Localization of hidden attractors in smooth Chua's systems

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Abstract:- The classical attractors of Lorenz, Rossler, Chua, Chen, and other widely-known attractors are those excited from unstable equilibria. From computational point of view this allows one to use numerical method, in which after transient process a trajectory, started from a point of unstable manifold in the neighborhood of equilibrium, reaches an attractor and identifies it. However there are attractors of another type: *hidden attractors, a basin of attraction of which does not contain neighborhoods of equilibria*. Recently such hidden attractors were discovered Chua's circuits by special analytical-numerical algorithm. In the present paper localization of hidden attractors in smooth Chua's systems is considered.

Key-Words:- Smooth Chua's circuit, Attractor localization, Hidden attractor, harmonic balance, describing function method

1 Introduction

The classical attractors of Lorenz [1], Rossler [2], Chua [3], Chen [4], and other widely-known attractors are those excited from unstable equilibria. From computational point of view this allows one to use numerical method, in which after transient process a trajectory, started from a point of unstable manifold in the neighborhood of equilibrium, reaches an attractor and identifies it. However there are attractors of another type [5]: *hidden attractors, a basin of attraction of which does not contain neighborhoods of equilibria*. The simplest examples of systems with such attractors are nested limit cycles in two-dimensional polynomial systems and hidden oscillations in counterexamples to widely-known Aizerman's and Kalman's conjectures on absolute stability (see, e.g., [9, 11]). Numerical localization, computation, and analytical investigation of such attractors are much more difficult problems.

Recently such hidden attractors were discovered [5] in classical Chua's circuit with continuous piecewise-linear nonlinearity saturation by special analytical-numerical algorithm

In this work we consider application of an analytical-numerical algorithm for localization of hidden attractor in smooth Chua's system.

2 Analytical-numerical method for attractors localization

Consider a system with vector nonlinearity nonlinearity

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x} + \boldsymbol{\psi}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{1}$$

Here $\mathbf{P}$ is a constant $(n \times n)$-matrix, $\boldsymbol{\psi}(\mathbf{x})$ is a continuous vector-function, and $\boldsymbol{\psi}(0) = 0$.

Define a matrix $\mathbf{K}$ such that the matrix in such a way that the matrix

$$\mathbf{P}_0 = \mathbf{P} + \mathbf{K} \tag{2}$$

has a pair of purely imaginary eigenvalues $\pm i\omega_0$ $(\omega_0 > 0)$ and the rest of its eigenvalues have negative real parts. Rewrite system (1) as

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}_0\mathbf{x} + \boldsymbol{\varphi}(\mathbf{x}), \tag{3}$$

where $\boldsymbol{\varphi}(\mathbf{x}) = \boldsymbol{\psi}(\mathbf{x}) - \mathbf{K}\mathbf{x}$.

Introduce a finite sequence of functions $\boldsymbol{\varphi}^0(\mathbf{x}), \boldsymbol{\varphi}^1(\mathbf{x}), \ldots, \boldsymbol{\varphi}^m(\mathbf{x})$ such that the graphs of neighboring functions $\boldsymbol{\varphi}^j(\mathbf{x})$ and $\boldsymbol{\varphi}^{j+1}(\mathbf{x})$ slightly differ from one another, the function $\boldsymbol{\varphi}^0(\mathbf{x})$ is small, and $\boldsymbol{\varphi}^m(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{x})$. Using a smallness of function $\varphi^0(\mathbf{x})$, we can apply and mathematically strictly justify [6, 7, 8, 9, 10, 11] the method of harmonic linearization (describing function method) for the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}_0\mathbf{x} + \boldsymbol{\varphi}^0(\mathbf{x}), \tag{4}$$

and determine a stable nontrivial periodic solution $\mathbf{x}^0(t)$. For the localization of attractor of original system (3), we shall follow numerically the transformation of this periodic solution (a starting *oscillating attractor* — an attractor, not including equilibria, denoted further by $\mathcal{A}_0$) with increasing j. Here two cases are possible: all the points of $\mathcal{A}_0$ are in an attraction domain of attractor $\mathcal{A}_1$, being an oscillating attractor of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}_0\mathbf{x} + \boldsymbol{\varphi}^j(\mathbf{x}), \tag{5}$$

with $j = 1$, or in the change from system (4) to system (5) with $j = 1$ it is observed a loss of stability (bifurcation) and the vanishing of $\mathcal{A}_0$. In the first case the solution $\mathbf{x}^1(t)$ can be determined numerically by starting a trajectory of system (5) with $j = 1$ from the initial point $\mathbf{x}^0(0)$. If in the process of computation the solution $\mathbf{x}^1(t)$ has not fallen to an equilibrium and it is not increased indefinitely (here a sufficiently large computational interval $[0, T]$ should always be considered), then this solution reaches an attractor $\mathcal{A}_1$. Then it is possible to proceed to system (5) with $j = 2$ and to perform a similar procedure of computation of $\mathcal{A}_2$, by starting a trajectory of system (5) with $j = 2$ from the initial point $\mathbf{x}^1(T)$ and computing the trajectory $\mathbf{x}^2(t)$.

Proceeding this procedure and sequentially increasing j and computing $\mathbf{x}^j(t)$ (being a trajectory of system (5) with initial data $\mathbf{x}^{j-1}(T)$) we either arrive at the computation of $\mathcal{A}_m$ (being an attractor of system (5) with $j = m$, i.e. original system (3)), either, at a certain step, observe a loss of stability (bifurcation) and the vanishing of attractor.

To determine the initial data $\mathbf{x}^0(0)$ of starting periodic solution, system (4) with nonlinearity $\boldsymbol{\varphi}^0(\mathbf{x})$ is transformed by linear nonsingular transformation $\mathbf{S}$ to the form

$$\begin{aligned} \dot{y}_1 &= -\omega_0 x_2 + \varepsilon\varphi_1(y_1, y_2, \mathbf{y}_3), \\ \dot{y}_2 &= \omega_0 x_1 + \varepsilon\varphi_2(y_1, y_2, \mathbf{y}_3), \\ \dot{\mathbf{y}}_3 &= \mathbf{A}\mathbf{x}_3 + \varepsilon\boldsymbol{\varphi}_3(y_1, y_2, \mathbf{y}_3) \end{aligned} \tag{6}$$

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Here y_1, y_2 are scalar values, $\mathbf{y}_3$ is $(n-2)$-dimensional vector; $\boldsymbol{\varphi}_3$ is an $(n-2)$-dimensional vector-function, φ_1, φ_2 are certain scalar functions; $\mathbf{A}_3$ is an $((n-2) \times (n-2))$-matrix, all eigenvalues of which have negative real parts. Without loss of generality, it can be assumed that for the matrix $\mathbf{A}_3$ there exists a positive number $d > 0$ such that

$$\mathbf{y}_3^*(\mathbf{A}_3 + \mathbf{A}_3^*)\mathbf{y}_3 \leq -2d|\mathbf{y}_3|^2, \quad \forall\, \mathbf{y}_3 \in \mathbb{R}^{n-2}. \tag{7}$$

Introduce the describing function

$$\Phi(a) = \int\limits_0^{2\pi/\omega_0} \Big[\varphi_1\left((\cos\omega_0 t)a, (\sin\omega_0 t)a, 0\right)\cos\omega_0 t + \\ +\varphi_2\left((\cos\omega_0 t)a, (\sin\omega_0 t)a, 0\right)\sin\omega_0 t\Big]\,dt.$$

and suppose, for the vector-function $\boldsymbol{\varphi}(\mathbf{x})$ the estimate

$$|\boldsymbol{\varphi}(\mathbf{x}') - \boldsymbol{\varphi}(\mathbf{x}'')| \leq L|\mathbf{x}' - \mathbf{x}''|, \quad \forall\, \mathbf{x}', \mathbf{x}'' \in \mathbb{R}^n \tag{8}$$

is satisfied.

Theorem 1 *[10] If it can be found a positive a_0 such that*

$$\Phi(a_0) = 0, \tag{9}$$

then for the initial data of periodic solution $\mathbf{x}^0(0) = \mathbf{S}(y_1(0), y_2(0), \mathbf{y}_3(0))^*$ at the first step of algorithm we have

$$y_1(0) = a_0 + O(\varepsilon),\ y_2(0) = 0,\ \mathbf{y}_3(0) = \mathbf{O_{n-2}}(\varepsilon), \tag{10}$$

where $\mathbf{O_{n-2}}(\varepsilon)$ is an $(n-2)$-dimensional vector such that all its components are $O(\varepsilon)$.

For the stability of $\mathbf{x}^0(t)$ (if the stability is regarded in the sense that for all solutions with the initial data sufficiently close to $\mathbf{x}^0(0)$ the modulus of their difference with $\mathbf{x}^0(t)$ is uniformly bounded for all $t > 0$), it is sufficient to require the satisfaction of the following condition

$$\left.\frac{d\Phi(a)}{da}\right|_{a=a_0} < 0.$$

3 Localization of hidden attractor in smooth Chua's system with vector nonlinearity

Consider the following modification of Chua's system

$$\begin{aligned} \dot{x} &= \alpha(y - x) - \alpha f_m(x), \\ \dot{y} &= x - y + z + g(y), \\ \dot{z} &= -(\beta y + \gamma z), \end{aligned} \tag{11}$$

where

$$f_m(x) = (k_1 x + k_3 x^3 + k_5 x^5), \quad g(x) = cy^2. \tag{12}$$

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We now apply the above algorithm. For this purpose, rewrite Chua's system (16) in the form (1)

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x} + \boldsymbol{\psi}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{13}$$

Here

$$\mathbf{P} = \begin{pmatrix} -\alpha(k_1+1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix},$$

$$\boldsymbol{\psi}(\mathbf{x}) = \begin{pmatrix} -\alpha(k_3x_1^3 + k_5x_1^5) \\ cx_2 \\ 0 \end{pmatrix},$$

Introduce a matrix $\mathbf{K}$ and small parameter ε, and represent system (18) as (4)

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \varepsilon\boldsymbol{\varphi}(\mathbf{r}^*\mathbf{x}), \tag{14}$$

where

$$\mathbf{P_0} = \mathbf{P} + \mathbf{K},\ \lambda_{1,2}^{\mathbf{P}_0} = \pm i\omega_0,\ \lambda_3^{\mathbf{P}_0} = -d,$$
$$\boldsymbol{\varphi}(\mathbf{x}) = \boldsymbol{\psi}(\mathbf{x}) - \mathbf{K}\mathbf{x}$$

By nonsingular linear transformation $\mathbf{x} = \mathbf{S}\mathbf{y}$ system (19) is reduced to the form (6)

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} + \varepsilon\mathbf{S}^{-1}\boldsymbol{\varphi}(\mathbf{y}), \tag{15}$$

where

$$\mathbf{A} = \mathbf{S}^{-1}\mathbf{P_0}\mathbf{S} = \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix}$$

Let $k_1 = -0.3092,\ k_3 = 0.6316,\ k_5 = -0.3$, then zero solution of system (11) is stable.

Taking $\omega_0 = 2.5, d = 10$ (so we define matrix $\mathbf{A}$, and one can obtain linearization matrix $\mathbf{K}$), the above procedure allows us to get initial data $x(0) = -1.5728,\ y(0) = 0,\ z(0) = 0$ for the first step of multistage procedure of construction of solutions. For $\varepsilon_1 = 0.1$ after transient process the computational procedure arrives at a almost periodic solution close to harmonic one. Further, with increasing parameter ε this periodic solution will be transformed into hidden attractor.

4 Localization of hidden attractor in smooth Chua's system with scalar nonlinearity

Consider the following smooth Chua's system:

$$\begin{aligned} \dot{x} &= \alpha(y - x) - \alpha f(x), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -(\beta y + \gamma z). \end{aligned} \tag{16}$$

Here the function

$$\begin{aligned} f(x) &= m_1 x + (m_0 - m_1)\tanh(x) = \\ &= m_1 x + (m_0 - m_1)\frac{e^\sigma - e^{-\sigma}}{e^\sigma + e^{-\sigma}} \end{aligned} \tag{17}$$

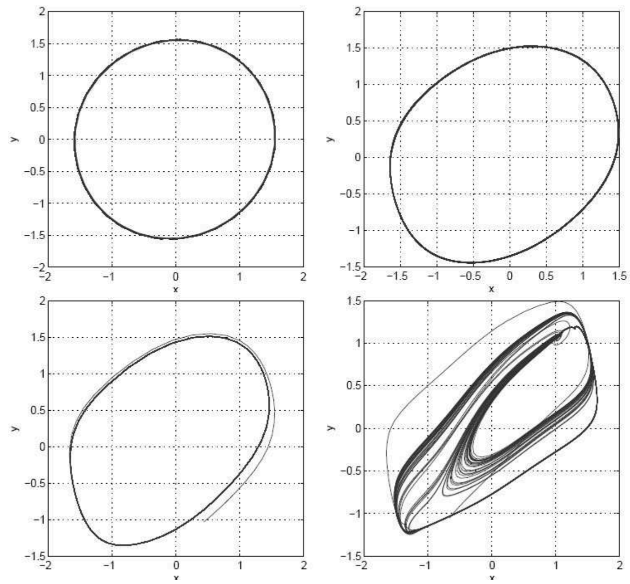

Fig. 1: $\varepsilon = 0.1$, $\varepsilon = 0.5$, $\varepsilon = 0.7$, $\varepsilon = 1$

characterizes a nonlinear element, of the system (here we consider smooth nonlinearity $\tanh(x)$ close to nonlinearity saturation(x) in the classical circuit); $\alpha, \beta, \gamma, m_0, m_1$ are parameters of the system.

We now apply the above algorithm to analysis of Chua's system. For this purpose, rewrite Chua's system (16) in the form (1)

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x} + \mathbf{q}\psi(\mathbf{r}^*\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{18}$$

Here

$$\mathbf{P} = \begin{pmatrix} -\alpha(m_1+1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix},$$

$$\mathbf{q} = \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\psi(\sigma) = (m_0 - m_1)\tanh(\sigma).$$

Introduce the coefficient k and small parameter ε, and represent system (18) as (4)

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \mathbf{q}\varepsilon\varphi(\mathbf{r}^*\mathbf{x}), \tag{19}$$

where

$$\mathbf{P_0} = \mathbf{P} + k\mathbf{q}\mathbf{r}^* = \begin{pmatrix} -\alpha(m_1+1+k) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix},$$

$$\lambda_{1,2}^{\mathbf{P}_0} = \pm i\omega_0, \ \lambda_3^{\mathbf{P}_0} = -d,$$

$$\varphi(\sigma) = \psi(\sigma) - k\sigma = (m_0 - m_1)\tanh(\sigma) - k\sigma.$$

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Consider system (19) with the parameters

$$\begin{aligned} &\alpha = 8.4562,\ \beta = 12.0732,\ \gamma = 0.0052, \\ &m_0 = 0.35,\ m_1 = -1.1468. \end{aligned} \tag{20}$$

Note that for the considered values of parameters there are three equilibria in the system: a locally stable zero equilibrium and two saddle equilibria.

Now we apply the above procedure of hidden attractors localization to Chua's system (18) with parameters (20). For this purpose, compute a starting frequency and a coefficient of harmonic linearization. We have

$$\omega_0 = 2.0392, \quad k = 0.2098\,.$$

Then, compute solutions of system (19) with nonlinearity $\varepsilon\varphi(x) = \varepsilon(\psi(x) - kx)$, sequentially increasing ε from the value $\varepsilon_1 = 0.1$ to $\varepsilon_{10} = 1$ with the step 0.1.

By (10) one can obtain the initial data

$$x(0) = 8.8200, y(0) = 0.5561, z(0) = -12.6008$$

for the first step of multistage procedure for the construction of solutions. For the value of parameter $\varepsilon_1 = 0.1$, after transient process the computational procedure reaches the starting oscillation $\mathbf{x}^1(t)$. Further, by the sequential transformation $\mathbf{x}^j(t)$ with increasing the parameter ε_j, using the numerical procedure, for original Chua's system (18) the set $\mathcal{A}_{\text{hidden}}$ is computed. This set is shown in Fig. 2.

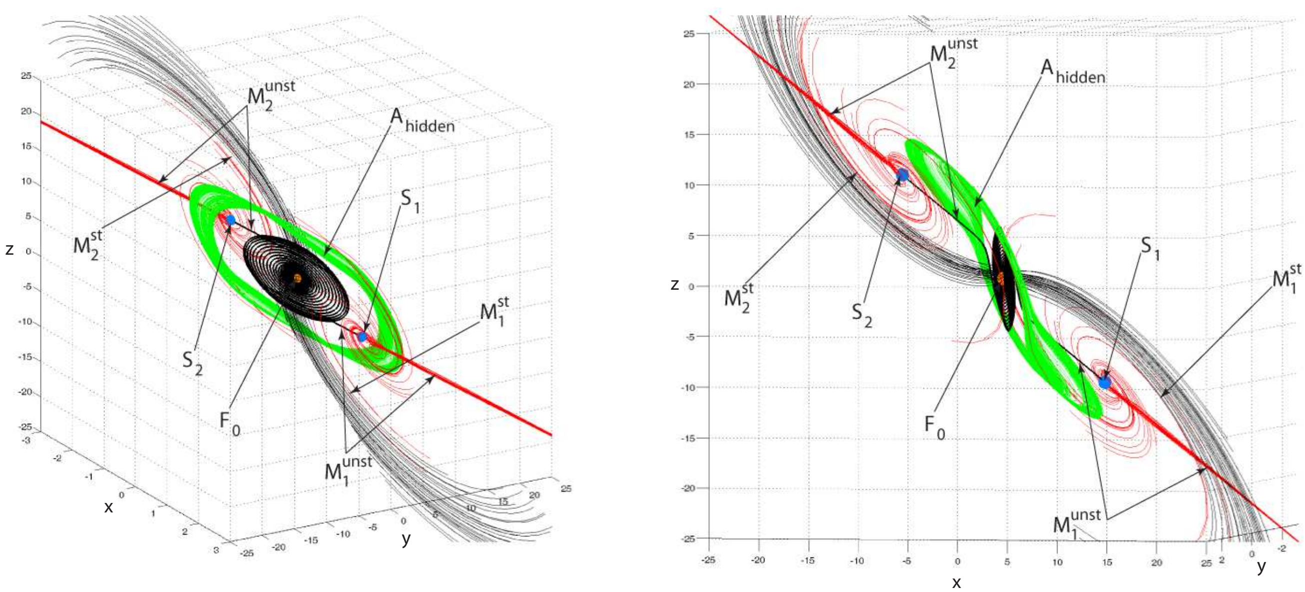

Fig. 2: Equilibrium, stable manifolds of saddles, and localization of hidden attractor.

We remark that for the computed trajectories it is observed Zhukovsky instability and the positiveness of Lyapunov exponent [17, 18].

5 Conclusions

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In the present work the application of special analytical-numerical algorithm for hidden attractor localization is discussed and the existence of such hidden attractor in smooth Chua's systems is demonstrated.

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References

[1] E.N.Lorenz, Deterministic nonperiodic flow, *J. Atmos. Sci.*, V.20, 1963, pp.130-141.

[2] O.E.Rossler, An Equation for Continuous Chaos, *Physics Letters*, V.57A, No.5, 1976, pp.397-398.

[3] L.O.Chua, G.N.Lin, Canonical Realization of Chua's Circuit Family, *IEEE Transactions on Circuits and Systems*, V.37, No.4, 1990, pp.885-902.

[4] G.Chen,& T.Ueta, Yet another chaotic attractor, /it Int. J. Bifurcation and Chaos No.9, 1999, pp.1465- 1466.

[5] G.A.Leonov, N.V.Kuznetsov, V.I.Vagaitsev, Localization of hidden Chua's attractors, *Physics Letters, Section A*, Vol. 375, is. 35, 2011, pp. 2230–2233 (doi:10.1016/j.physleta.2011.04.037).

[6] G.A.Leonov, On harmonic linearization method, *Doklady Akademii Nauk. Physcis*, 424(4), 2009, pp. 462–464.

[7] G.A.Leonov, On harmonic linearization method, *Automation and remote controle*, 5, 2009, pp. 65–75.

[8] G.A.Leonov, On Aizerman problem, *Automation and remote control*, No.7, 2009, pp. 37–49.

[9] G.A.Leonov, Effective methods for periodic oscillations search in dynamical systems, *App. math. & mech.*, 74(1),2010, pp. 37–73.

[10] G.A.Leonov, V.I.Vagaitsev, N.V.Kuznetsov, Algorithm for localizing Chua attractors based on the harmonic linearization method, *Doklady Mathematics*, 82(1),2010, pp.663–666.

[11] G.A.Leonov, V.O.Bragin, N.V.Kuznetsov, Algorithm for Constructing Counterexamples to the Kalman Problem, *Doklady Mathematics*, 82(1), 2010, pp. 540–542.

[12] L.O.Chua, A Zoo of Strange Attractors from the Canonical Chua's Circuits, *Proceedings of the IEEE 35th Midwest Symposium on Circuits and Systems (Cat. No.92CH3099-9)*, V.2, 1992, pp.916-926.

[13] L.O.Chua, A Glimpse of Nonlinear Phenomena from Chua's Oscillator, *Philosophical Transactions: Physical Sciences and Engineering*, V.353, No.1701, 1995, pp.3-12.

[14] E.Bilotta, P.Pantano, A gallery of Chua attractors, *World scientific series on nonlinear science, Series A*, V.61, 2008.

[15] Z.Shi, S.Hong and K.Chen, Experimental study on tracking the state of analog Chua's circuit with particle filter for chaos synchronization, *Physics Letters A*, Vol. 372, Iss. 34, 2008, pp. 5575–5580.

[16] V.S.Anishchenko, T.Kapitaniak, M.A.Safonova and O.V.Sosnovzeva, Birth of double-double scroll attractor in coupled Chua circuits, *Physics Letters A*, Vol.192, Iss.2-4, 1994, pp. 207–214.

[17] G.A.Leonov, *Strange attractors and classical stability theory*, St.Petersburg university Press, 2008.

[18] G.A.Leonov, N.V.Kuznetsov, Time-Varying Linearization and the Perron effects, *International Journal of Bifurcation and Chaos*, Vol. 17, No.4, 2007, pp. 1079-1107.