Fractional Order Calculus in Control Theory

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Abstract: - The main aim of this contribution is to introduce the fundamentals of the Fractional Order Calculus (FOC) and outline its possible application to analysis and synthesis of control systems. The basic theoretical concepts of FOC are followed by techniques for potential fractional order systems description and stability investigation. Moreover, the paper offers the overview of the existing fractional order controllers and highlights the benefits of the fractional approach in comparison with the classical integer one.

Key-Words: - Fractional Order Calculus, Differintegration, Fractional Order Controllers, Control Theory, Control Systems

1 Introduction

The Fractional Order Calculus (FOC) constitutes the branch of mathematics dealing with differentiation and integration under an arbitrary order of the operation, i.e. the order can be any real or even complex number, not only the integer one [1], [2], [3]. Although the FOC represents more than 300-year-old issue [4], [5], its great consequences in contemporary theoretical research and real world applications have been widely discussed relatively recently. The idea of non-integer derivative was mentioned for the first time probably in a letter from Leibniz to L'Hospital in 1695. Later on, the pioneering works related to FOC have elaborated by personalities such as Euler, Fourier, Abel, Liouville or Riemann. The interested reader can find the more detailed historical background of the FOC e.g. in [1].

According to [4], [6], the reason why FOC remained practically unexplored for engineering applications and why only pure mathematics was "privileged" to deal with it for so long time can be seen in multiple definitions of FOC, missing simple geometrical interpretation, absence of solution methods for fractional order differential equations and seeming adequateness of the Integer Order Calculus (IOC) for majority of problems. However, nowadays the situation is going better and the FOC provides efficient tool for many issues related to fractal dimension, "infinite memory", chaotic behaviour, etc. Thus, the FOC has already came in useful in engineering areas such as bioengineering, viscoelasticity, electronics, robotics, control theory and signal processing [6]. Several control applications are available e.g. in [7], [8], [9].

The paper is not intended to bring any novel theoretical knowledge nor application results. Its main purpose is to aggregate the FOC theory and introduce its utilization in control theory on the basis of literature from "References" section.

2 Basic Concepts of Fractional Order Calculus

The FOC is based on generalization of differentiation and integration to an arbitrary order, which can be rational, irrational or even complex. This generalization has led to the introduction of basic continuous differintegral operator [1], [2], [4], [6]:

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & \operatorname{Re}\alpha > 0 \\ 1 & \operatorname{Re}\alpha = 0 \\ \displaystyle\int_a^t (d\tau)^{-\alpha} & \operatorname{Re}\alpha < 0 \end{cases} \tag{1} $$

where α is the order of the differintegration (usually $\alpha \in \mathbb{R}$) and a is a constant connected with initial conditions.

There is an array of definitions of differintegral in the literature. The three most frequent definitions bear the names of Riemann-Liouville, Grünwald-Letnikov and Caputo. The most known and used Riemann-Liouville version has the form [4]:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{2} $$

under condition $(n-1<\alpha<n)$. The term $\Gamma(\cdot)$ represents so-called Gamma function.

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Alternatively, the Grünwald-Letnikov definition is given by [4], [8]:

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{j=0}^{\left[\frac{t-a}{h}\right]}(-1)^j\binom{\alpha}{j}f(t-jh) \quad (3)$$

where

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)} \quad (4)$$

and where h is the time increment and [.] means the integer part.

Finally, Caputo has defined the differintegral as [5]:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau \quad (5)$$

Each of the definitions of an interpolation of the integer order operations sequence has its advantages and drawbacks and the user choice depends mainly on the purpose and the area of application [3], [10].

The automatic control theory widely exploits the Laplace transform for the sake of analysis and synthesis simplicity. The Laplace transform (denoted as *L*) of the differintegral can be written as [4], [8]:

$$\begin{aligned} L\left\{{}_aD_t^\alpha f(t)\right\} &= \int_0^\infty e^{-st}\,{}_0D_t^\alpha f(t)dt = \\ &= s^\alpha F(s) - \sum_{m=0}^{n-1} s^m (-1)^j\,{}_0D_t^{\alpha-m-1} f(t)\Big|_{t=0} \end{aligned} \quad (6)$$

where integer n lies within $(n-1<\alpha\le n)$.

3 Description of Fractional Order Systems

A fractional order continuous-time linear time-invariant dynamical system can be described by a fractional order differential equation [3], [4], [5]:

$$\begin{aligned} &a_nD^{\alpha_n}y(t) + a_{n-1}D^{\alpha_n-1}y(t)+\cdots+a_0D^{\alpha_0}y(t) = \\ &= b_mD^{\beta_m}u(t)+b_{m-1}D^{\beta_{m-1}}u(t)+\cdots+b_0D^{\beta_0}u(t) \end{aligned} \quad (7)$$

where $u(t)$ is the input signal, $y(t)$ is the output signal, $D^\gamma \equiv {}_0D_t^\gamma$ represents fractional derivative, a_k with $(k=0,\dots,n)$ and b_k with $(k=0,\dots,m)$ denote constants, and α_k with $(k=0,\dots,n)$ and β_k with $(k=0,\dots,m)$ are arbitrary real numbers. According to [4], [5], one can assume inequalities $\alpha_n>\alpha_{n-1}>\cdots>\alpha_0$ and $\beta_m>\beta_{m-1}>\cdots>\beta_0$ without loss of generality.

Another option for fractional order system description is in the form of incommensurate real orders transfer function [3], [5]:

$$G(s) = \frac{B(s^{\beta_k})}{A(s^{\alpha_k})} = \frac{b_ms^{\beta_m}+b_{m-1}s^{\beta_{m-1}}+\cdots+b_0s^{\beta_0}}{a_ns^{\alpha_n}+a_{n-1}s^{\alpha_{n-1}}+\cdots+a_0s^{\alpha_0}} \quad (8)$$

The symbols in (8) have the same meaning as in (7).

It has been shown (e.g. in [5], [11]) that every incommensurate order system (8) can be expressed as a commensurate one by means of a multivalued transfer function.

4 Stability of Fractional Order Systems

Obviously, the stability is the very fundamental and critical requirement during control system design. It is widely known that an integer order continuous-time linear time-invariant system is stable if and only if all roots of its characteristic polynomial have negative real parts. In other words, the roots must lie in the left half of the complex plane. Investigation of stability of the fractional order systems represents the more complicated issue [5], [12].

For example, the stability of commensurate fractional order systems can by analyzed via the theorem of Matington [12] or the definition from [5], which describes the way of mapping the poles from s^α-plane into the *w*-plane. An interesting result is that the poles of the stable fractional order system can be located even in the right half of such complex plane. This is illustrated in fig. 1 where the stability region for a commensurate fractional order linear time-invariant system with order $0<\alpha<1$ is depicted [4], [5].

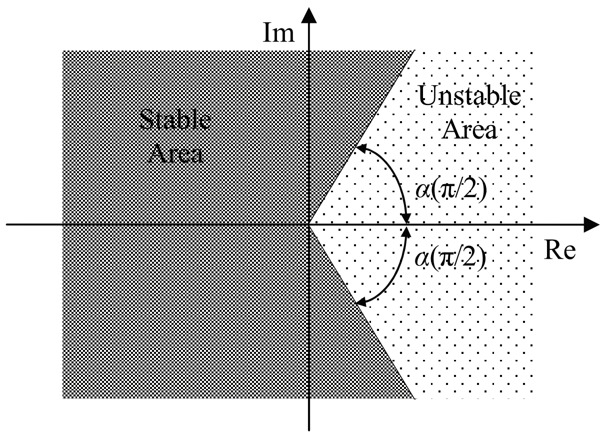

Fig. 1: Region of stability for the commensurate fractional order system with $0<\alpha<1$

5 Fractional Order Controllers

The nice survey on fractional order control is given e.g. in [13]. This paper has distinguished among four typical combinations of integer/fractional order controlled system vs. integer/fractional order of controller and shown that the fractional algorithms have better results from many points of view.

Usually, the four basic approaches to fractional order control, i.e. four different fractional order controllers are reviewed in the literature [4], [6], [14]. Their overview can be found in the following subsections.

5.1 Tilted Proportional and Integral (TID) Controller

First, the TID controller has the same structure as classical PID controller, but the proportional gain is replaced with a function $s^{-\alpha}$ with $\alpha \in \mathbb{R}$, which allows wider tuning options and better control behaviour in comparison with the integer order PID controller [15].

5.2 CRONE

Next popular controller is CRONE. The abbreviation CRONE stands for French "Commande Robuste d'Ordre Non Entier" (non-integer order robust control) and represents approach inspired by the fractal robustness [16], [17], [18]. The CRONE controllers have been already applied to many real plants. Besides, the approach has its own Matlab toolbox.

5.3 Fractional Order PID Controllers

The elegant and efficient fractional order modification of conventional PID controllers has been introduced in [10]. They are known as $PI^{\lambda}D^{\mu}$ controllers and can be described by transfer function:

$$C(s) = K_P + K_I s^{-\lambda} + K_D s^{\mu} \tag{9}$$

where λ and μ are positive real numbers, and K_P, K_I and K_D denote the proportional, integral and derivative constant, respectively. This embellishment of PID algorithm offers much wider selection of tuning parameters which can consequently improve the control performance. However, there is a relative lack of rigorous tuning techniques for this type of controllers so far.

5.4 Fractional Lead-Lag Controller

Finally, the paper [19] has introduced the extension of classical lead-lag controllers to its fractional version. Furthermore, self-tuning approach for fractional lead-lag compensators can be found in [20].

6 Conclusion

The paper has been focused on introduction to FOC with emphasis to potential application to engineering, especially analysis and synthesis of control systems. It has offered the basic theoretical aspects of FOC, dealt with description and stability of fractional order systems and overviewed the possible fractional order control approaches.

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