A Design of Parameter Optimal Iterative Learning Control for Linear Discrete-Time Systems

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Abstract: In this paper, the iterative learning control algorithm proposed by Owens and Feng, which guarantees the monotonic convergence of the tracking error norms along with the trial, will be modified. The learning gain of the proposed method is determined through a quadratic cost function. Numerical simulations will be presented to confirm the validity of the proposed design.

Key–Words: iterative learning control, discrete-time systems, parameter optimization, monotonic convergence.

1 Introduction

The iterative learning control (ILC) proposed by Arimoto et al [1] is a method to control systems operating in a repetitive mode. Examples such systems include robot manipulators, chemical batch processes reliability testing rigs. The control purpose of ILC is to follow a specified trajectory with high precision. Unlike model matching method [2], [3], it might be useful for the plant with non-minimum phase property.

There are many approaches to ILC in the literatures [4]-[6], i.e., the method based on the PD control [1], the inverse system [7]-[9], $H_\infty$ control [10], [11] and so on. Although the convergence properties of these algorithms have been analyzed, it is not always clear how to choose the free parameters of the algorithms to attain fast or monotonic convergence.

Owens and Feng [12] used parameter optimization through a quadratic cost function as a method to establish the iterative learning control law. The important feature of the algorithm is that the learning gain is to be varied from each trial. The method guarantees monotonic convergence of the error to zero, if a given plant satisfies a definite condition [12]. In the case of non-definite plants, the behavior of the method were discussed in [13].

In this paper, the method by Owens and Feng will be modified for non-positive definite plants. The learning gain is varied not only each trial, but also varied each steps. With such modification, it can be useful for non-definite plants. Moreover, a special analysis for the tracking error will be shown.

2 Problem Statement

Consider the following single-input, single-output linear discrete-time system:

\[ x(t+1) = Ax(t) + bu(t) \]
\[ y(t) = cx(t) \]  \hspace{1cm} (1)

where \( u(t), y(t) \) and \( x(t) \) are the input, output and state vector respectively. \( t \) is the time instant. The transfer function \( G(z) \) for the above system is given by

\[ G(z) = c(zI - A)^{-1}b, \]  \hspace{1cm} (2)

where \( z \) is the time-shift operator. Since

\[ (zI - A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \cdots \]

The input-output relation of the system (1) can be also written by

\[ y(t) = cbu(t-1) + cAbu(t-2) + cA^2bu(t-3) + \cdots. \]  \hspace{1cm} (3)
Define the output error \(e_k(t)\) by

\[
e_k(t) = r(t) - y_k(t),
\]

where \(r(t)\) is the reference signal, and the subscript \(k\) denotes the trial index, i.e., \(y_k(t)\) means the output signal at time instance \(t\) in the \(k\)-th control trial. \(u_k(t)\) is defined by the same way. Then, set the column vectors \(Y_k\), \(U_k\) and \(E_k\) by

\[
Y_k = \begin{bmatrix} y_k(1) \\ y_k(2) \\ \vdots \\ y_k(\tau) \end{bmatrix}, \quad U_k = \begin{bmatrix} u_k(0) \\ u_k(1) \\ \vdots \\ u_k(\tau-1) \end{bmatrix}, \quad E_k = \begin{bmatrix} e_k(1) \\ e_k(2) \\ \vdots \\ e_k(\tau) \end{bmatrix}
\]

\[
E_k = \begin{bmatrix} r(1) - y_k(1) \\ r(2) - y_k(2) \\ \vdots \\ r(\tau) - y_k(\tau) \end{bmatrix}.
\]

The purpose of control is to calculate the uniformly bounded control input vector \(U_{k+1}\) of the form

\[
U_{k+1} = U_k + G_k E_k
\]

which make \(E_k \to 0\) as \(k \to \infty\), where \(G_k\) is a gain matrix to be determined \((G_1 = I)\).

From eqns.(4) and (5),

\[
Y_k = T_{\tau-1} U_k = \begin{bmatrix} cb & 0 & \cdots & 0 \\ cAb & cb & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ cA^{\tau-1}b & cA^{\tau-2}b & \cdots & cb \end{bmatrix} U_k
\]

\[
= \begin{bmatrix} u_k(0) \\ u_k(1) \\ u_k(\tau-1) \\ \vdots \\ u_k(\tau-2) \\ u_k(0) \end{bmatrix} \begin{bmatrix} cb \\ cAb \\ \vdots \\ \vdots \\ \vdots \\ cA^{\tau-1}b \end{bmatrix}
\]

If \(T_{\tau-1}\) is unknown, an initial values of the Markov parameters can be found by solving eqn.(9) after the first trial. In general, the Markov parameters can be updated by

\[
\Delta Y_k = T_{\tau-1}(U_k - U_{k-1}),
\]

\[
\Delta u_k(0) \\ \Delta u_k(1) \\ \Delta u_k(\tau-1) \\ \vdots \\ \Delta u_k(\tau-2) \\ \Delta u_k(0) \end{bmatrix} \begin{bmatrix} cb \\ cAb \\ \vdots \\ \vdots \\ \vdots \\ cA^{\tau-1}b \end{bmatrix},
\]

\[
\Delta Y_k := Y_k - Y_{k-1}, \quad \Delta u_k(i) := u_k(i) - u_{k-1}(i).
\]

If \(|\Delta u_k(0)|\) is smaller than some tolerance values which will be given by the designer, then stop the update of the Markov parameters.

Owens and Feng [12] considered the case where \(G_k = g_k I\). Then,

\[
Y_{k+1} = T_{\tau-1}(U_k + g_k E_k) = Y_k + g_k T_{\tau-1} E_k
\]

Set the cost function \(J(g_k)\) by

\[
J(g_k) = ||E_{k+1}||^2 + \rho g_k^2, \rho > 0
\]

\[
= ||(I - g_k T_{\tau-1}) E_k||^2 + \rho g_k^2
\]

\[
= (\rho + ||T_{\tau-1} E_k||^2)g_k^2 - 2E_k^T T_{\tau-1} E_k g_k + ||E_k||^2.
\]

Minimizing with respect \(g_k\) gives

\[
g_k = \frac{E_k^T T_{\tau-1} E_k}{\rho + ||T_{\tau-1} E_k||^2}.
\]

Then,

\[
E_{k+1}^T E_k + E_k^T T_{\tau-1} (I - g_k T_{\tau-1}) E_k - E_k^T T_{\tau-1} E_k
\]

\[
= E_k^T (I - g_k T_{\tau-1}^T) (I - g_k T_{\tau-1}) E_k - E_k^T T_{\tau-1} E_k
\]

\[
= E_k^T \{g_k^2 T_{\tau-1} (T_{\tau-1} - g_k (T_{\tau-1}^T + T_{\tau-1})) E_k
\]

\[
= g_k^2 ||T_{\tau-1} E_k||^2 - 2(\rho + ||T_{\tau-1} E_k||^2)
\]

\[
= -g_k^2(2\rho + ||T_{\tau-1} E_k||^2) \leq 0,
\]

and thus \(E_{k+1}^T E_k \geq 0\) converges to a non-negative value. Since \(2\rho + ||T_{\tau-1} E_k||^2 > 0\), \(\lim_{k \to \infty} g_k = \lim_{k \to \infty} E_k^T T_{\tau-1} E_k = 0\).

Therefore, \(E_k \to 0\) if \(T_{\tau-1}^T + T_{\tau-1} > 0\) or \(T_{\tau-1}^T + T_{\tau-1} < 0\). In the following sections, it will be considered the gain which makes \(E_k \to 0\) without the above definite conditions.

3 Derivation of Diagonal Gain Matrix

Applying the control input (7) to eqn.(8),

\[
Y_{k+1} = T_{\tau-1} U_{k+1}
\]

\[
= T_{\tau-1}(U_k + G_k E_k)
\]

\[
= Y_k + T_{\tau-1} E_k G_k,
\]

\[
E_k := \begin{bmatrix} e_k(1) \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad G_k := \begin{bmatrix} g_k(1) \\ \vdots \\ \vdots \\ \vdots \\ g_k(\tau) \end{bmatrix}.
\]

Therefore,

\[
E_{k+1} = E_k - T_{\tau-1} E_k G_k.
\]

Set the cost function \(J(\tilde{G}_k)\) by

\[
J(\tilde{G}_k) = ||E_{k+1}||^2 + \rho ||\tilde{G}_k||^2, \rho > 0.
\]

Then, using eqn.(8),

\[
J(\tilde{G}_k) = ||E_k - T_{\tau-1} E_k \tilde{G}_k||^2 + \rho ||\tilde{G}_k||^2
\]

\[
= \tilde{G}_k^T \Gamma_k \tilde{G}_k - \tilde{G}_k^T E_k^T T_{\tau-1} E_k
\]

\[
= \tilde{G}_k^T E_k^T T_{\tau-1} E_k + ||E_k||^2
\]

\[
= \tilde{G}_k^T \Gamma_k \tilde{G}_k - ||E_k||^2 + ||E_k||^2
\]

\[
= \tilde{G}_k^T \Gamma_k \tilde{G}_k - ||E_k||^2 + ||E_k||^2.
\]
where
\[ \Gamma_k = \rho I + E_k^T T_{\tau-1}^k T_{\tau-1}^k E_k. \] (15)

Therefore, \( J(\mathcal{G}_k) \) is minimized if
\[ \mathcal{G}_k = \Gamma_k^{-1} E_k^T T_{\tau-1}^k E_k \]
\[ = E_k^T T_{\tau-1}^k (\rho I + T_{\tau-1}^k E_k E_k^T T_{\tau-1}^k)^{-1} E_k \] (16)

Substituting eqn.(16) to eqn.(12),
\[ E_{k+1} = (I - T_{\tau-1}^k E_k \Gamma_k^{-1} E_k^T T_{\tau-1}^k) E_k. \] (17)

Consider the following candidate of Lyapunov function \( V_k \):
\[ V_k = E_k^T E_k \geq 0. \] (18)

Then,
\[ \Delta V_{k+1} = V_{k+1} - V_k \]
\[ = E_k^T ((I - T_{\tau-1}^k E_k \Gamma_k^{-1} E_k^T T_{\tau-1}^k)^2 - I) E_k \]
\[ = E_k^T (I - \Gamma_k^{-1} E_k^T T_{\tau-1}^k E_k)^2 \]
\[ = -2T_{\tau-1}^k E_k \Gamma_k^{-1} E_k^T T_{\tau-1}^k E_k \]
\[ = \mathcal{G}_k^T (E_k^T T_{\tau-1}^k T_{\tau-1}^k E_k - 2 \Gamma_k) \mathcal{G}_k \]
\[ = -\mathcal{G}_k^T (2\rho I + E_k^T T_{\tau-1}^k T_{\tau-1}^k E_k) \mathcal{G}_k \leq 0. \] (19)

Therefore, \( V_k \) becomes the Lyapunov function and \( \Delta V_k \rightarrow 0 \) as \( k \rightarrow \infty \). Since
\[ 2\rho I + E_k^T T_{\tau-1}^k T_{\tau-1}^k E_k = \rho I + \Gamma_k > 0, \]
it follows
\[ \lim_{k \rightarrow \infty} \mathcal{G}_k = \lim_{k \rightarrow \infty} E_k^T T_{\tau-1}^k E_k = 0. \] (20)

Since
\[ E_k^T T_{\tau-1}^k E_k \]
\[ = \begin{bmatrix} e_k(1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_k(\tau) \end{bmatrix} \begin{bmatrix} cb & \cdots & cA^{r-1}b \\ \vdots & \ddots & \vdots \\ 0 & \cdots & cb \end{bmatrix} \begin{bmatrix} e_k(1) \\ \vdots \\ e_k(\tau) \end{bmatrix} \]
\[ = \begin{bmatrix} cbe_k(1) + e_k(1) \sum_{i=2}^\tau cA^{r-1}be_k(i) \\ \vdots \\ cbe_k(\tau) + e_k(\tau) \sum_{i=2}^\tau cA^{r-2}be_k(i) \end{bmatrix} \]
\[ = \begin{bmatrix} cbe_k(\tau - 2) + e_k(\tau - 2) \sum_{i=2}^\tau cA^{r-\tau+1}be_k(i) \\ \vdots \\ cbe_k(\tau - 1) + cAbe_k(\tau)e_k(\tau - 1) \end{bmatrix} \]
\[ = \begin{bmatrix} cbe_k(\tau) \\ \vdots \\ cbe_k(\tau - 1) \end{bmatrix} \] (21)

From the bottom element of the above matrix, it follows
\[ \lim_{k \rightarrow \infty} e_k(\tau) = 0 \] (22)

If \( cb \neq 0 \). Substituting the above relation to eqn.(21), it follows
\[ \lim_{k \rightarrow \infty} E_k^T T_{\tau-1}^k E_k \]
\[ = \begin{bmatrix} cbe_k(1) + e_k(1) \sum_{i=2}^\tau cA^{r-1}be_k(i) \\ \vdots \\ cbe_k(\tau - 2) + cAbe_k(\tau - 1)e_k(\tau - 2) \end{bmatrix} \]
\[ = \begin{bmatrix} cbe_k(\tau - 1) \end{bmatrix} \]

and thus
\[ \lim_{k \rightarrow \infty} e_k(\tau - 1) = 0 \] (23)

and so on. Therefore,
\[ \lim_{k \rightarrow \infty} E_k = \lim_{k \rightarrow \infty} \begin{bmatrix} e_k(1) \\ \vdots \\ e_k(\tau) \end{bmatrix} = 0. \] (24)

Even if \( cb = 0 \), there exists the integer \( d \) such that \( cb = \cdots = cA^{d-1}b = 0 \) and \( cA^d b \neq 0 \), and thus the above discussions is not loss of generality.

4 Non-Diagonal Gain Matrix

Define the following matrices:
\[ \mathcal{E}_k = \begin{bmatrix} E_k^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_k^T \end{bmatrix}, \quad \mathcal{G}_k = \text{rs}[\mathcal{G}_k], \] (25)

where \( \text{rs}[:] \) denotes the row string of matrix. Then,
\[ \mathcal{G}_k E_k = \mathcal{E}_k \mathcal{G}_k \]
holds, and the discussions from eqn.(11) to eqn.(20) also hold. Moreover,
\[ \mathcal{E}_k^T T_{\tau-1}^k E_k \]
\[ = \begin{bmatrix} E_k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_k \end{bmatrix} \begin{bmatrix} cb & \cdots & cA^{r-1}b \\ \vdots & \ddots & \vdots \\ 0 & \cdots & cb \end{bmatrix} \begin{bmatrix} e_k(1) \\ \vdots \\ e_k(\tau) \end{bmatrix} \]
\[ = \begin{bmatrix} \sum_{i=1}^\tau cA^{i-1}be_k(i) \\ \vdots \\ cbe_k(\tau) + e_k(\tau) \sum_{i=1}^\tau cA^{\tau-2}be_k(i) \end{bmatrix} \]
\[ = \begin{bmatrix} \sum_{i=\tau-1}^\tau cA^{i-\tau+2}be_k(i) \\ cbE_k e_k(\tau) \end{bmatrix} \]
\[ \rightarrow 0. \] (26)

If \( cb \neq 0 \), it follows
\[ \lim_{k \rightarrow \infty} E_k e_k(\tau) = \lim_{k \rightarrow \infty} \begin{bmatrix} e_k(1)e_k(\tau) \\ e_k(2)e_k(\tau) \\ \vdots \\ e_k(\tau) \end{bmatrix} = 0. \] (27)
from the bottom element in eqn.(26), and thus

$$\lim_{k \to \infty} e_k(\tau) = 0.$$  \hfill (28)

Substituting the above relation to eqn.(26), it follows

$$\lim_{k \to \infty} E_k^T T_{\tau-1} E_k = \left[ \begin{array}{c} \sum_{i=2}^{\tau-1} c A^{i-1} b E_k e_k(i) \\ \vdots \\ cb E_k e_k(\tau-2) + c Ab E_k e_k(\tau-1) \\ cb E_k e_k(\tau-1) \\ \end{array} \right]$$

and thus $\lim_{k \to \infty} e_k(\tau-1) = 0$ and so on. Therefore,

$$\lim_{k \to \infty} E_k = \lim_{k \to \infty} \left[ \begin{array}{c} e_k(1) \\ e_k(2) \\ \vdots \\ e_k(\tau) \end{array} \right] = 0$$  \hfill (29)

As is in the previous discussion, it is not loss of generality if $cb = 0$.

It is worth noting that the expression of the second equality of eqn.(16) for $\hat{G}_k$ is useful, it needs the inversion of $\tau \times \tau$ matrix while the first expression needs the inversion of $\tau^2 \times \tau^2$.

## 5 High Order Algorithm

Owens and Feng also proposed a high order algorithm as follows:

$$U_{k+1} = U_k + \sum_{i=0}^{M} g_i E_{k-i}$$  \hfill (30)

for an integer $M > 0$. In this section, this type control law is extended like the previous section, i.e.,

$$U_{k+1} = U_k + \sum_{i=0}^{M} G_{ki} E_{k-i}$$  \hfill (31)

where

$$G_{ki} = \text{diag}\{g_{ki}(1), \ldots, g_{ki}(\tau)\}.$$  

It can be easily obtained

$$E_{k+1} = E_k - T_{\tau-1} \sum_{i=0}^{M} G_{ki} E_{k-i}.$$  \hfill (32)

Define

$$\bar{E}_k = \text{diag}\{e_k(1), \ldots, e_k(\tau)\}, \quad \bar{G}_k = \left[ \begin{array}{c} g_{k0}(1) \\ \vdots \\ g_{k0}(\tau) \\ \vdots \\ g_{kM}(1) \\ \vdots \\ g_{kM}(\tau) \end{array} \right].$$  \hfill (33)

Then, eqn.(32) can be rewritten by

$$E_{k+1} = E_k - T_{\tau-1} \left[ \begin{array}{c} \bar{E}_k \\ \vdots \\ \bar{E}_{k-M} \end{array} \right] \bar{G}_k.$$  \hfill (34)

Set the cost function $J(\bar{G}_k)$ by

$$J(\bar{G}_k) = ||E_{k+1}||^2 + \sum_{i=0}^{M} \rho_i ||G_{ki}||^2$$  \hfill (35)

$$= ||\Gamma^{-1/2} \bar{G}_k - \Gamma^{-1/2} ||^2 \left[ \begin{array}{c} \bar{E}_k^T \\ \vdots \\ \bar{E}_{k-M}^T \end{array} \right] T_{\tau-1} E_k^T ||^2$$

$$+ ||E_k||^2 - ||\Gamma^{-1/2} ||^2 \left[ \begin{array}{c} \bar{E}_k^T \\ \vdots \\ \bar{E}_{k-M}^T \end{array} \right] T_{\tau-1} E_k^T ||^2$$

where

$$\Gamma = \left[ \begin{array}{cccc} \rho_0 I + F_{k,k} & F_{k,k-1} & \cdots & F_{k,k-M} \\ F_{k-1,k} & \rho_1 I + F_{k-1,k-1} & \cdots & F_{k-1,k-M} \\ \vdots & \vdots & \ddots & \vdots \\ F_{k-M,k} & F_{k-M,k-1} & \cdots & \rho_M I + F_{k-M,k-M} \end{array} \right].$$

$$F_{ij} := \bar{E}_i^T T_{\tau-1} \bar{E}_j.$$  \hfill (36)

Therefore, $J(\bar{G}_k)$ is minimized if

$$\bar{G}_k = \Gamma^{-1} \left[ \begin{array}{c} \bar{E}_k^T \\ \vdots \\ \bar{E}_{k-M}^T \end{array} \right] T_{\tau-1} E_k.$$  \hfill (37)

Error analysis can be done similarly as in the previous section.

## 6 Numerical Simulations

Set $(A, b, c)$ in eqn.(1) by

$$A = \left[ \begin{array}{cc} 0 & 1 \\ -0.5 & -1 \end{array} \right], \quad b = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right], \quad c = \left[ \begin{array}{c} 1.2 \\ 1 \end{array} \right].$$

In this case, the control objective is non-minimum phase plant and the definite conditions are not satisfied.

Reference signal $r(t)$ is given by

$$r(t) = \left\{ \begin{array}{ll} 1 & (1 \leq t < 20) \\ -1 & (t \geq 20) \end{array} \right.$$  

and $\rho = 1$. Figure 1 shows the output tracking errors of scalar gain ILC proposed by Owens and Feng [12], where errors of the 5th, 10th and 40th trial are shown. Since the learning does not proceed, it is hard to distinguish these three lines.
Figure 1: Output Tracking Error with Scalar Gain

Figure 2: Output Tracking Error with Diagonal Gain Matrix

Figure 3: Square Sum of Output Tracking Errors

Figure 4: Output Tracking Error with Non-Diagonal Gain Matrix

7 Conclusions

In this paper, the method proposed by Owens and Feng [12] was modified, and the assumption to confirm the tracking error convergence to zero was removed. It was also developed the high order algorithm proposed by Owens and Feng. Simulation results was shown for validity of the proposed method.

In this paper, it was only considered the ideal case linear systems and will necessary to extend for the systems having uncertain or time-varying parameters and non-linearity. It will be also considered the feedback control of the proposed method. Numerical simulations will be shown on-site, to confirm the validity of the proposed design.
References:


A Appendix

In the followings, it is assumed that the initial values of the state $x_0$ are identical in each trial. Considering $x_0$, eqn(8) can be rewritten by

$$Y_k = T_{\tau-1}U_k + X_0, \quad X_0 := \begin{bmatrix} cA \\ cA^2 \\ \vdots \\ cA^{\tau-1} \end{bmatrix} x_0. \quad (38)$$

Applying the above relation for the first and second trials, it gives

$$Y_2 - Y_1 = T_{\tau-1}(U_2 - U_1).$$

Using eqn.(7) and $G_1 = I$, it becomes

$$Y_2 - Y_1 = T_{\tau-1} E_1 \begin{bmatrix} e_1(1) & 0 & \cdots & 0 \\ e_1(2) & e_1(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_1(\tau) & e_1(\tau-1) & \cdots & e_1(1) \end{bmatrix} \begin{bmatrix} cb \\ cAb \\ \vdots \\ cA^{\tau-1}b \end{bmatrix}. \quad (39)$$

Thus, the Markov parameters can be easy to obtain from the above equation unless $e_1(1) \neq 0$. (In the case of $|e_1(1)| \ll 1$, it may be useful to set the reference signal temporary.)