Hamiltonian Cycle within Extended OTIS-Cube Topology

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Abstract: This paper introduces a complete description of constructing a Hamiltonian cycle in the Extended OTIS-$n$-Cube topology. The recently proposed network has many good topological features such as regular degree, semantic structure, low diameter, and ability to embed graphs and cycles. Constructing a Hamiltonian cycle is an important feature for any topology due to the importance of broadcast messages within networks. This paper proposes an algorithm to form a Hamiltonian cycle in the extended OTIS-$n$-Cube Interconnection network. Examples are also presented in different network sizes to show how Hamiltonian cycles are constructed.

Key-Words: Interconnection Networks, OTIS-$n$-Cube, Topological Properties, Routing Algorithm.

1 Introduction
In the last decade, there has been an increasing interest in a class of interconnection networks called Optical Transpose Interconnection Systems “OTIS-networks” [1, 16, 18, 19]. Marsden et al were the first to propose the OTIS-networks [12]. Extensive studies and modelling results for the OTIS have been reported in [5, 6, 11, 21]. The achievable terabit throughput at a reasonable cost makes the OTIS a strong competitor to the electronic alternatives [9, 12]. These encouraging findings prompt the need for further testing of the suitability of the OTIS for real-world parallel applications.

The advantage of using the OTIS as optoelectronic architecture lies in its ability to manoeuvre the fact that free space optical communication is superior in terms of speed and power consumption when the connection distance is more than a few millimetres [9]. In the OTIS, shorter (intra-chip) communication is realised by electronic interconnects while longer (inter-chip) communication is realised by free space interconnects.

OTIS technology processors are partitioned into groups, where each group is realised on a separate chip with electronic inter-processor connects. Processors on separate chips are interconnected through free space interconnects. The philosophy behind this separation is to utilise the benefits of both the optical and electronic technologies.

Processors within a group are connected by a certain interconnecting topology, while transposing group and processor indexes achieve inter-group links. Using $n$-cube as a factor network will yield the OTIS-$n$-Cube in denoting this network.

OTIS-$n$-Cube is basically constructed by "multiplying" a cube topology by itself. The set of vertices is equal to the Cartesian product on the set of vertices in the factor cube network. The set of edges $E$ in the OTIS-$n$-Cube consists of two subsets, one is from the factor cube, called cube-type edges, and the other subset contains the transpose edges. The OTIS approach suggests implementing cube-type edges by electronic links since they involve intra-chip short links and implementing transpose edges by free space optics. Throughout this paper the terms “electronic move” and the “OTIS move” (or “optical move”) will be used to refer to data transmission based on electronic and optical technologies, respectively.

Although the OTIS-$n$-Cube network has many attractive topological properties but it suffers from having limited optical links between the different groups. When source and destination nodes are in two different groups, the fact that only one optical link connects two distinguished groups directly create a congestion problem to most of the shortest paths that have to pass through this particular optical link. Furthermore, alternative paths are too long compared to the short path because they have to be routed via a third group which required passing via two optical links in addition to the electronic moves in each group to reach the destination.

The Extended OTIS-$n$-Cube is a recently proposed interconnection network based on the “OTIS-$n$-Cube” network [2]. This new topology has many attractive properties such as the regular degree, the small diameter, embedding structure nature, etc. The Extended OTIS-$n$-Cube network outperforms the
OTIS-\(n\)-Cube in many feature including semantic structure, regularity, smaller diameter, and other exceptional properties [2].

Embedding of topologies with regular structure and also irregular structure has been broadly investigated in the literature, e.g [3, 7, 8, 22]. Embedding structures and other topologies is one of the key features of interest in interconnection networks. The load of an embedding is the maximum number of nodes in a graph assigned to any node in the embedded graph. We are interested in this research only in one-to-one mappings to embed a Hamiltonian cycle, so the load of any embedding is one [20].

In the mathematical field of graph theory, a Hamiltonian path is a path in an undirected graph which visits each node exactly once. A Hamiltonian cycle is a path in an undirected graph which visits each node exactly once and also returns to the starting node. Determining whether such paths and cycles exist in graphs is the Hamiltonian path problem [8, 22].

The Hamiltonian path seeks whether there is a route in a directed network from a beginning node to an ending node, visiting each node exactly once. The Hamiltonian path problem is a NP complete, achieving astonishing computational complexity. This challenge has inspired researchers to broaden the definition of computer computations. The Hamiltonian problem arises in many real world applications including DNA applications [22].

This paper proposes a theoretical study on the routing properties in general and embedding Hamiltonian cycle in specific for the Extended OTIS-\(n\)-Cube due to its attractive properties. Section 2 presents notations and preliminary definitions. Details of embedding a Hamiltonian cycle in the Extended OTIS-\(n\)-Cube topology will be discussed in section 3. Section 4 concludes the paper.

## 2 Preliminary Definitions

The \(n\)-dimensional undirected graph binary \(n\)-cube is one of the well known networks which have been used in real life systems [10, 13, 14, 17].

**Definition 1:** The undirected graph \(n\)-cube with \(2^n\) vertices, representing nodes, which are labelled by the \(2^n\) binary digits of length \(n\). The binary system consists of two bits; 0 and 1. Two nodes are connected by a direct edge if, and only if, their labels differ in exactly one bit position.

The Extended OTIS-\(n\)-Cube is constructed by "multiplying" a cube topology by itself. The vertex set is equal to the Cartesian product on the original vertex set in the factor cube network. The initial step is similar to OTIS-\(n\)-Cube construction; this is why we name it Extended OTIS-\(n\)-Cube.

**Definition 2:** Let \(\langle g_1, p_1 \rangle\) be group and processor addresses of a node in an Extended OTIS-\(n\)-Cube labelled as series of bits \((x_1, \ldots, x_n)\), \((y_1, \ldots, y_n)\) consequently where each bit is either 0 or 1. A node \(\langle g_2, p_2 \rangle\) is called an opposite of node \(\langle g_1, p_1 \rangle\) if and only if they differ only in the first bit position of \(g_1\) and \(g_2\) labels, and also in the first bit position of \(p_1\) and \(p_2\) labels; they differ only in \(x_1\) and \(y_1\), e.g. node \(\langle 00, 00 \rangle\) is an opposite node of \(\langle 01, 01 \rangle\).

**Definition 3:** The two nodes \(\langle g_1, p_1 \rangle\) and \(\langle g_2, p_2 \rangle\) are connected via a transpose edge if and only if \(g_1 = p_2\) and \(g_2 = p_1\).

The edge set consists of electronic edges from the factor network and two new types of edges called the transpose and opposite edges, both types are considered optical edges. The formal definition of the Extended OTIS-\(n\)-Cube is given below.

**Definition 4:** Let \(n\)-cube = \((V_0, E_0)\) be an undirected graph representing an \(n\)-cube network where \(n\) is the cube degree. The Extended OTIS-\(n\)-Cube = \((V, E)\) network is represented by an undirected graph obtained from \(n\)-cube as follows \(V = \{(g, p)\mid g, p \in V_0\}\) and \(E = \{(g, p_1), (g, p_2)\}\) if \((p_1, p_2)\in E_0\) \(\cup\) \(\{(g, p), (p, g)\}\) if \(g, p \in V_0\) \(\cup\) \(\{(g, p), (p, p)\}\) if \(g, p \in V_0\) and \(g\) is an opposite of \(p\).

In the Extended OTIS-\(n\)-Cube, the address of a node \(u = \langle g, p \rangle\) from \(V\) is composed of two components.

![Fig1. 16-processor Extended OTIS-2-cube](image-url)
Figure 1 shows a 16 processor Extended OTIS-2-Cube, solid arrows represent transpose edges while dashes arrows represent opposite edges. The notation \((g, p)\) is used to refer to the group and processor addresses respectively, two nodes \((g_1, p_1)\) and \((g_2, p_2)\) are connected by a direct edge if one of the following cases occurs:

1. If \(g_1 = g_2\) and \((p_1, p_2)\) \(\in E_0\) where \(E_0\) is the set of edges in \(n\)-cube network, in this case the two nodes are connected by an electronic edge if their labels differ only by one bit position.
2. If \(g_1 = p_2\) and \(p_1 = g_2\), in this case the two nodes are connected by a transpose edge.
3. If \(g_1 = p_1\), \(g_2 = p_2\), and \(g_1\) is an opposite of \(g_2\), then the two nodes are connected by an opposite edge.

### 3 Hamiltonian Cycle Structure in the Extended OTIS-n-Cube

This section presents a Hamiltonian cycle structure within the recently proposed interconnection topology. First, we introduce some routing topological properties of the Extended OTIS-n-Cube which are needed to show the Hamiltonian cycle formation in this topology.

**Theorem 1.** If the cube factor degree is \(n\), then any node in the Extended OTIS-n-Cube is regular and the node degree is \(n + 1\).

**Proof.** Every node has \(n\) electronic edges based on the properties of the \(n\)-cube factor. Also every node; \((g, p)\), has an additional optical edge based on the Extended OTIS-n-Cube topology rule: \(\{(g, p), (p, g)\}\) \(\mid g, p \in V_0 \cup \{(g, g), (p, p)\}\) \(\mid g, p \in V_0 \cap g\) is an opposite of \(p\)

so if \(g = p\) then \((g, p) \xrightarrow{O} (g_{op}, g_{op})\) else \((g, p) \xrightarrow{O} (p, g)\).

Since every node has an \(n\) number of electronic, in addition to one optical edge, then by definition the topology is regular.

**Theorem 2.** Let \((g_1, p_1)\) and \((g_2, p_2)\) be two different nodes in the Extended OTIS-n-Cube. The length of shortest path from the source node \((g_1, p_1)\) to the destination node \((g_2, p_2)\) is defined mutually exclusive as in the following order:

\[
\text{Length} = \begin{cases} \\
d(p_1, p_2) \\
d(p_1, g_1) + d(g_1, \text{opposite}, p_2) + 1 \\
d(p_1, g_1) + d(p_1, \text{opposite}, g_2) + 2 \\
d(p_1, p_2) + d(g_1, g_2) \\
\min(d(p_1, g_2) + d(p_2, g_1) + 1, d(p_1, p_2) + d(g_1, g_2) + 2) \end{cases}
\]

Where \(d(p_1, p_2)\) is the number of bit positions differ between \(p_1\) and \(p_2\) labels.

**Proof.** By following one of the five possible paths shown in sections; \(i, ii, iii, iv,\) and \(v\). The length of the shortest path between the nodes \((g_1, p_1)\) and \((g_2, p_2)\) can be as follows:

- If \(g_1 = g_2\) \(\text{op}\) it means that one optical move is needed to move toward the destination group via a group opposite edge. To reach the destination, some electronic moves might be needed first at one source group to reach \((g_1, g_1)\), then one optical move to reach the destination group; finally other electronic moves at the destination group might be needed to reach the destination node.
- If \(p_1 = p_2\) \(\text{op}\) it means that two optical moves are needed to reach the destination group through an intermediate group equal to \(p_1, p_2\). This requires some electronic moves to perform the two optical moves, and finally to reach the destination node at minimal distance.
- If \(p_1 = p_2\), \(g_1 = g_2\), and \(d(p_1, p_2) = n\) it means that two optical moves in addition to some electronic moves are needed to reach the destination group through an intermediate group \(g_1, g_1\). First an opposite move is required to reach \((g_{1op}, p_{1op})\), then \(n-1\) electronic moves to reach \((g_{1op}, g_2)\), then an optical move to reach \((g_2, g_{1op})\), and finally other \(n-1\) electronic moves to reach the destination node \((g_2, p_2)\) at minimal distance.
- Otherwise we choose the shortest path based on the factor OTIS moves [4].

Since Hamiltonian is a cycle in an undirected graph which visits each node exactly once and finally returns to the starting node, the following steps are the description of the proposed algorithm proofing that the Extended OTIS-n-Cube topology is Hamiltonian:

1. Let assume that the start node of a path is \(<0,1>\), and \(p = 2\), where \(p\) represents the bit position of the label.

\[
\begin{align*}
\text{if } g_1 &= g_2 \\
\text{if } g_1 &= g_2, \text{opposite} \\
\text{if } p_1 &= p_2, \text{opposite} \\
\text{if } g_1 &= p_1, \text{and } g_2 &= p_2, \text{and } d(g_1, g_2) &= n \\
\text{Otherwise}
\end{align*}
\]
2. do $2^n-1$ factor moves towards a potential target node by complementing the $p^{th}$ bit in the factor label, if the target factor address matches an already visited group or matches the start node group address then increase $p$ by 1 modulus $n$, if all label bits were tested an no move is performed then perform a nand; not and; operation between group address and factor address of the current node. The outcome will be the factor address of the node target node.

3. do an optical move from $<g,p>$ to $<p,g>$

4. increase $p$ by 1 modulus $n$

5. Repeat steps 2, 3, and 4 as long as the move will not lead to the group label of the start node until the $2^n$ groups are visited.

6. Finally, construct an optical move back toward the start node.

In the following examples, the dots represent $n-1$ factor moves of the corresponding nodes; every arrow represents an optical move.

Example 1: Hamiltonian cycle within an Extended OTIS-2-Cube topology, Figure 2 shows a representation of such a Hamiltonian cycle.

![Fig.2: A Hamiltonian cycle in an Extended OTIS-2-Cube.](image)

Example 2: Hamiltonian cycle within an Extended OTIS-3-Cube graph, Figure 3 shows a representation of such a Hamiltonian cycle.

![Fig.3: A Hamiltonian Cycle in an Extended OTIS-3-Cube.](image)

Example 3: Hamiltonian cycle within an Extended OTIS-2-Cube graph, Figure 4 shows a representation of such a Hamiltonian cycle.

![Fig.4: A Hamiltonian Cycle in an Extended OTIS-4-Cube.](image)

4 Conclusion

This paper introduced the attractive routing topological properties for the Extended OTIS-$n$-Cube interconnection network by structuring Hamiltonian cycles within the new topology. An algorithm of forming a Hamiltonian cycle within the Extended OTIS-$n$-Cube is presented in details showing the good communication properties of the new network. Examples to show the structure of a Hamiltonian cycle in different network sizes of Extended OTIS-$n$-Cube are also presented in this paper.

References:


