}{2}} \cdot \|z_p\|_A.$$ 

Since

$$z_n - z_n^\delta = \theta \cdot \sum_{k=0}^{n-1} (E - \theta \cdot A)^k (u_p - u_b),$$

then the upper bound for $\|z_p - z_n^\delta\|^2$ is

$$\|z_n - z_n^\delta\|^2 = \left(\sum_{k=0}^{n-1} (1 - \theta \cdot \lambda)^k \right)^2 d((E - \theta \cdot A)^k (u_p - u_b), u_p - u_b) = \left(\sum_{k=0}^{n-1} (1 - \theta \cdot \lambda)^k \right)^2 \sum_{k=0}^{n-1} d((E - \theta \cdot A)^k (u_p - u_b), u_p - u_b).$$

Introducing the notation

$$g_n(\lambda) = \lambda^{-1} \cdot \left(1 - (1 - \theta \cdot \lambda)^n\right)^2$$

we obtain the upper bound for $g_n(\lambda)$ under the condition $0 < \theta \leq \frac{4}{3 \cdot \|A\|}.$

For $n = 2$ we have

$$g_2(\lambda) = \theta^2 \cdot (2 - \theta \cdot \lambda)^2.$$ 

Hence

$$\lambda = \lambda^{**} = \frac{2}{3 \cdot \theta}$$

is a stationary point. Since

$$g_2(\lambda^{**}) = -4 \cdot \theta^3 < 0,$$

then $\lambda^{**}$ is the point of a local maximum and

$$g_2(\lambda^{**}) = \frac{32}{27} \cdot \theta.$$ 

It can easily be shown by induction that

$$g_n(\lambda) \leq \frac{4}{3} \cdot \theta, \quad g_n(\lambda) \leq \frac{13}{20} \cdot n \cdot \theta, \quad n \geq 2.$$ 

Thus,

$$\|z_p - z_n\| \leq \left(\frac{13}{20} \cdot n \cdot \theta\right)^{\frac{1}{2}} \cdot \delta \quad \text{if} \quad n \geq 2,$$

$$\|z_p - z_n\| \leq \left(\frac{4}{3} \cdot n \cdot \theta\right)^{\frac{1}{2}} \cdot \delta \quad \text{if} \quad n \geq 1.$$ 

Moreover, it follows from the inequality

$$\|z_p - z_n\| \leq \|z_p - z_n^\delta\| + \|z_n - z_n^\delta\|$$

that

$$\|z_p - z_n\| \leq \left(2 \cdot n \cdot \theta \cdot e\right)^{-\frac{1}{2}} \cdot \|z_p\|_A +$$

$$\left(\frac{4}{3} \cdot n \cdot \theta\right)^{\frac{1}{2}} \cdot \delta, \quad n \geq 1,$$

$$\|z_p - z_n\| \leq \left(2 \cdot n \cdot \theta \cdot e\right)^{-\frac{1}{2}} \cdot \|z_p\|_A +$$

$$\left(\frac{13}{20} \cdot n \cdot \theta\right)^{\frac{1}{2}} \cdot \delta, \quad n \geq 2.$$ 

It can be seen from the above inequalities that the first terms tend to zero as $n \to \infty$, and it is sufficient to require $\sqrt{n} \cdot \delta \to 0$ as $n \to \infty$, $\delta \to 0$ in order to get

$$\|z_p - z_n\|_A \to 0.$$ 

Theorem 4 is proved.

3 Determination of the Optimal Number of Iterations

Let us find such $n = n(\delta)$, depending on the given error $\delta$ in the right-hand side of (1), for which the
estimates (6) become minimal. Differentiating the right-hand side of (6) and equating it to zero we obtain
\[
- \frac{1}{2} (2 \cdot \theta \cdot n)^{\frac{1}{2}} \cdot n^{\frac{3}{2}} \cdot \| \mathbf{p} \|_A + \\
+ \frac{1}{2} \left( \frac{13}{20} \cdot \theta \right)^{\frac{1}{2}} \cdot \mathbf{\delta} \cdot n^{\frac{1}{2}} = 0.
\]
Thus,
\[
n_{\text{opt}} = \left( \frac{13}{10} \right)^{\frac{1}{2}} \cdot \theta^{-1} \cdot e^{\frac{1}{2}} \cdot \mathbf{\delta}^{-1} \cdot \| \mathbf{p} \|_A .
\] (7)
Substituting (7) into (6) we obtain
\[
\| \mathbf{z}_p - \mathbf{z}_n^{\delta \text{opt}} \|_A \leq \left( \frac{13}{10} \right)^{\frac{1}{2}} \cdot (2 \cdot \mathbf{\delta} \cdot \| \mathbf{z}_p \|_A)^{\frac{1}{2}} .
\] (8)
It is seen from (8) that the optimal error estimate does not depend on the iteration parameter \( \theta \). However, as can be seen from (7), the optimal number of iterations \( n_{\text{opt}} \) depends on \( \theta \).

Therefore, in order to reduce the number of iterations \( n \), i.e. in order to reduce calculation cost, one needs to choose \( \theta \) as large as possible from the condition
\[
0 < \theta \leq \frac{4}{3 \| A \|}
\]
so that \( n_{\text{opt}} \) would be an integer, for example,
\[
n_{\text{opt}} = \left[ 0.4 \cdot \left( \| \mathbf{z}_p \|_A \cdot \mathbf{\delta} \right)^{-1} \right],
\]
where the square bracket \( [ \cdot ] \) represents the integer part of \( t \).

Thus, we have proved the following theorem:

**THEOREM 5.** Optimal error estimate for iterative process (3) under the conditions of Theorem 4 has the form
\[
\| \mathbf{z}_p - \mathbf{z}_n^{\delta \text{opt}} \|_A \leq \left( \frac{13}{10} \right)^{\frac{1}{2}} \cdot (2 \cdot \mathbf{\delta} \cdot \| \mathbf{z}_p \|_A)^{\frac{1}{2}}
\]
and is reached
\[
n_{\text{opt}} = \left[ \frac{10 \cdot \| \mathbf{z}_p \|_A}{13 \cdot e \cdot \| \mathbf{\delta} \|} \right].
\]

### 4 The Degree of Convergence Estimation in the Norm of Initial Hilbert Space

Let us show that the results obtained above also hold in the norm of initial Hilbert space \( H \) if we shall require the following additional assumptions:
\[
E_{m_0} \mathbf{z}_p = 0 \text{ and } E_{m_0} \mathbf{z}^\delta = 0,
\]
where \( E_{m_0} \equiv \int_0^m \mathbf{d} E_{s} , \ m_0 \in (0, \| A \|) \) is some fixed number.

Indeed, since
\[
\| \mathbf{z}_p - \mathbf{z}^\delta \|_H = (A(\mathbf{z}_p - \mathbf{z}^\delta),\mathbf{z}_p - \mathbf{z}^\delta) \leq \\
\leq M \cdot (\mathbf{z}_p - \mathbf{z}^\delta,\mathbf{z}_p - \mathbf{z}^\delta)_n = \\
= M \cdot \| \mathbf{z}_p - \mathbf{z}^\delta \|_H ,
\]
\[
\| \mathbf{z}_p - \mathbf{z}^\delta \|_H = (A(\mathbf{z}_p - \mathbf{z}^\delta),\mathbf{z}_p - \mathbf{z}^\delta)_n \geq \\
\geq m_0 \cdot (\mathbf{z}_p - \mathbf{z}^\delta,\mathbf{z}_p - \mathbf{z}^\delta)_n = m_0 \cdot \| \mathbf{z}_p - \mathbf{z}^\delta \|_H ,
\]
where
\[
M = \sup \left( A \mathbf{x}, \mathbf{x} > 0 \right) ,
\]
\[
m_0 \in (0, \| A \|) ,
\]
then
\[
\sqrt{m_0 \cdot \| \mathbf{z}_p - \mathbf{z}^\delta \|_H } \leq \| \mathbf{z}_p - \mathbf{z}^\delta \|_H \leq \sqrt{M} \cdot \| \mathbf{z}_p - \mathbf{z}^\delta \|_H .
\] (9)

Two-sided inequality (9) allows one to assert that the sequence \( \| \mathbf{z}_p - \mathbf{z}^\delta \|_H \) is convergent to zero if and only if the sequence \( \| \mathbf{z}_p - \mathbf{z}^\delta \|_H \) converges to zero.

Since the convergence of \( \| \mathbf{z}_p - \mathbf{z}^\delta \|_H \) is proved in Theorem 4, the following theorem holds:

**THEOREM 6.** Iterative process (3) converges in the norm of the given Hilbert space \( H \), if
1) \( E_{m_0} \mathbf{z}_n = 0 \) , \( E_{m_0} \mathbf{z}^\delta = 0 \),

where \( E_{m_0} \equiv \int_0^m \mathbf{d} E_{s} , \ m_0 \in (0, \| A \|) \) is some fixed number.

2) the number of iterations \( n \) is chosen from the condition
\[
\sqrt{n \cdot \delta} \to 0 \text{ as } n \to \infty , \delta \to 0 .
\]
Moreover, under the condition 0 < \( \theta \leq \frac{4}{3 \| A \|} \) the following error estimates hold for iterative process (3)
\[
\| \mathbf{z}_p - \mathbf{z}^\delta \|_H \leq (2 \cdot n \cdot m_0 \cdot 0 \cdot e)^{\frac{1}{2}} \cdot \| \mathbf{z}_p \|_H + \\
\left( \frac{4}{3} \cdot m_0 \cdot 0 \right)^{\frac{1}{2}} \cdot \delta , \ n \geq 1 ,
\]
\[
\| \mathbf{z}_p - \mathbf{z}^\delta \|_H \leq (2 \cdot n \cdot m_0 \cdot 0 \cdot e)^{\frac{1}{2}} \cdot \| \mathbf{z}_p \|_H + \\
\left( \frac{13}{20} \cdot m_0 \cdot 0 \right)^{\frac{1}{2}} \cdot \delta , \ n \geq 2 ,
\]


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where $m_0 \in (0,\|A\|)$ is some fixed number.

5 Some Remarks Concerning Obtained Results

A. The operator $A$ is assumed to be self-conjugate and positive. If the operator $A$ is not self-conjugate or is not positive, then equivalent equation

$$A^*Az = A^*u,$$

$$z \in H, \ u \in AH,$$

(10)

should be used instead of equation (1), where the operator $A^*$ is conjugate to $A$. All the results obtained above are valid in this case.

B. All the results obtained above take place if zero does not belong to the spectrum of the operator $A$. If the point zero belongs to the spectrum of the operator $A$, then equation (1) (or equation (10), if zero belongs to the spectrum of the operator $A^*A$) can have infinitely many solutions. The approach described above and all the results obtained in the paper are valid also in this case. The method described above guarantees convergence to normal solution, i.e. to the solution with minimal norm ([1]).

References:
