Mixed Number and Clifford Algebra

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Abstract: - Scalar and vector quantities are well known. Here we presented another type of quantity which is the sum of a scalar and a vector which we called mixed number. We have presented the Geometric product of Clifford algebra and compared it with the mixed product.

Key-Words: Mixed number**,** Geometric product, Mixed product

1. INTRODUCTION

In Clifford algebra, a new virtual unit j has been introduced. $j^2 = 1$, $j \neq 1$, $j^* = -j$ and be named as hyperbolic virtual unit [1, 2]. We also known that in Clifford algebra, taking an equation [3-6]

$$\mathbf{A}\ \mathbf{B} = \mathbf{A.B} + \mathbf{A} \wedge \mathbf{B} \qquad \ldots\ldots(1)$$

Where **A** and **B** are two vectors, **A.B** is a dot product $\mathbf{A} \wedge \mathbf{B}$ i.e. **A** wedge **B** which is different from the usual cross-product in the sense that it has magnitude ABsinθ and shares its skew property $\mathbf{A} \wedge \mathbf{B} = -\mathbf{B} \wedge \mathbf{A}$, but it is not a scalar or a vector: it is directed area, or bivector, oriented in the plane containing **A** and **B**. This product (equation 1) is called geometric product.

In the second section we introduce another quantity which we called mixed number. In the third section we compare the mixed number algebra with Clifford algebra.

2. MIXED NUMBER

Mixed number [7 -11] α is the sum of a scalar x and a vector **A** like quaternion [12-14] i.e. $\alpha = x + \mathbf{A}$

The product of two mixed numbers is defined as [15, 16]

$$\alpha\beta = (x + \mathbf{A})(y + \mathbf{B}) = xy + \mathbf{A.B} + x\mathbf{B} + y\mathbf{A} + i\mathbf{A}\times\mathbf{B} \qquad \ldots\ldots\ (2)$$

The product of Mixed numbers is associative, that is if α, β and γ are three Mixed numbers defined by $\alpha = x + \mathbf{A}$, $\beta = y + \mathbf{B}$ and $\gamma = z + \mathbf{C}$ we can write

$$(\alpha\beta)\gamma = \alpha(\beta\gamma) \qquad \ldots\ldots(3)$$

Basis Mixed number **i**, **j** and **k** are defined in terms of mixed number α by

$\alpha = x + \mathbf{A} = (x + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$ where x, A_1, A_2 and A_3 are scalars and **i**, **j** and **k** are basis Mixed numbers with the properties that

$\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = 1$ i.e. $\mathbf{i}^2 = 1$, $\mathbf{j}^2 = 1$, $\mathbf{k}^2 = 1$ and

$\mathbf{ij} = i\,\mathbf{k}$, $\mathbf{jk} = i\,\mathbf{i}$, $\mathbf{ki} = i\,\mathbf{j}$, $\mathbf{ji} = -i\,\mathbf{k}$, $\mathbf{kj} = -i\,\mathbf{i}$, $\mathbf{ik} = -i\,\mathbf{j}$

where $i = \sqrt{(-1)}$.

Rule of multiplication of mixed numbers has been shown in the table – 1.

	1	**i**	**j**	**k**
1	1	**i**	**j**	**k**
i	**i**	1	i**k**	– i**j**
j	**j**	– i**k**	1	i**i**
k	**k**	i**j**	–i**i**	1

Table-1: Multiplication table for Mixed numbers.

Taking x = y = 0 we get from equation (2)

$$\mathbf{A} \otimes \mathbf{B} = \mathbf{A.B} + i\mathbf{A} \times \mathbf{B} \qquad \ldots\ldots(4)$$

This product is called mixed product and the symbol ⊗ is chosen for it.

2.1 Consistency of mixed product with Pauli matrix algebra.

It can be shown that [17]

$$(\sigma.\mathbf{A})(\sigma.\mathbf{B}) = \mathbf{A.B} + i\sigma.(\mathbf{A} \times \mathbf{B}) \qquad \ldots\ldots(5)$$

where **A** and **B** are two vectors and σ is the Pauli matrix. If we consider σ as an unit matrix the equation (5) will be same as equation (4).

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Therefore from equation (4) and (5) we can say that the mixed product is directly consistent with Pauli matrix algebra.

2.2 Consistency of mixed product with Dirac equation.

Dirac equation $(E - \alpha.\mathbf{P} - \beta m)\psi = 0$ can be operated by the Dirac operator $(t - \alpha.\mathbf{V} - \beta n)$ then we get

$(t - \alpha.\mathbf{V} - \beta n)\ \{(E - \alpha.\mathbf{P} - \beta m)\psi\} = 0$ ……(6)

For mass-less particles i.e. for m = n = 0 we get

$(t - \alpha.\mathbf{V})(E - \alpha.\mathbf{P})\psi = [\{tE + \mathbf{V}.\mathbf{P} + i\sigma.(\mathbf{V}\times\mathbf{P})\} + \{-(t\sigma.\mathbf{P} + E\sigma.\mathbf{V})\}]\psi = 0$ ……(7)

where ψ is the wave function.

putting t = 0 and E = 0 in the equation (7) we get

$(\alpha.\mathbf{V})(\alpha.\mathbf{P})\psi = \{\mathbf{V}.\mathbf{P} + i\sigma.(\mathbf{V}\times\mathbf{P})\}\psi = 0$

or, $(\alpha.\mathbf{V})(\alpha.\mathbf{P}) = \{\mathbf{V}.\mathbf{P} + i\sigma.(\mathbf{V}\times\mathbf{P})\}$ ……(8)

If we consider α and σ are both unit matrices the equation (8) will be same form as equation (4). Therefore from equation (4) and (8) it is clear that Mixed product is consistent with Dirac equation.

2.3 Applications of mixed products in dealing with differential operators

In a region of space where there is no charge or current, Maxwell's equation can be written

as

$$\left.\begin{array}{ll} \text{(i)}\ \nabla.\mathbf{E} = 0 & \text{(ii)}\ \nabla\times\mathbf{E} = -(\partial\mathbf{B})/(\partial t) \\ \text{(iii)}\ \nabla.\mathbf{B} = 0 & \text{(iv)}\ \nabla\times\mathbf{B} = \mu_0\varepsilon_0(\partial\mathbf{E})/(\partial t) \end{array}\right| \quad \text{…..(9)}$$

From these equations it can be written as [18]

$$\left.\begin{array}{l} \nabla^2\mathbf{E} = \mu_0\varepsilon_0(\partial^2\mathbf{E})/(\partial t^2) \\ \nabla^2\mathbf{B} = \mu_0\varepsilon_0(\partial^2\mathbf{B})/(\partial t^2) \end{array}\right| \quad \text{…..(10)}$$

Using equation (4) and (9) we can write

$$\begin{aligned} \nabla \otimes \mathbf{E} &= \nabla.\mathbf{E} + i\nabla \times \mathbf{E} \\ &= 0 + \{-i(\partial\mathbf{B})/(\partial t)\} \end{aligned}$$

or, $\nabla \otimes \mathbf{E} = -i(\partial\mathbf{B})/(\partial t)$ …..(11)

$$\begin{aligned} \text{or,}\nabla \otimes(\nabla \otimes \mathbf{E}) &= \nabla \otimes \{-i(\partial\mathbf{B})/(\partial t)\} \\ &= -i(\partial/\partial t)\ \{\nabla \otimes \mathbf{B}\} \\ &= -i(\partial/\partial t)\ \{\nabla.\mathbf{B} + i\nabla \times \mathbf{B}\} \\ &= -i(\partial/\partial t)\{ 0 + i\ \mu_0\varepsilon_0(\partial\mathbf{E})/(\partial t)\} \end{aligned}$$

or, $\nabla \otimes(\nabla \otimes \mathbf{E}) = \mu_0\varepsilon_0(\partial^2\mathbf{E})/(\partial t^2)$ ……(12)

It can be shown that $\nabla \otimes(\nabla \otimes \mathbf{E}) = \nabla^2\mathbf{E}$ …..(13)

From equation (12) and (13) we can write

$\nabla^2\mathbf{E} = \mu_0\varepsilon_0(\partial^2\mathbf{E})/(\partial t^2)$

which is exactly the same as shown in equation (10)

Similarly using mixed product it can also be shown that

$\nabla^2\mathbf{B} = \mu_0\varepsilon_0(\partial^2\mathbf{B})/(\partial t^2)$

Therefore mixed product can be used successfully in dealing with differential operators.

2.4 Elementary properties of mixed product

(i) Mixed product of two perpendicular vectors is equal to the imaginary of the vector product of the vectors.

(ii) Mixed product of two parallel vectors is simply the scalar product of the vectors.

(iii) It satisfies the distribution law of multiplication.

(iv) It is associative.

3. COMPARISION OF CLIFFORD AND MIXED NUMBER ALGEBRA

The algebra of 3-dimensional space, the Pauli algebra, is central to physics, and deserves further emphasis. It is an 8-dimensional linear space of multivectors, which we can written as [19]

$$\mathbf{M} = \underset{\text{scalar}}{\boldsymbol{\alpha}} + \underset{\text{vector}}{\mathbf{a}} + \underset{\text{bivector}}{\mathbf{ib}} + \underset{\text{pseudoscalar}}{\mathbf{i}\boldsymbol{\beta}} \quad \text{……(14)}$$

The space of even-grade elements of this algebra,

$\boldsymbol{\psi} = \boldsymbol{\alpha} + \mathbf{ib},$ ……(15)

is closed under multiplication and forms a representation of the quarternion algebra. Explicitly, identifying **i**, **j**, **k** with $i\sigma_1$, $-i\sigma_2$, $i\sigma_3$ respectively, we have the usual quarternion relations, including the famous formula

$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$ ……(16)

Finally it can be shown that [19]:

$\mathbf{a} \times \mathbf{b} = -\boldsymbol{i}\ \mathbf{a} \wedge \mathbf{b}.$ ……(17)

Thus $\mathbf{a} \wedge \mathbf{b}$ is taken before the multiplication by **i**. The duality operation in three dimensions

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interchanges a plane with a vector orthogonal to it (in a right-handed sense). In the mathematical literature this operation goes under the name of the `Hodge dual'. Quantities like **a** or **b** would conventionally be called `polar vectors', while the `axial vectors' which result from cross-products can now be seen to be disguised versions of bivectors.

Using equation (4) and (17) we can write

$$\mathbf{A} \otimes \mathbf{B} = \mathbf{A}.\mathbf{B} + i\,(-i\,\mathbf{A} \wedge \mathbf{B})$$

$$\text{Or, } \mathbf{A} \otimes \mathbf{B} = \mathbf{A}.\mathbf{B} + \mathbf{A} \wedge \mathbf{B} \qquad \ldots\ldots(18)$$

The right hand side of equation (1) and (18) are same. Therefore, we can say that mixed product is directly consistent with the geometric product of Clifford algebra.

4. CONCLUSION

We have clearly explained the mixed number algebra. Mixed product is derived from mixed number algebra. It has observed that mixed product is directly consistent with Pauli matrix algebra, Dirac equation and Geometric product of Clifford algebra.

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