The Reconstruction of Gaussian Processes Realizations with an Arbitrary Set of Samples

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Abstract: - The method of the reconstruction of Gaussian processes realizations based on the Conditional Mean Rule is described. The optimal reconstruction algorithms of stationary and non stationary processes are considered. In result of the investigation, two principal characteristics (the reconstruction function and the error reconstruction function) are obtained for all variants of processes. There are two features of studied algorithms: the reconstruction function is a linear function of samples and the error reconstruction function does not depend on the values of samples. These features are valid for the sampling-reconstruction procedure of Gaussian processes only.

Key-Words: - Sampling-Reconstruction Algorithm, Gaussian Process, Error Reconstruction

1 Introduction
The classical Sampling Theorem usually associated with the names of Whittaker, Kotelnikov, and Shannon (or WKS theorem) has been proved for deterministic functions with the limited spectrum. The WKS theorem has been generalized on stochastic stationary processes by A. Balakrishnan [1]. Balakrishnan’s theorem (BT) transfers the signification of the WKS theorem of deterministic functions on all stationary stochastic processes with the finite power spectrum. Following BT [1], it is sufficiently to know a boundary frequency of the power spectrum of a process in order to reconstruct any realizations of any process with zero error. The reconstruction function is determined as the linear sum of the given set of infinite samples multiplied by the basic functions of the type \( \sin x / x \) for any process as well.

Unfortunately, BT leaves without any considerations the influence on the reconstruction procedure of such very important characteristics of the sampled stochastic process like the probability density function (pdf), the type of the covariance function, the type of the power spectrum, the types of moment or cumulant functions of high orders, the finite number of samples, etc.

Some authors (see, for instance, [2], [3], etc.) omit the condition of the restricted spectrum and use the infinite power spectrum in order to describe the SRP of stochastic processes. Such description of stochastic processes is indefinite because a lot of different stochastic processes can have the same power spectrum or the same covariance function. Like in BT, the authors of the mentioned papers do not use the information about pdf and postulate the linear estimation in the Sampling-Reconstruction Procedures (SRP). As a result, the reconstruction function is always determined as a linear function of samples for stochastic processes with an arbitrary pdf. The real situation is another. The optimal algorithms for the reconstruction of realizations of non Gaussian processes are no linear. The error reconstruction function does not equal to zero when the number of samples is finite. Generally, each random process must have its own optimal reconstruction function and its own error reconstruction function.

In order to overcome above mentioned difficulties, connected with BT, we suggest to use the conditional mean rule (CMR) in the statistical description of the SRP of stochastic processes of various types. Below, we shall generally consider this rule and then we restrict our consideration by SRP of Gaussian processes realizations.

The main goal of the present paper is to discuss the optimal statistical SRP description of random processes if the number of samples is limited and arbitrary. It is naturally that the pdf of processes must be compulsory used. The conditional mean rule (CMR) [4], [5] will be applied in order to get this goal. This approach has been used in some of the author’s papers devoted to the statistical optimal
SRP description of some stochastic processes (see, for instance, [6] - [10]).

We suppose that the given stochastic process \( x(t) \) is characterized by its multidimensional pdf \( w_m(x(t_1), x(t_2), \ldots, x(t_m)) \). One realization of this process is sampled in times \( T = T_1, T_2, \ldots, T_N \). There is a set of samples \( X, T = x(T_1), x(T_2), \ldots, x(T_N) \). Number \( N \) of samples and their times of occurrence \( T = T_1, T_2, \ldots, T_N \) are arbitrary and known. It is necessary to reconstruct the sampled realization on the basis of the set \( X, T \) and the known pdf \( w_m(x(t_1), x(t_2), \ldots, x(t_m)) \), \( N < m \). The infinite set of many realizations of the conditional stochastic process \( \tilde{x}(t) \) pass through all samples \( x(T_1), x(T_2), \ldots, x(T_N) \). The sampled realization is one among an infinite number of other realizations of the conditional stochastic process \( \tilde{x}(t) \). We cannot principally know it. We need to find the statistical estimation \( \tilde{x}(t) \) in any current time \( t \) applying the conditional mean (mathematical expectation) rule (see n. 21.5 in the classical book [4] or Chapter 19 in [5]). The CMR guarantees the estimation of a random variable with minima of the mean square error. Following this rule we use the conditional mean function \( \tilde{m}(t) = \left\{ x(t) \mid X, T \right\} \) as the reconstruction function. (From here the angle parentheses mean the mathematical expectation operation.) The quality of the reconstruction will be evaluated by the conditional variance function \( \tilde{\sigma}^2(t) = \left\{ x(t) - \tilde{m}(t) \right\}^2 \). This function is the error reconstruction function. One can emphasize that the both principal SRP characteristics \( \tilde{m}(t) \) and \( \tilde{\sigma}^2(t) \) can be found on the basis of the conditional multidimensional pdf \( w_N(x(t) \mid X, T) \) of the given processes.

When \( t > T_N \) there is the extrapolation procedure, when \( T_i < t < T_{i+1} \), \( i = 1, \ldots, N - 1 \) there is the interpolation procedure. In spite of the limited number \( N \) this rule allows us to reconstruct an unknown realization on the all time axis. In this case it is necessary to use the series of shifts of \( N \) samples, renewing them with the occurrence of each new sample or a new group of samples. The CMR algorithm is valid for processes with an arbitrary pdf, but below we restrict our consideration by the SRP description of Gaussian processes realizations.

Owing to the knowledge of the Gaussian multidimensional pdf, the general expressions for the conditional mean and the conditional variance are known. We rewrite them following [11]:

\[
\tilde{m}(t) = m(t) + \sum_{i=1}^{N} \sum_{l=1}^{N} K(t, T_i) a_{il} x(T_i) - m(T_i),
\]

(1)

\[
\tilde{\sigma}^2(t) = \sigma^2(t) - \sum_{i=1}^{N} \sum_{l=1}^{N} K(t, T_i) a_{il} K(T_i, t).
\]

(2)

where \( m(t), \sigma^2(t), K(t_i, t_j) \) are the mean, the variance and the covariance function of the sampled process; \( \parallel a_{il} \parallel \) is the inverse matrix of the covariance matrix.

### 2 The Stationary Case

In the stationary case we put: \( m(t) = m = 0 \), \( \sigma^2(t) = \sigma^2 = 1 \) and \( K(t_i, t_j) = K(t_i - t_j) \). Then the expressions (1) - (2) will be simpler:

\[
\tilde{m}(t) = \sum_{i=1}^{N} \sum_{l=1}^{N} K(t, T_i) a_{il} x(T_i),
\]

(3)

\[
\tilde{\sigma}^2(t) = 1 - \sum_{i=1}^{N} \sum_{l=1}^{N} K(t, T_i) a_{il} K(T_i - t).
\]

(4)

The reconstruction function can be represented in the form:

\[
\tilde{m}(t) = \sum_{i=1}^{N} x(T_i) b_{il} t
\]

(5)

where

\[
b_{il}(t) = \sum_{i=1}^{N} K(t, T_i) a_{il},
\]

(6)

is the basic function.

Let us comment the formulae (3) – (6).

1) The reconstruction function \( \tilde{m}(t) \) is the linear function with respect of samples.

2) The error reconstruction function \( \tilde{\sigma}^2(t) \) does not depend on the values of samples, it depends on the location of samples \( T = T_1, T_2, \ldots, T_N \).

3) The basic function \( b_{il}(t) \) depends on the covariance function, the elements of the inverse
covariance function, the number of samples, and the location of samples.
4) The location of samples is arbitrary, i.e. the periodic case is particular.
5) The type of the covariance function (or the power spectrum) has not any restrictions. For instance, the power spectrum can be finite or infinite.

3 Examples

3.1 The Gaussian Markov Process
This process is described by the covariance function:

\[ K(\tau) = \exp(-\alpha |\tau|). \tag{7} \]

This process is formed on the output of an integrated RC circuit driven by white noise. The parameter the circuit is: \( \alpha = 1/RC \). The covariance time is: \( \tau_c = 1/\alpha \). We put \( \alpha = \tau_c = 1 \). The process under consideration is Markov process. It means that following Markov property, the reconstruction function \( f(t_i) \) depends on the values of neighbor samples \( x(T_i), x(T_{i+1}) \) only. So, in this case one can consider \( N=2 \) in the interpolation regime and \( N=1 \) in the extrapolation regime. The graph of the error reconstruction function is presented in Fig. 1. As one can see, the sampling intervals are different. When the interval duration is increased the maxima of the error reconstruction function is increased as well.

![Fig. 1. Error reconstruction function of the Gaussian Markov process.](image)

3.2 Some other Models with Low–Pass Spectrums
We consider some other models of Gaussian processes on the output of various linear filters driven by white noise. We add one and two integrated RC circuits in the series connection with the first RC circuit. Then, the output process of two series RC filters is characterized by the covariance function:

\[ K(\tau) = (1 + \alpha |\tau|)\exp(-\alpha |\tau|). \tag{8} \]

The process on the output of three RC series filters has another covariance function:

\[ K(\tau) = \left[1 + \alpha |\tau| + (\alpha \tau)^2 / 3\right]\exp(-\alpha |\tau|). \tag{9} \]

![Fig. 2. The basic function of the process on the output of the three integrated RC circuits.](image)

Physically, the processes with the covariance functions (8) and (9) have forms of their realizations more and more smoothly. In order to provide \( \tau_c = 1 \), it is necessary to choose \( \alpha = 2 \) and \( \alpha = 8/3 \)
correspondingly. We illustrate the SRP of the process with the covariance function (9) by two figures. In Fig. 2, one can see the form of some basic functions, and Fig. 3 gives us the graph of the error reconstruction function. The comparison of Fig. 1 and Fig. 3 provides a possibility to make the following conclusion: The smooth process is characterized by smaller error reconstruction function, when other parameters are equals.

![Fig. 3. The reconstruction error function of the process on the output of the two integrated RC circuits.](image)

In the same manner one can analyze SRP of Gaussian processes with various covariance functions:

\[ K(\tau) = \sin \frac{\Delta \omega \tau}{\Delta \omega}, \quad (10) \]

\[ K(\tau) = (1 - \alpha |\tau|) \exp(-\alpha |\tau|), \quad (11) \]

\[ K(\tau) = \exp -\alpha |\tau| \left( \cos \frac{\alpha_0 \tau}{\alpha_0} + \frac{\alpha}{\alpha_0} \sin \frac{\alpha_0 |\tau|}{\alpha} \right). \quad (12) \]

The first covariance function (10) corresponds to the process with an ideal rectangular spectrum, the second (11) describes the speech model, and the third is related with the process with a narrow-band spectrum. It is quite possible to analyze SRP of processes with finite spectrums as well. In this case it is necessary to use the Wiener-Khinchin transform in the limited interval of the frequency in order to obtain the corresponding expression for the covariance function. After this the application of the expressions (3) – (6) gives us a required result.

4 The non Stationary Case

It is clear that SRP of non stationary processes can be described by the general expressions (1) and (2). In this case all or some of the main statistical characteristics of the sampled process must depend on time: \( m(t), \sigma^2(t), K(t_1, t_2) \). Usually but not always, the covariance function determination of non stationary processes connected with some difficulties.

4.1 Simple examples

Here we give some rather simple examples of non stationary Gaussian processes:

1) The Wiener process. This process is the Markov process with statistical characteristics:

\[ m(t) = 0, \sigma^2 t = ct, K_{t_1, t_2} = c \min t_1, t_2. \quad (13) \]

The Wiener process is principally non stationary process.

2) The process on the output of the integrated RC circuit in the transition regime. There is the stochastic differential equation for such process:

\[ \frac{dx}{dt} = -\alpha x + \alpha n t, \quad (14) \]

where \( n t \) is white noise with the spectral density \( 0.5N_0 \). The initial condition is \( x_0, t = 0 = x_0 \). On the basis of (14) one can find the mathematic expectation, the covariance function and the variance:

\[ m(t) = x_0 \exp -\alpha t, \quad (15) \]

\[ K(t, t + \tau) = \sigma^2 t \exp -\alpha |\tau|, \quad (16) \]

\[ \sigma^2(t) = 0.25N_0 1 - \exp -2\alpha t. \quad (17) \]

3) The process on the output of the two series integrated RC circuits with different parameters \( \alpha, \beta \) in the transition regime [12]:

\[ K(t_1, t_2) = \frac{\beta \sigma_1^2}{\beta^2 - \alpha^2} \alpha + \beta \exp -\beta(t_1 + t_2) - \]

\[ -\alpha \exp -\beta(t_2 - t_1) + \beta \exp -\alpha(t_2 - t_1) - \]

\[ -\exp -\alpha t_1 - \beta t_1 \exp -\alpha t_1 - \beta t_2. \quad (18) \]
where $\sigma_1^2$ is the variance on the output of the first RC circuit. Using expressions (1), (2), (13) – (18) one can calculate the both principal SRP characteristics of mentioned non stationary processes.

4.2 The Process on the Output of the Time Varying System

This case is more difficult. We consider the integrated RC circuit when its parameter $\alpha t$ depends on time deterministically. The stochastic differential equation has a view:

$$\frac{dx}{dt} = -\alpha t x + \alpha t n t,$$  \hspace{1cm} (19)

with the initial condition $x_0, t_0$.

The solution of (19) is

$$x(t) = \int_{t_0}^{t} \alpha y n y dy + x_0 \exp \left[ -\int_{t_0}^{t} \alpha z dz \right].$$  \hspace{1cm} (20)

On the basis of (20) one can find the mathematic expectation and the covariance function of the output process:

$$\langle x(t) \rangle = x_0 \exp \left[ -\int_{t_0}^{t} \alpha z dz \right],$$  \hspace{1cm} (21)

$$K_{t_1, t_2} = \exp \left[ -\int_{t_1}^{t_2} \alpha z dz \right] \times$$

$$\times \int_{t_0}^{t_1} \alpha^2(y) \exp \left[ -\int_{y}^{t_1} \alpha z dz \right] dy, t_2 > t_1.$$  \hspace{1cm} (22)

The general formulas (21) and (22) can be concretized for any deterministic function $\alpha t$. For sake of simplicity we chose two variants: linear and harmonic functions. In the first case the parameter $\alpha t$ is described by the expressions:

$$\alpha t = \begin{cases} \alpha_0, 0 < t < \alpha_t \\ \alpha_0 + V_\alpha t - t_0, t > \alpha_t. \end{cases}$$  \hspace{1cm} (23)

In the second case we have:

$$\alpha t = \begin{cases} \alpha_0, 0 < t < \alpha_t \\ \alpha_0 + V_\alpha \sin \Omega t - t_0, t > \alpha_t. \end{cases}$$  \hspace{1cm} (24)

In (23) and (24), the parameters $\alpha_0, V_\alpha, \alpha_t, \Omega$ are constants.

Using (1), (2), (22) – (24), one can obtain the principal SRP characteristics of such type processes. Here we do not give calculation results. In order to emphasize a non trivial character of the sampled processes, we illustrate two possible realizations of such non stationary processes in Fig. 4 and Fig. 5. As one can see, the sampled processes change not only their mean and variance, but their time structure.

5 Some generalized remarks

The above written method of SRP description of Gaussian processes can be applied for some other problems in this scientific area. We mark some of them:

1) SRP of continuous non Gaussian processes.
2) SRP of non Gaussian processes with jumps.
3) SRP of multidimensional Gaussian processes.
4) SRP of random fields.
5) SRP of random processes and fields with jitter effect.

6 Conclusions
Using CMR method it is possible to obtain the principal SRP characteristics: 1) the optimal reconstruction algorithm of sampled realization, 2) the error reconstruction function for all above mentioned problems.

References: