There are more locally Brunovsky systems than constant ones

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Abstract: This paper is devoted to give a procedure to obtain locally Brunovsky linear systems over commutative rings. We prove that for any partition of the state-space into direct summands verifying some relations there are a feedback class of locally Brunovsky systems.

Key–Words: Linear systems, commutative rings, feedback, classification, projective module

1 Introduction

This brief paper is devoted to give a method to obtain locally Brunovsky systems (i.e. pairs of matrices \((A, B)\) having a local canonical form.

The Brunovsky canonical form for linear systems over a field is well known (see [2]). We focus linear systems having a local Brunovsky form. Of course constant linear systems will be locally Brunovsky, but how about the converse?

Our examples will be performed mainly over rings of continuous functions \(R = C(\Lambda)\) defined over topological space \(\Lambda\), but the results are stated in general, hence valid over any commutative ring.

**Definition 1** Let \(R\) be a commutative ring. A linear system over \(R\) is a pair of matrices \((A, B) \in R^{n \times n} \times R^{n \times m}\).

The feedback actions on a given linear system are:

1. Isomorphisms (e.g. basis changes) \(P \in R^n\). Transforming \((A, B) \mapsto (PAP^{-1}, PB)\)
2. Isomorphisms (e.g. basis changes) \(Q \in R^m\). Transforming \((A, B) \mapsto (A, BQ)\)
3. Feedback loops transforming \((A, B) \mapsto (A + BF, B)\)

Two linear systems are said to be feedback equivalent if there exists a finite composition of feedback actions bringing one system into the other.

The feedback equivalence is an equivalence relation, hence the space of linear systems are partitioned into disjoint union of feedback classes. Consequently two linear systems are equivalent if and only if they lie in the same class.

For constant linear systems \((A, B)\) one has the Kalman decomposition and Brunovsky Theorem. Hence if system \((A, B)\) is reachable then feedback equivalence class are determined by Kronecker invariants associated (see [2] or [4]). A canonical representative of each class is fixed: The Brunovsky canonical form. Consequently two linear systems are feedback equivalent if and only if they have the same Brunovsky canonical form.

For reachable linear systems over a commutative ring the situation is harder. No classification are available (and it is unlikely been solved, because the feedback classification problem contains wild classification problems not yet solved).

If \(R\) is a projective-free ring (i.e. every projective \(R\)-module is free) then a linear system \((A, B)\) is equivalent to a Brunovsky canonical form if and only if all invariants

\[
I_i(A, B) = \frac{\text{Im}(B, AB, \ldots, A^i B)}{\text{Im}(B, AB, \ldots, A^{i-1} B)}
\]

are free (see [4]). This is actually equivalent to say that the determinantal ideals of matrices \((B, AB, \ldots, A^{i-1} B)\) are either zero or the whole ring \([3], [4]\).

We deal with the case of locally Brunovsky systems, that is, linear systems having a local Brunovsky form.

**Definition 2** Let \(R\) be a commutative ring. A linear system \((A, B)\) over \(R\) is locally Brunovsky if and only if all localizations \((A_p, B_p)\) are Brunovsky systems over local rings \(R_p\) for each prime ideal \(p\) of \(R\).
Because Brunovsky character is given by the freeness of invariant modules $I_i$ it would follow that locally Brunovsky character comes from the local freeness of that modules. This result is proven in [3]: Local Brunovsky character is equivalent to invariants $I_i$ being projective for all $i$.

Now it is clear that there are more locally Brunovsky systems than Brunovsky ones, but to find out one of them we need to consider a non-projective-free ring (in the case $R = C(L)$, a non-contractible topological space $\Lambda$).

Perhaps the simplest example is given over $\Lambda = S^2$ the unit sphere: The following linear system defined on $R[x, y, z]/(x^2 + y^2 + z^2 - 1)$ (the coordinate ring of sphere $S^2$)

$$\begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix}$$ (2)

Above linear system is locally Brunovsky because its invariants are trivial ($I_1 = R$ and $I_i = 0$ for $i \geq 1$) but it is not equivalent to a Brunovsky canonical form because matrix $(x, y, z)$ is not equivalent to $(1, 0, 0)$ (you can’t comb the radial vector bundle on the sphere).

Our main result in the next section is to show that some partitions of free $n$-dimensional based $R$-module $R^n$ into direct sum gives locally Brunovsky linear systems. We also give a nice combinatorial consequence: Whereas the account of Brunovsky canonical forms is actually equivalent to the account of partitions of $n$ in sums of integers, the account of locally Brunovsky feedback classes will be equivalent to give the partitions of $n$ in sums in a semigroup only depending on $R$.

## 2 Main result

**Theorem 3** Let $R$ be a commutative ring. There exists a linear system $(A, B)$ with invariants $I_i(A, B) = P_i$ for each decomposition of $R^n$ into direct sum

$$R^n \cong P_1 \oplus P_2 \oplus \cdots \oplus P_s$$ (3)

with the property of $P_i$ maps onto $P_{i+1}$, for all $i$.

**Proof:**

Assume $R^n \cong P_1 \oplus P_2 \oplus \cdots \oplus P_s$. We use Bass [1] matrix notation to set a linear map

$$P_1 \oplus P_2 \oplus \cdots \oplus P_s \xrightarrow{(a_{ij})} P_1 \oplus P_2 \oplus \cdots \oplus P_s$$ (4)

given by the matrix of linear maps $(a_{ij})$ where $a_{ij} : P_j \rightarrow P_i$.

Let $\alpha_{i+1,i} : P_i \rightarrow P_{i+1}$ any surjection (existing by hypothesis). Let the linear map given by matrix of linear maps

$$F = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \alpha_{21} & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & \alpha_{s,s-1} & 0 \end{pmatrix}$$ (5)

Because $P_1$ is projective then it is direct summand of a free module $P_1 \oplus Q \cong R^m$. Let $G : P_1 \oplus Q \rightarrow P_1 \oplus P_2 \oplus \cdots \oplus P_s$ the linear map given by matrix of linear maps

$$G = \begin{pmatrix} 1_{P_1} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$$ (6)

It is clear that $\text{Im}(G) = P_1$. Hence $I_1(F, G) = P_1$. Now by performing matrix multiplications $FG, F(FG), F(F(FG)), \ldots$ one has that $\text{Im}(FG, F(FG), \ldots) = P_1 \oplus \cdots \oplus P_s$. Consequently it is clear that $I_i(F, G) = P_i$. Hence if we use the isomorphisms $R^n \cong P_1 \oplus P_2 \oplus \cdots \oplus P_s$ and $R^m \cong P_1 \oplus Q$ we obtain a (usual) matrix pair $(A, B)$ having the same invariants $I_i(A, B) = I_i(F, G) = P_i$ (because isomorphisms are feedback actions).

Let $R = \mathbb{R}[\sin(\theta), \cos(\theta)]$ be the ring of trigonometric polynomials. Linear system

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} \cos(\theta) - 1 & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) + 1 \end{pmatrix}$$ (7)

has the partition $R^2 \cong M \oplus M$ where $M$ is the (unique) non-trivial one dimensional vector bundle over the circle (the Möbius strip). This is a locally Brunovsky linear systems that is not equivalent to a Brunovsky canonical form.

Main consequence of above theorem is that there are as many feedback classes of systems over $R^n$ as partitions of the free module $R^n$ into the monoid of finitely generated projective modules. In the particular case of $R = C(\Lambda)$, by Swan’s Theorem [1] or [5], there are as many feedback classes of $n$-dimensional systems as decompositions of the trivial $n$-dimensional vector bundle over $\Lambda$. In both cases $i$-th direct summand must map into $(i + 1)$-th.
3 Conclusion

As announced in the title of the paper there are more locally Brunovsky linear systems than constant ones. We have written down two examples defined on the unit circle and the unit sphere.

A future work is to compute effectively all feedback classes of locally Brunovsky linear systems. This is a combinatorial problem because we have established in Theorem 3 that there are at least many classes as partitions of $\mathbb{R}^n$ into direct summands $P_i$ with the property of $i$-th summand maps onto $(i+1)$-th. Hence a study of partitions of elements in the monoid $P(R)$ of finitely generated projective $R$-modules is needed.

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