Analytical solution of an eddy current problem for a two-layer tube with varying properties

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Abstract: - Analytical solution for the change in impedance of a coil located inside a two-layer cylindrical tube is obtained in the present paper. The electric conductivity and magnetic permeability of the inner layer of the tube depend on the radial coordinate. The properties of the outer layer are assumed to be constant. The solution is expressed in terms of improper integrals containing modified Bessel's functions of complex index. Extension of the proposed model for multilayer tubes is discussed.

Key-Words: - vector potential, change in impedance, Bessel function

1 Introduction

Analytical solutions to eddy current testing problems for the case where the properties of a conducting medium (the electric conductivity and magnetic permeability) are constant can be found in the literature (see, for example, [1] and [2]). In many applications, however, the magnetic and electric properties of the medium vary with respect to the depth of a conducting layer. Examples include thermal processing, decarbonization and spot welding [3], [4]. From a mathematical point of view the problem is related to the solution of the Maxwell's equations with variable coefficients. Some analytical solutions for the case of variable electric conductivity and/or magnetic permeability are given in [2].

Cylindrical tubes are often tested with eddy current method (see, for example, [5], [6]). Theoretical models for eddy current testing of long cylindrical tubes with constant properties are presented in [7]-[9].

In this paper we present an analytical solution of an eddy current problem for a coil located inside a two-layer conducting tube. The properties on the inner layer of the tube are assumed to vary with respect to the radial coordinate. The solution obtained in the present paper generalizes the results presented in [10] where the radius of a conducting cylindrical region is assumed to be infinite. Such an approximation is valid only for high frequencies. The solution presented here can be used for wide range of frequencies since more detailed description of the conducting tube is considered below (it is assumed that the tube has a two-layer structure).

The solution is found in closed form in terms of an improper integral containing modified Bessel functions of the first and second kind.

2 Formulation of the problem

Suppose that a single-turn coil of radius r_c is located inside a two-layer conducting tube at the height z_0 (the outer radius of the second layer is assumed to be infinite). The radius of the inner tube, r_1, is chosen as the measure of length. The dimensionless radii of the inner and outer cylindrical layers are 1 and R, respectively. We denote by R_0, R_1 and R_2 the following three regions: (a) the free space $R_0 : 0 \le r \le 1, 0 \le \varphi \le 2\pi, -\infty < z < +\infty$; (b) the inner cylindrical layer $R_1 : 1 \le r \le R, 0 \le \varphi \le 2\pi, -\infty < z < +\infty$ with variable magnetic permeability $\mu = \mu_* r^{\alpha}$ and electric conductivity $\sigma = \sigma_0 r^{\beta}$, where α and β are given constants; (c) the outer cylindrical layer $R_2 : r > R, 0 \le \varphi \le 2\pi, -\infty < z < +\infty$ with constant magnetic permeability μ_2 and electric conductivity σ_2.

The current in the coil is assumed to be of the form

$$i(t)\vec{e}_{\varphi} = I \exp(j\omega t)\vec{e}_{\varphi}, \qquad (1)$$

where I is the amplitude of the current, ω is the frequency and $\vec{e}_{\varphi}$ is the unit vector in the φ-direction (here (r, φ, z) is the system of cylindrical polar coordinates with the origin at the coil's center).

Since the vector potential is independent on φ it can be represented in the form

$$\vec{A}_i(r, \varphi, z, t) = A_i(r, z)\exp(j\omega t)\vec{e}_{\varphi}, i = 0,1, \qquad (2)$$

where the subscripts 0,1 and 2 correspond to regions R_0, R_1 and R_2, respectively.

Substituting (1) and (2) into the Maxwell's equations we obtain the following system of equations for the amplitudes, $A_i(r,z),\ i = 0,1,2$ of the vector potential in regions R_0, R_1 and R_2, respectively:

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I r_1^2 \delta(r - r_0)\delta(z - z_0), \qquad (3)$$

$$\frac{\partial^2 A_1}{\partial r^2} + \left(\frac{1}{r} - \frac{1}{\mu}\frac{d\mu}{dr}\right)\frac{\partial A_1}{\partial r} - \left(\frac{1}{r^2} + \frac{1}{r\mu}\frac{d\mu}{dr} + r_1^2 j\omega\sigma\mu_0\mu\right)A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \qquad (4)$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r}\frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} + \frac{\partial^2 A_2}{\partial z^2} + \tilde{r}_1^2 j\omega\sigma_2\mu_0\mu_2 A_2 = 0, \qquad (5)$$

where $\delta(z)$ is the Dirac delta function, μ_0 is the magnetic constant, σ_2 and μ_2 are the electric conductivity and magnetic permeability of region R_2, respectively.

Using magnetic permeability and electric conductivity profiles of the form

$$\mu = \mu_* r^{\alpha},\ \sigma = \sigma_0 r^{\beta}, \qquad (6)$$

we rewrite equation (4) in region R_1 as follows

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{(1-\alpha)}{r}\frac{\partial A_1}{\partial r} - \left(\frac{1+\alpha}{r^2} + p_1^2 r^{\alpha+\beta}\right)A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \qquad (7)$$

where $p_1 = \eta_1\sqrt{j}$, $\eta_1 = \tilde{r}_1\sqrt{\omega\sigma_0\mu_*\mu_0}$.

The boundary conditions have the form

$$A_0|_{r=1} = A_1|_{r=1}, \quad \frac{\partial A_0}{\partial r}|_{r=1} = \frac{1}{\mu_*}\frac{\partial A_1}{\partial r}|_{r=1}, \qquad (8)$$

$$A_1|_{r=R} = A_2|_{r=R}, \quad \frac{1}{\tilde{\mu}}\frac{\partial A_1}{\partial r}|_{r=R} = \frac{1}{\mu_2}\frac{\partial A_2}{\partial r}|_{r=R}, \qquad (9)$$

$$A_2 \to 0, \quad r \to \infty, \qquad (10)$$

$$A_0 \to 0, \quad A_1 \to 0, \quad A_2 \to 0, \quad z \to \pm\infty, \qquad (11)$$

where $\tilde{\mu} = \mu_* R^{\alpha}$.

3 Mathematical analysis

The solution to (3)-(11) can be found by the method of Fourier transform. Applying the transform of the form

$$\tilde{A}_i(r, \lambda) = \int_{-\infty}^{\infty} A_i(r, z)e^{-i\lambda z}dz, \qquad i = 0,1,2 \qquad (12)$$

to (3)-(11) we obtain

$$\frac{d^2\tilde{A}_0}{dr^2} + \frac{1}{r}\frac{d\tilde{A}_0}{dr} - \frac{\tilde{A}_0}{r^2} - \lambda^2\tilde{A}_0 = -\mu_0 I\tilde{r}_1^2 e^{-i\lambda z_0}\delta(r - r_0), \qquad (13)$$

$$\frac{d^2\tilde{A}_1}{dr^2} + \frac{(1-\alpha)}{r}\frac{d\tilde{A}_1}{dr} - \left(\frac{1+\alpha}{r^2} + p^2 r^{\alpha+\beta} + \lambda^2\right)\tilde{A}_1 = 0, \qquad (14)$$

$$\frac{d^2\tilde{A}_2}{dr^2} + \frac{1}{r}\frac{d\tilde{A}_2}{dr} - \frac{\tilde{A}_2}{r^2} - q^2\tilde{A}_2 = 0, \qquad (15)$$

where

$q = \sqrt{\lambda^2 + p_2^2}, p_2 = \eta_2\sqrt{j}, \eta_2 = \tilde{r}_1\sqrt{\omega\mu_0\mu_2\sigma_2}$.

The transformed boundary conditions are as follows

$$\tilde{A}_0|_{r=1} = \tilde{A}_1|_{r=1}, \quad \frac{d\tilde{A}_0}{dr}|_{r=1} = \frac{1}{\mu_*}\frac{d\tilde{A}_1}{dr}|_{r=1}, \qquad (16)$$

$$\widetilde{A}_1|_{r=R} = \widetilde{A}_2|_{r=R}, \quad \frac{1}{\widetilde{\mu}}\frac{d\widetilde{A}_1}{dr}|_{r=R} = \frac{1}{\mu_2}\frac{d\widetilde{A}_2}{dr}|_{r=R} \quad (17)$$

$$\widetilde{A}_2 \to 0, \quad r \to \infty. \quad (18)$$

The solution to (13) is found in two subregions of region R_0, namely, $0 < r < r_0$ and $r_0 < r < 1$. We denote the solutions in these regions by $\widetilde{A}_{00}(r,\lambda)$ and $\widetilde{A}_{01}(r,\lambda)$, respectively. The bounded solution to (11) in region $0 < r < r_0$ is

$$\widetilde{A}_{00}(r,\lambda) = C_1 I_1(\lambda r) \quad , \quad (19)$$

where $I_1(\lambda r)$ is the modified Bessel function of the first kind of order 1.
The solution to (11) in region $r_0 < r < 1$ is

$$\widetilde{A}_{01}(r,\lambda) = C_2 I_1(\lambda r) + C_3 K_1(\lambda r) \quad (20)$$

where $K_1(\lambda r)$ is the modified Bessel function of the second kind of order 1.
Closed-form solution to (14) can be found for different combinations of the parameters α and β. The solution for the case $\alpha = -1, \beta = -1$ is (see[10]):

$$\widetilde{A}_1(r,\lambda) = C_4 \frac{I_\nu(\lambda r)}{\sqrt{r}} + C_5 \frac{K_\nu(\lambda r)}{\sqrt{r}}, \quad (21)$$

where $K_\nu(\lambda r)$ is the modified Bessel function of order ν and $\nu = \sqrt{p_1^2 + 1/4}$.
The solution to (15) satisfying condition (18) is

$$\widetilde{A}_2(r,\lambda) = C_6 K_1(qr) \quad (22)$$

Two conditions at $r = r_0$ are required in order to obtain to determine the constants of integration:

$$\widetilde{A}_{00}|_{r=r_0} = \widetilde{A}_{01}|_{r=r_0}, \quad (23)$$

$$\frac{d\widetilde{A}_{01}}{dr}|_{r=r_0} - \frac{d\widetilde{A}_{00}}{dr}|_{r=r_0} = -\mu_0 I \widetilde{r}_1^2 e^{-i\lambda z_0}. \quad (24)$$

Condition (23) represents continuity of the vector potential at $r = r_0$. Condition (24) is obtained if one integrates (13) with respect to r from $1-\varepsilon$ to $1+\varepsilon$ and considers the limit in the resulting equation as $\varepsilon \to +0$.
Using (19), (20), (23) and (24) we obtain the constant C_3 in the form

$$C_3 = \mu_0 I \widetilde{r}_1^2 e^{-i\lambda z_0} r_0 I_1(\lambda r_0). \quad (25)$$

Using (19)-(22) and the remaining boundary conditions (16)-(17) we obtain all constants of integration. In particular,

$$C_2 = -\gamma_2 C_3, \quad (26)$$

where $\gamma_2 = D_2 / E_2$ and

$$D_2 = K_1(\lambda)[\lambda I_\nu'(\lambda) - I_\nu(\lambda)/2 - \gamma_1(\lambda K_\nu'(\lambda) - K_\nu(\lambda)/2)] - \mu_* \lambda K_1'(\lambda)[I_\nu(\lambda) - \gamma_1 K_\nu(\lambda)],$$

$$E_2 = I_1(\lambda)[\lambda I_\nu'(\lambda) - I_\nu(\lambda)/2 - \gamma_1(\lambda K_\nu'(\lambda) - K_\nu(\lambda)/2)] - \mu_* \lambda I_1'(\lambda)[I_\nu(\lambda) - \gamma_1 K_\nu(\lambda)].$$

Here

$$\gamma_1 = \frac{D_1}{E_1}$$

and

$$D_1 = \widetilde{\mu} q K_1'(qR) I_\nu(\lambda R) - \mu_2 K_1(qR)[\lambda I_\nu'(\lambda R) - I_\nu(\lambda R)/(2R)],$$

$$E_1 = \widetilde{\mu} q K_1'(qR) K_\nu(\lambda R) - \mu_2 K_1(qR)[\lambda K_\nu'(\lambda R) - K_\nu(\lambda R)/(2R)].$$

The induced vector potential in free space is given by the formula

$$\widetilde{A}_0^{ind}(r,\lambda) = C_2 I_1(\lambda r). \quad (27)$$

Applying the inverse Fourier transform to (27) we obtain

$$A_0^{ind}(r,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widetilde{A}_0^{ind}(r,\lambda) e^{i\lambda z} d\lambda. \quad (28)$$

Substituting (25)-(27) into (28) we obtain the induced vector potential in the form (it is assumed here that $z_0 = 0$)

$$A_0^{ind}(r,z) = -\frac{\mu_0 I \tilde{r}_1^{\,2} r_0}{\pi} \int_0^\infty \gamma_2 I_1(\lambda r_0) I_1(\lambda r) \cos \lambda z d\lambda. \tag{29}$$

One of the characteristics that is often used in applications is the induced change in impedance in the coil. It is computed by the formula

$$Z^{ind} = \frac{j\omega}{I} \oint_L A_0^{ind}(r,z) dl, \tag{30}$$

where L is the contour of the coil. Substituting (29) into (30) we obtain the change in impedance of the form

$$Z^{ind} = -2 j\omega \mu_0 \tilde{r}_1^{\,2} r_0 \int_0^\infty \gamma_2 I_1^2(\lambda r_0) d\lambda. \tag{31}$$

4 Conclusion

Analytical solution of the change in impedance of a coil located inside a two-layer conducting tube is found in the present paper. The electric conductivity and magnetic permeability of the inner layer depend on the radial coordinate. The properties of the outer layer are assumed to be constant. The solution is obtained by the method of Fourier integral transform. The change in impedance is obtained in the form of an improper integral. The solution obtained in the present paper can be generalized for the case of a multilayer tube where the properties of each layer vary with respect to the radial coordinate. In addition, the coil can also be located either inside the tube or outside the tube. Finally, only one particular combination of the parameters α and β is considered in the paper while other combinations are also possible so that the solution can be expressed in terms of other special functions. This topic is currently investigated by the authors.

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