Regularization of the hypersingular integrals in 3-D fracture mechanics. Rectangular BE and piecewise linear approximations.

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Abstract: - This paper deals with the hypersingular integrals, which arise when the boundary integral equation (BIE) methods are used for solution of fracture mechanics problems. For hypersingular integral regularization the methodology based on theory of distribution and Green’s theorems have been used. This methodology is applied for regularization of the hypersingular integrals over rectangular boundary elements (BE) for the case of piecewise constant and piecewise linear approximations. The hypersingular integrals are transformed into the regular contour integrals that can be easily calculated analytically.

Key-Words: - Crack, hypersingular integral, BIE, fracture.

1 Introduction
Approach to regularization of the hypersingular integrals based on the theory of distributions and finite part integrals has been developed in [5]. Then it was further developed for regularization of the hypersingular integrals in static and dynamic problems of fracture mechanics in [10] and [11] respectively. More applications of this regularization method can be found in review articles [2, 3, 9]. Further development of this approach and application of the Green’s theorems in the sense of theory of distribution has been done in [6]. The equations presented in [7, 8] permit transform divergent hypersingular integrals into the regular ones. The developed approach can be applied not only for hypersingular integrals regularization but also for a wide class of divergent integral regularizations.

In the present paper, the above mentioned approach for the divergent integral regularization is further developed and applied for the case of 3-D elastostatic crack problems. We consider the 2-D hypersingular integrals over arbitrary convex polygon for piecewise constant approximation and over rectangular BE for piecewise linear approximation as Hadamard’s finite part integrals (F.P.). It is important to mention that in resentened equations all calculations can be done analytically, no numerical integration is needed.

2 Boundary integral equations
Let us consider an infinite elastic medium occupying the whole space $\mathbb{R}^3$ which contain arbitrarily oriented plane crack. The crack is described by a corresponding oriented surfaces $\Omega^+ \cup \Omega^-$, where $\Omega^+$ and $\Omega^-$ are opposite edges. The crack surfaces $\Omega^+$ and $\Omega^-$ are locally parallel, and their curvatures are relatively small. In $V := \mathbb{R}^3 \setminus \Omega^+ \cup \Omega^-$ we consider the behavior of the medium governed by the linear Lame equations of elastostatics for the displacement field $u_i(x)$, i.e.,

$$A_{ij} u_j(x) = 0, \quad A_{ij} = \mu \delta_{ij} \partial_i \partial_j + (\lambda + \mu) \partial_i \partial_j, \quad x \in V,$$

subject to the boundary conditions

$$p_i(x) = p_i^+ \text{ for } x_i \in \Omega^+, \quad p_i(x) = p_i^- \text{ for } x_i \in \Omega^-.$$ 

Because we consider infinite region must satisfy additional conditions in the form

$$u_j(x) = O(r^{-1}) \quad \sigma_{ij}(x) = O(r^{-2}) \quad \text{for } r \to \infty$$

where $\lambda$ and $\mu$ are Lame constants, $\mu > 0$ and $\lambda > -\mu$, $\delta_{ij}$ is a Kronecker’s symbol, $\partial_i = \partial / \partial x_i$ denotes the partial derivatives with respect to space, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the distance in the 3-D Euclidian space. Throughout this paper we use the Einstein summation convention.

We introduce Cartesian coordinates system, with $x_1$ and $x_2$ axes in the plane of the crack, and the $x_3$
axis perpendicular to this plane. Following [1-3] we suppose the opposite crack edge surfaces are identified \((\Omega^+ = \Omega^- = \Omega)\) and are distinguished only by the direction of the external normal vectors \((\mathbf{n}^+ = -\mathbf{n}^- = \mathbf{n})\). Then deformation of the crack edges is defined by crack opening \(\Delta u_i(x) = u_i^+(x) - u_i^-(x), \forall x \in \Omega\), since we suppose that only small deformations occur.

In [1-3] it was shown that in this case the BIE that relate load \(p_i(y)\) on the crack faces \(\Omega^+\) and \(\Omega^-\) and crack opening \(\Delta u_i(x)\) may be written in the following form

\[ p_i(y) = - \int_\Omega F_{ij}(x,y) \Delta u_j(x) \, dS. \]  

The kernels \(F_{ij}(x,y)\) in the BIE (5) may be presented in the form

\[
F_{11} = \frac{\mu}{4\pi(1-\nu)} \left[ \frac{(1-2\nu)}{r^3} + 3\nu \frac{(x_1-y_1)^2}{r^5} \right], \\
F_{12} = \frac{\mu\nu}{4\pi(1-\nu)} \frac{(x_1-y_1)(x_2-y_2)}{r^5}, \\
F_{22} = \frac{\mu}{4\pi(1-\nu)} \left[ \frac{(1-2\nu)}{r^3} + 3\nu \frac{(x_2-y_2)^2}{r^5} \right], \\
F_{33} = \frac{\mu}{4\pi(1-\nu)} \frac{1}{r^3}, 
\]

where \(\mu\) and \(\nu\) are elastic modulus and Poison ration.

Simple observation shows that kernels in the BIE (5) tend to infinity when \(r \to 0\). More detailed analysis of the equations (5) and kernels (6) give us the following result, with \(x \to y\)

\[ F_{ij}(x,y) \to r^{-3} \]  

Integrals with these kernels are divergent and therefore need special consideration. Usually such integrals are considered in the sense of finite part according to Hadamard.

To construction finite dimensional equations we shall apply approximation by finite functions and splitting \(\Omega\) into finite elements

\[ \Omega = \bigcup_{n=1}^{N} \Omega_n, \quad \Omega_n \cap \Omega_k = \emptyset, \text{ if } n \neq k. \]  

We will introduce systems of shape functions \(\varphi_q(x)\) and \(\psi_p(x)\) in the finite domains \(\Omega_n\). Then the vectors of displacements and traction on the boundary element \(\Omega_n\) will be represented approximately in the form

\[
\Delta u_i(x) \approx \sum_{q=1}^{Q} \Delta u_i^q(\mathbf{x}_q) \varphi_{nq}(x), \quad x \in \Omega_n \\
p_i(x) \approx \sum_{q=1}^{Q} p_i^q(\mathbf{x}_q) \varphi_{nq}(x), \quad x \in \Omega_n
\]  

and on the whole crack surface \(\Omega\) in the form

\[
\Delta u_i(x) \approx \sum_{n=1}^{N} \sum_{q=1}^{Q} \Delta u_i^q(\mathbf{x}_q) \varphi_{nq}(x), \quad x \in \bigcup_{n=1}^{N} \partial \Omega_n \\
p_i(x) \approx \sum_{n=1}^{N} \sum_{q=1}^{Q} p_i^q(\mathbf{x}_q) \varphi_{nq}(x), \quad x \in \bigcup_{n=1}^{N} \partial \Omega_n
\]

Substitution of the expressions (10) in (5) gives us the finite-dimensional representations for the vectors of displacements and traction on the boundary in the form

\[
p_i^n(y) = - \sum_{n=1}^{N} \sum_{q=1}^{Q} F_{ij}(y,x_q) \Delta u_j^q(x_q) \\
\]  

More detailed information on transition from the BIE to the BEM equations can be found in [1].

### 3 Piecewise constant approximation

In order to simplify situation we transform global system of coordinates such that the origins of global and local systems of coordinates coincide. The coordinate axes \(x_1\) and \(x_2\) are located in the plane of the element and coincide with the local ones \(\xi_1\) and \(\xi_2\), while the axis \(x_3\) is perpendicular to that plane. In this case \(x_3 = 0\) and \(n_1 = 0, \ n_2 = 0, \ n_3 = 1\). Let us consider rectangle BE that is shown in Fig. 1.
The quadrilateral BE is defined by its angular nodes and its shape functions are
\[ \phi_1 = \frac{1}{4}(1 - \xi_1)(1 - \xi_2), \quad \phi_2 = \frac{1}{4}(1 + \xi_1)(1 - \xi_2), \quad \phi_3 = \frac{1}{4}(1 + \xi_1)(1 + \xi_2), \quad \phi_4 = \frac{1}{4}(1 - \xi_1)(1 + \xi_2) \] (13)

Then global coordinates can be expressed as functions of local ones in the form
\[ x_i(\xi_1, \xi_2) = \sum_{q=1}^{4} x_i^q \phi_q(\xi_1, \xi_2), \quad y_i(\xi_1, \xi_2) = \sum_{q=1}^{4} y_i^q \phi_q(\xi_1, \xi_2). \] (14)

Derivatives of the shape functions are
\[ \frac{\partial \phi_1(\xi)}{\partial \xi_1} = -\frac{1}{4}(1 - \xi_2), \quad \frac{\partial \phi_1(\xi)}{\partial \xi_2} = -\frac{1}{4}(1 - \xi_1), \quad \frac{\partial \phi_2(\xi)}{\partial \xi_1} = -\frac{1}{4}(1 - \xi_2), \quad \frac{\partial \phi_2(\xi)}{\partial \xi_2} = -\frac{1}{4}(1 - \xi_1), \quad \frac{\partial \phi_3(\xi)}{\partial \xi_1} = \frac{1}{4}(1 + \xi_2), \quad \frac{\partial \phi_3(\xi)}{\partial \xi_2} = \frac{1}{4}(1 + \xi_1), \quad \frac{\partial \phi_4(\xi)}{\partial \xi_1} = \frac{1}{4}(1 - \xi_2), \quad \frac{\partial \phi_4(\xi)}{\partial \xi_2} = \frac{1}{4}(1 - \xi_1), \quad \frac{\partial \phi_1(\xi)}{\partial \eta_1} = -r_1(\xi_1) - n_1 - n_2, \quad \frac{\partial \phi_2(\xi)}{\partial \eta_2} = -r_2(\xi_2) + n_1 - n_2, \quad \frac{\partial \phi_3(\xi)}{\partial \eta_1} = r_1(\xi_1) + n_1 + n_2, \quad \frac{\partial \phi_4(\xi)}{\partial \eta_2} = -r_2(\xi_2) - n_1 + n_2. \] (15)

Coordinates of the nodal points are
\[ (x_1 = -\Delta_1, x_2 = -\Delta_2), \quad (x_1 = \Delta_1, x_2 = -\Delta_2), \quad (x_1 = \Delta_1, x_2 = \Delta_2), \quad (x_1 = -\Delta_1, x_2 = \Delta_2). \]

We introduce also some more useful notations that will be used bellow
\[ r(\xi, \eta) = \sqrt{(\Delta_1(1 + \xi_1) - y_1^0)^2 + (\Delta_2(1 + \xi_2) - y_2^0)^2}, \]
\[ r_x = (x_1 - y_1^0)\hat{n}_x, \quad r_y = (x_1 - y_1^0)\hat{n}_y + (x_2 - y_2^0)\hat{n}_x, \quad r_z = (x_2 - y_2^0)\hat{n}_y - (x_1 - y_1^0)\hat{n}_z. \] (16)

Regular representations for these integrals can be found in our previous publications [7, 8]. They have the form
\[ J_i^{0,0} = F.P. \int_{S_5} \phi_i(\xi) \frac{r_x}{r^3} dS = -\int_{\partial S_5} \left( \phi_i(\xi) \frac{r_x}{r^3} + \frac{1}{r} \frac{\partial}{\partial n} \phi_i(\xi) \right) d\ell \]
\[ J_i^{2,0} = F.P. \int_{S_5} \phi_i(\xi) \frac{r_x^2}{r^3} dS = \]
\[ = \int_{\partial S_5} \left( \phi_i(\xi) \frac{x_1^2 r_x}{r^5} - \frac{2 x_1 n_1}{3 r^3} - \frac{x_2^2}{3 r^3} - \frac{2 r}{3 r^3} \frac{\partial}{\partial n} \phi_i(\xi) \right) d\ell \]
\[ J_i^{0,2} = F.P. \int_{S_5} \phi_i(\xi) \frac{x_1^2 r_x}{r^5} dS = \]
\[ = \int_{\partial S_5} \left( \phi_i(\xi) \frac{x_1^2 r_x}{r^5} - \frac{2 x_1 n_2}{3 r^3} - \frac{x_2^2}{3 r^3} - \frac{2 r}{3 r^3} \frac{\partial}{\partial n} \phi_i(\xi) \right) d\ell \]
\[ J_i^{1,1} = F.P. \int_{S_5} \phi_i(\xi) \frac{x_1 x_2}{r^5} dS = \]
\[ = \int_{\partial S_5} \left( \phi_i(\xi) \frac{x_1 x_2}{r^5} - \frac{r_x}{3 r^3} - \frac{r_y}{3 r^3} - \frac{\partial}{\partial n} \phi_i(\xi) \right) d\ell \]. (17)

In the case of rectangular BE equation (12) has the following type
\[ F_{p,1}^q(\eta, x) \Delta u_p^*(\eta, x) = \int_{\partial S_5} F_{p,1}^q(\eta, x) \phi_i(\eta) \Delta u_p^*(\eta, x) d\ell = \sum_{k=1}^{N} \int_{S_5} F_{p,1}^q(\eta, x) \phi_i(\eta) \Delta u_p^*(\eta, x) d\ell \] (18)

Analysis of this equation and representations (17) shows that we have to calculate the sum of integrals of the following type
\[ J_{i,1}^{0,0} = \int_{\partial S_5} \phi_i(\xi) \frac{x_1^2 r_x}{r^5} d\ell(\xi) \] (19)

Details of the calculations are presented in the Appendix A. Final results side by side of the calculations are presented below.

**Side 1-2.** In this case the sums of the integrals (19) are
\[ J_{1,1}^{0,0}(1) = -\frac{1}{2 \Delta_1}, \quad J_{2,1}^{0,0}(1) = -\frac{1}{4 \Delta_1}, \quad J_{3,1}^{0,0}(1) = \frac{1}{4 \Delta_1}, \quad J_{4,1}^{0,0}(1) = \frac{1}{2 \Delta_1}, \quad J_{5,1}^{0,0}(1) = \frac{1}{4 \Delta_1}, \]
\[ J_{1,2}^{0,0}(1) = -\frac{1}{2 \Delta_1}, \quad J_{2,2}^{0,0}(1) = -\frac{1}{2 \Delta_1}, \quad J_{3,2}^{0,0}(1) = \frac{1}{2 \Delta_1}, \quad J_{4,2}^{0,0}(1) = \frac{1}{4 \Delta_1}, \quad J_{5,2}^{0,0}(1) = \frac{1}{2 \Delta_1}, \]
\[ J_{1,3}^{0,0}(1) = -\frac{1}{12 \Delta_1}, \quad J_{2,3}^{0,0}(1) = -\frac{1}{12 \Delta_1}, \quad J_{3,3}^{0,0}(1) = \frac{1}{12 \Delta_1}, \quad J_{4,3}^{0,0}(1) = \frac{1}{4 \Delta_1}, \quad J_{5,3}^{0,0}(1) = \frac{1}{2 \Delta_1}.

**Side 2-3.** In this case the sums of the integrals (19) are
\[ J_{1,1}^{0,0}(2) = \frac{1}{4}(J_{1,0}^1 - J_{1,1}^{1,0}), \]
\[ J_{1,2}^{0,0}(2) = \frac{1}{4}(J_{1,0}^2 - J_{1,2}^{1,0}), \quad J_{2,1}^{0,0}(2) = \frac{1}{4}(J_{2,0}^1 - J_{1,2}^{1,0}), \quad J_{3,1}^{0,0}(2) = \frac{1}{4}(J_{3,0}^1 - J_{1,2}^{1,0}), \quad J_{4,1}^{0,0}(2) = \frac{1}{4}(J_{4,0}^1 - J_{1,2}^{1,0}), \quad J_{5,1}^{0,0}(2) = \frac{1}{4}(J_{5,0}^1 - J_{1,2}^{1,0}). \]
\[
J_{2,3}^{s,0}(2) = \Delta_1 \Delta_2 (I_{3,0} - I_{3,1}) - \frac{1}{4}(I_{1,0} - I_{1,1}) - \frac{1}{4}(I_{1,0} + I_{1,1}),
\]
\[
J_{2,3}^{s,0}(2) = \Delta_1 \Delta_2 (I_{3,0} + I_{3,1}) - \frac{1}{4}(I_{1,0} + I_{1,1}),
\]
\[
J_{2,5}^{s,0}(2) = \frac{1}{4}(I_{1,0} + I_{1,1}), \quad (21)
\]
\[
J_{2,5}^{s,2}(2) = \frac{1}{4}(I_{1,0} - I_{1,1}) + \Delta_1^2 (I_{3,0} - I_{3,1}),
\]
\[
J_{2,5}^{s,2}(2) = 4\Delta_1^2 \Delta_2 (I_{5,0} - I_{5,1}) - \frac{\Delta_1^2 + 4\Delta_1 \Delta_2}{3}(I_{3,0} - I_{3,1}) - \frac{1}{4}(I_{1,0} - I_{1,1}),
\]
\[
J_{2,5}^{s,0}(2) = 4\Delta_1^2 \Delta_2 (I_{5,0} + I_{5,1}) - \frac{\Delta_1^2 + 4\Delta_1 \Delta_2}{3}(I_{3,0} + I_{3,1}) - \frac{1}{4}(I_{1,0} + I_{1,1}),
\]
\[
J_{2,5}^{s,2}(2) = \frac{1}{4}(I_{1,0} + I_{1,1}) + \Delta_1^2 (I_{3,0} + I_{3,1}),
\]
\[
J_{2,5}^{s,2}(2) = \frac{1}{2}(I_{1,0} - I_{1,1}) - \Delta_2^2 (I_{3,0} - I_{3,1}),
\]
\[
J_{2,5}^{s,0}(2) = 4\Delta_1 \Delta_2 (I_{5,0} - I_{5,1}) - \frac{3\Delta_1^2 + \Delta_2^2}{6}(I_{3,0} - I_{3,1}) - \frac{1}{2}(I_{1,0} - I_{1,1}),
\]
\[
J_{2,5}^{s,0}(2) = 4\Delta_1 \Delta_2 (I_{5,0} + I_{5,1}) - \frac{3\Delta_1^2 + \Delta_2^2}{6}(I_{3,0} + I_{3,1}) - \frac{1}{2}(I_{1,0} + I_{1,1}),
\]
\[
J_{4,5}^{s,2}(2) = \frac{1}{2}(I_{1,0} + I_{1,1}) - \Delta_1^2 (I_{3,0} + I_{3,1}),
\]

**Side 3-4**. In this case the sums of the integrals (19) are

\[
J_{1,3}^{s,0}(3) = \frac{1}{4}(I_{1,0} - I_{1,1}), \quad J_{2,3}^{s,0}(3) = \frac{1}{4}(I_{1,0} + I_{1,1}),
\]
\[
J_{0,3}^{1,0}(3) = \Delta_1 \Delta_2 (I_{5,0} + I_{5,1}) - \frac{1}{4}(I_{1,0} + I_{1,1}),
\]
\[
J_{0,3}^{1,0}(3) = \Delta_1 \Delta_2 (I_{5,0} - I_{5,1}) - \frac{1}{4}(I_{1,0} - I_{1,1}),
\]
\[
J_{1,5}^{s,3}(3) = \frac{1}{2}(I_{1,0} - I_{1,1}) - \Delta_1^2 (I_{3,0} - I_{3,1}),
\]
\[
J_{1,5}^{s,3}(3) = \frac{1}{2}(I_{1,0} + I_{1,1}) - \Delta_1^2 (I_{3,0} + I_{3,1}), \quad (22)
\]
\[
J_{2,3}^{s,0}(3) = 4\Delta_1 \Delta_2 (I_{5,0} + I_{5,1}) - \frac{3\Delta_1^2 + \Delta_2^2}{3}(I_{3,0} + I_{3,1}) - \frac{1}{2}(I_{1,0} + I_{1,1}),
\]
\[
J_{2,5}^{s,2}(3) = 4\Delta_1 \Delta_2 (I_{5,0} - I_{5,1}) - \frac{3\Delta_1^2 + \Delta_2^2}{3}(I_{3,0} - I_{3,1}) - \frac{1}{2}(I_{1,0} - I_{1,1}),
\]
\[
J_{2,5}^{s,0}(3) = \frac{1}{4}(I_{1,0} - I_{1,1}) + \Delta_1^2 (I_{3,0} - I_{3,1}),
\]
\[
J_{2,5}^{s,2}(3) = \frac{1}{4}(I_{1,0} + I_{1,1}) + \Delta_1^2 (I_{3,0} + I_{3,1}),
\]
\[
J_{2,5}^{s,0}(3) = 4\Delta_1 \Delta_2 (I_{5,0} - I_{5,1}) - \frac{3\Delta_1^2 + \Delta_2^2}{3}(I_{3,0} - I_{3,1}) - \frac{1}{2}(I_{1,0} - I_{1,1}),
\]
\[
J_{2,5}^{s,2}(3) = 4\Delta_1 \Delta_2 (I_{5,0} + I_{5,1}) - \frac{3\Delta_1^2 + \Delta_2^2}{3}(I_{3,0} + I_{3,1}) - \frac{1}{2}(I_{1,0} + I_{1,1}),
\]
\[
J_{2,5}^{s,0}(3) = 4\Delta_1 \Delta_2 (I_{5,0} - I_{5,1}) - \frac{3\Delta_1^2 + \Delta_2^2}{3}(I_{3,0} - I_{3,1}) - \frac{1}{2}(I_{1,0} - I_{1,1}),
\]
\[
J_{2,5}^{s,2}(3) = 4\Delta_1 \Delta_2 (I_{5,0} + I_{5,1}) - \frac{3\Delta_1^2 + \Delta_2^2}{3}(I_{3,0} + I_{3,1}) - \frac{1}{2}(I_{1,0} + I_{1,1}),
\]

**Side 4-1**. In this case the sums of the integrals (19) are

\[
J_{1,3}^{s,0}(4) = \frac{1}{2\Delta_2}, \quad J_{2,3}^{s,0}(4) = \frac{1}{2\Delta_2}, \quad J_{3,5}^{s,0}(4) = \frac{1}{4\Delta_2},
\]
\[
J_{2,3}^{s,0}(4) = \frac{1}{4\Delta_2}, \quad J_{3,5}^{s,0}(4) = \frac{1}{4\Delta_2},
\]
\[
J_{1,3}^{s,0}(4) = \frac{1}{2\Delta_2}, \quad J_{2,3}^{s,0}(4) = \frac{1}{2\Delta_2}, \quad J_{3,5}^{s,0}(4) = \frac{1}{2\Delta_2},
\]
\[
J_{2,5}^{s,0}(4) = \frac{1}{4\Delta_2}, \quad J_{3,5}^{s,0}(4) = \frac{1}{4\Delta_2}, \quad J_{3,5}^{s,0}(4) = \frac{1}{4\Delta_2},
\]
\[
J_{2,3}^{s,0}(4) = \frac{1}{2\Delta_2}, \quad J_{3,5}^{s,0}(4) = \frac{1}{2\Delta_2}, \quad J_{3,5}^{s,0}(4) = \frac{1}{2\Delta_2},
\]
\[
J_{3,5}^{s,0}(4) = \frac{1}{4\Delta_2}, \quad J_{1,3}^{s,0}(4) = 0, \quad J_{2,3}^{s,0}(4) = 0, \quad J_{1,5}^{s,0}(4) = 0.
\]
All integrals of the type $I_{p,i}$ in (20)-(23) are represented by the following formulae

\[ I_{p,0} = \Delta_i \int \frac{1}{r(x)} \xi^{i} \, d\xi = \frac{\Delta_i \xi + r_i(k)}{(r^2(k) - r_i^2(k))r(\xi)^3}, \]

\[ I_{p,1} = \Delta_i \int \frac{1}{r(x)} \xi \, d\xi = -\frac{r_i(k)\Delta_i \xi + r^3(k)}{(r^2(k) - r_i^2(k))r(\xi)^3}, \]

\[ I_{p,2} = \Delta_i \int \frac{1}{r(x)} \xi^2 \, d\xi = \frac{(\Delta_i \xi + r_i(k))^3}{3(3r^2(k) - r_i^2(k))r(\xi)^3} \]

\[ -2r_i(k)I_{p,1} - r^2(k)I_{p,0} \]

\[ I_{p,3} = \Delta_i \int \frac{1}{r(x)} \xi^3 \, d\xi = \frac{2r^2(k)r(\xi)^2 + r^2(k)\Delta_i \xi \xi^2 + 3\Delta_i \xi^2 r_i^2(k)}{3\Delta_i r(\xi)^3} I_{p,1} \]

Here we use the following notation for the corresponding integrals

\[ I_{p,i} = (\Delta_i)^{i-1} \int \frac{1}{r(x)} \xi^i \, d\xi \quad (25) \]

We have taken into account that integration in (20)-(23) has to be done in the way as it is shown on Fig. 2.

Finally sums of the integrals in (19) with taking into account above calculations are

\[ J_{i,0}^{l,m} = \sum_{k=1}^{4} J_{i,0}^{l,m}(k). \]

All integrals of the type $J_{i,0}^{l,m}(k)$ here have already been calculated above side by side for $k=1,\ldots,4$ and represented by the equations (20)-(23).

Substituting in (16) all obtained for each side integrals and take into account (26) finally we have

\[ F_{11}^{r}(y_r, x_q) = \frac{\mu}{4\pi(1-\nu)} \left( 1 - 2\nu \right) \sum_{k=1}^{4} J_{i,0}^{l,m}(k) \]

\[ F_{12}^{r}(y_r, x_q) = \frac{\mu}{4\pi(1-\nu)} \left( 1 - 2\nu \right) \sum_{k=1}^{4} J_{i,0}^{l,m}(k) \]

\[ F_{33}^{r}(y_r, x_q) = \frac{\mu}{4\pi(1-\nu)} \sum_{k=1}^{4} J_{i,0}^{l,m}(k), \]

\[ F_{12}^{r}(y_r, x_q) = \frac{\mu\nu}{4\pi(1-\nu)} \sum_{k=1}^{4} J_{i,1}^{l,m}(k). \]

It is important to mention that here all calculations can be done analytically, no numerical integration is needed.

4 Numerical calculations

Let a 3-D elastic unbounded body has a penny-shaped crack which is located in the plane $R^2 = \{x: x_3 = 0\}$ and its surface has coordinates $\Omega = \{x_1^2 + x_2^2 \leq R, x_3 = 0\}$ as it is shown in Fig. 3.

\[ \Delta \mu_i(x) = \frac{2(1-\nu^2)pR}{\pi E} \sqrt{1 - r^2 / R^2}, \]

where $R$ is the crack radio and $r$ is polar coordinate.
The BIE that related load on the crack edges and their displacements discontinuity (5) in this case has the form

$$p_3(y) = -\int_{\Omega} F_{33}(x,y) \Delta u_3(x) \, dS.$$  \hspace{1cm} (29)

The kernel $$F_{33}(x,y)$$ is defined by the equation (2.6).

Results of analytical and numerical calculations for this case are shown in Fig. 4 and Fig.5 for 300 BEs and 600 BEs respectively.

![Fig. 4. Penny-shaped crack opening versus radius in 3-D elastic space for 300 BEs.](image)

![Fig. 5. Penny-shaped crack opening versus radius in 3-D elastic space for 600 BEs.](image)

From the presented diagrams follow that obtained numerical results good coincide with analytical one on the outside of small area near the crack tip.

5 Conclusion

Based on the theory of distribution approach for the divergent hypersingular integrals regularization is developed here and applied for the case of 3-D elastostatic crack problems. We consider the 2-D hypersingular integrals over arbitrary convex polygon for piecewise constant approximation and over rectangular BE for piecewise linear approximation find regular formulas for their calculation. It is important to mention that in presented equations all calculations can be done analytically, no numerical integration is needed.

References: