Tracking and Disturbance Attenuation for Unstable Systems: Algebraic

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Abstract: The paper is aimed to the design of linear continuous controllers for unstable single input – output systems. The controller design is studied in the ring of (Hurwitz) stable and proper rational functions R_{PS}. All stabilizing feedback controllers are given by a general solution of a Diophantine equation in R_{PS}. Then asymptotic tracking and disturbance attenuation is obtained through the divisibility conditions in this ring. The attention of the paper is focused on a class of unstable systems. Both, one and two degree of freedom (1DOF, 2DOF) control structure. The methodology brings a scalar parameter for tuning and influencing of controller parameters. As a result, a class of PI, PID controllers are developed but the approach generates also complex controllers. Simulations and verification are performed in the Matlab+Simulink environment.

Keywords: Unstable systems, Diophantine equation, Asymptotic tracking, Disturbance attenuation.

1 Introduction

The dynamics of many technological plants exhibit unstable behavior. Probably, the reason can be seen in nonlinearity of many industrial processes and plants. Such nonlinear systems exhibit multiple steady states and some of them may be unstable. The situation where linear systems have unstable poles may occur e.g. in a continuous-time stirred exothermic tank reactor, in distillation columns, in polymerization processes or in a class of biochemical processes where the processes must operate at an unstable steady state. Moreover, a time delay can be also an inherent part of many technological plants.

The most frequent tool for feedback industrial control has been still PID controller. It is believed that more than 90 % feedback loops are equipped with this controller. Also, a great amount for PID assessing and tuning rules has been developed. The traditional engineering design approach of PID like controllers was performed either in the frequency domain or in polynomial representation (see e.g. [1], [2], [3]). Most of them are scheduled for stable systems without or with time delay, see e.g. [1], [2].The unstable cases are studied e.g. in [4], [5].

In this contribution, a general technique for a class of unstable systems is proposed. The control design is performed in the ring of proper and Hurwitz stable rational functions R_{PS}. All stabilizing controllers are given by all solutions of Diophantine equation in this ring and asymptotic tracking and disturbance attenuation is then formulated by additional conditions of divisibility. This fractional approach proposed in [6], [10], [17] enables a deeper insight into control tuning and a more elegant derivation of all suitable controllers. The situation and details for stable and time-delay free systems can be found in [12] - [16] for various control problems. This technique introduces a scalar parameter $m > 0$ which influences a control responses and also robust behaviour. The R_{PS} ring also enables to utilize the H_∞ norm as a tool for perturbation evaluation.

2 Descriptions over Rings

Linear continuous-time dynamic systems have been traditionally described by the Laplace transform. So polynomials became a basic tool for the stability analysis and controller design. Since the characteristic feedback polynomial has two known (plant) and two unknown (controller) polynomials, the Diophantine equations began to penetrate into synthesis method, see e.g. [9]. However, the ring of polynomials induces some drawbacks with solutions of Diophantine equations. Almost all from the infinite number of solutions cannot be used for controller transfer functions because they are not proper, see e.g. [10], [15]. These problems were overcome by introducing of the different ring of proper and stable rational functions. The pioneering work in the so called fractional approach is the work [17], further extension can be found in [10], [15]. Simply speaking, a ratio of polynomials is replaced by a ratio of two Hurwitz stable and proper rational functions. In this paper, the following ring $R_{PS}(m)$ is utilized.

The ring $R_{PS}(m)$ denotes the set of rational functions having no poles in the plane Re(s)≥-m. Generally, polynomial transfer functions in the ring $R_{PS}(m)$ take the form:

$$G(s)=\frac{b(s)}{a(s)}=\frac{\frac{b(s)}{(s+m)^n}}{\frac{a(s)}{(s+m)^n}}; \qquad (1)$$
$$n=\max(\deg\ a,\ \deg\ b)$$

where $m>0$. Also, signals in control systems can be expressed similarly. The stepwise reference signal w and harmonic disturbance v are in the rational description given by ratios:

$$w=\frac{1}{s}=\frac{G_w}{F_w}=\frac{\frac{1}{s+m}}{\frac{s}{s+m}} \qquad (2)$$

$$v=\frac{G_v}{F_v}=\frac{\frac{1}{\left(s+m\right)^2}}{\frac{s+\omega^2}{\left(s+m\right)^2}} \qquad (3)$$

The load disturbance n is supposed also in the form of (2), (3). The divisibility of elements in R_{ps} is defined through the all unstable zeros (including infininy) of the rational functions, see [18] for details.

The basic control problem is then formulated as follows within the context of Fig.1: Consider the known transfer function (1), the reference and disturbance (2), (4). The task is to design a proper transfer function C(s) so that the closed loop system is asymptotic stable and the tracking error $e(t)=w(t)-y(t)$ tends to zero. Moreover, a step-wise disturbance n(t) has to be eliminated without a non-zero steady-state error (disturbance attenuation).

3 Control and Disturbance Rejection Design in R_{PS}

Suppose a general closed loop control system depicted in Fig.1. The controller C(s) generates the control variable u according the equation:

$$P\,u=R\,w-Q\,y+n \qquad (4)$$

where n is a load disturbance. Note that a traditional one degree-of-freedom (1DOF) feedback controller operating on the tracking error is obtained for $Q = R$.

Basic relations following from Fig. 1 are

$$y=\frac{B}{A}u+v \qquad u=\frac{R}{P}w-\frac{Q}{P}y+n \qquad (5)$$

and w, v, n are independent external inputs into the closed loop system.

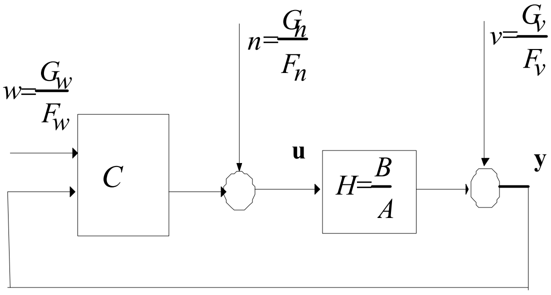

Fig. 1 General closed loop system

Further, the following equations hold:

$$y=\frac{BR}{AP+BQ}\frac{G_w}{F_w}+\frac{AP}{AP+BQ}\frac{G_v}{F_v}+\frac{BP}{AP+BQ}\frac{G_n}{F_n} \ (6)$$

The 1DOF (FB) structure is obtained for $R=Q$ (depicted in Fig.2) and the last relation gives the controlled error $e=w-y$:

$$e=\frac{AP}{AP+BQ}\frac{G_w}{F_w}+\frac{AP}{AP+BQ}\frac{G_v}{F_v}+\frac{BP}{AP+BQ}\frac{G_n}{F_n} \ (7)$$

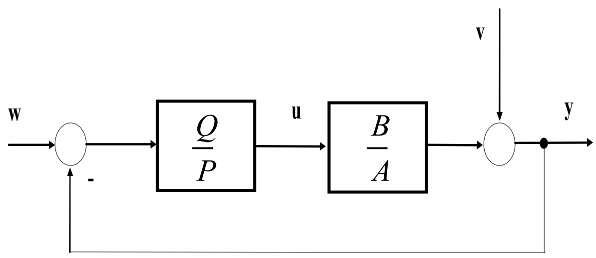

Fig. 2 Structure 1DOF (FB) of the close loop system

The first step of the control design is to stabilize the system by a proper feedback loop. It can be formulated in an elegant way in R_{PS} by the Diophantine equation:

$$AP+BQ=1 \qquad (8)$$

with a general solution for SISO systems $P=P_0+BT$, $Q=Q_0-AT$; where T is free in R_{PS} and P_0, Q_0 is a pair of particular solutions (Youla – Kučera parameterization of all stabilizing controllers). Details and proofs can be found e.g.[10], [15], [16], [18]. Then control error for FBFW structure:

$$e=\left(1-BR\right)\frac{G_w}{F_w}+AP\frac{G_v}{F_v}+BP\frac{G_n}{F_n} \qquad (9)$$

Now, it is necessary to solve both structures 1DOF and 2DOF separately. For asymptotic tracking and the 2DOF (FBFW) structure, the second Diophantine equation gets the form:

$$F_wZ+BR=1 \qquad (10)$$

where $Z \in R_{PS}$ is not used in the control law.
The tracking error e tends to zero if

a) F_w divides AP for 1DOF (11)
b) F_w divides $1-BR$ for 2DOF (12)

Another control problem of practical importance is disturbance rejection and disturbance attenuation. In both cases, the effect of disturbances v and n should be asymptotically eliminated from the plant output. Since the both disturbances are external inputs into the feedback part of the system, the effect must be processed by a feedback controller. It means that the second and third parts in (11) and (12) are

$$\frac{AP}{AP+BQ}\frac{G_v}{F_v} \qquad (13)$$

$$\frac{BP}{AP+BQ}\frac{G_n}{F_n} \qquad (14)$$

must belong to $R_{PS}(s)$, i.e. all $AP+BQ$, F_v, F_n should cancel. In other words, a multiple F_v, F_n must divide P. More precisely F_v, must divide the multiple AP and F_n the multiple BP. When define relatively prime elements A_0, F_{v0} and B_0, F_{n0} in $R_{PS}(s)$

$$\frac{A}{F_v}=\frac{A_0}{F_{v0}}, \qquad \frac{B}{F_n}=\frac{B_0}{F_{n0}} \qquad (15)$$

then the problem of disturbance rejection and attenuation is solvable if and only if the pairs F_v, B and F_n, B are relatively prime and the feedback controller is given by

$$C_b=\frac{Q}{P}=\frac{Q}{P_1F_{v0}F_{n0}} \qquad (16)$$

where P_1, Q is any solution of the equation

$$AF_{v0}F_{n0}P_1+BQ=1 \qquad (17)$$

4 Simple controllers

The fractional approach performed in the ring R_{ps} enables a control design in a very elegant way. Probably, the simplest unstable system is an integrator with the transfer function:

$$G(s)=\frac{b_0}{s} \qquad (18)$$

The basic stabilizing equation (8) takes the form

$$\frac{s}{s+m,}p_0+\frac{b_0}{s+m}q_0=1 \qquad (19)$$

and all solutions can be expressed by

$$P=1+\frac{b_0}{s+m}T;\;\; Q=\frac{m}{b_0}-\frac{s}{s+m}T \qquad (20)$$

with T free in R_{ps}. For the integrator, the condition of divisibility between stepwise F_w and A is generically fulfilled because they are the same and the simplest controller is proportional with the gain m/b_0. The influence of tuning parameter m is shown in Fig.3 where responses for three various parameters are depicted (b_0=1). Naturally, this controller is not able to compensate any load disturbance.
Now, it is necessary to find such a free parameter T in (20) so that controller $C_b=\dfrac{Q}{P}$ ensures asymptotic tracking for a stepwise load disturbance (2). So, the condition (11) is achieved for $t_0=-\dfrac{m}{b_0}$ and the 1 DOF controller takes the form of PI one:

$$\frac{Q}{P}=\frac{\frac{2m}{b_0}s+\frac{m^2}{b_0}}{s} \qquad (21)$$

It is clear that tuning parameter m is incorporated into controller parameters in a nonlinear way. The influence for control behaviour is then demonstrated in Fig. 4 (also for b_0=1).

A bit more complex situation occurs for disturbance rejection with harmonic signal (3). Then the parameterization (20) leads to the expression:

$$P=1+\frac{b_0}{s+m}\frac{t_1s+t_0}{s+m}\approx\frac{s^2+\omega^2}{(s+m)^2} \qquad (22)$$

It is necessary to find parameters t_0, t_1 satisfying the identity in (22). Equating of coefficients in (22), the following linear equations for t_0, t_1 are:

$$\begin{aligned} 2m+b_0t_1&=0 \\ m^2+t_0&=\omega^2 \end{aligned} \qquad (23)$$

with the solution

$$\begin{aligned} t_1&=-\frac{2m}{b_0} \\ t_0&=\omega^2-m^2 \end{aligned} \qquad (24)$$

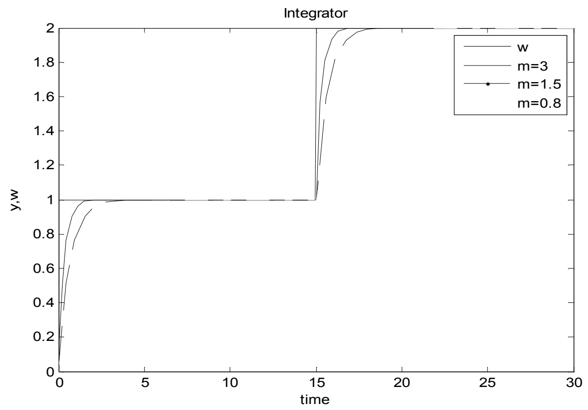

Fig. 3 Simple integrator with P controller

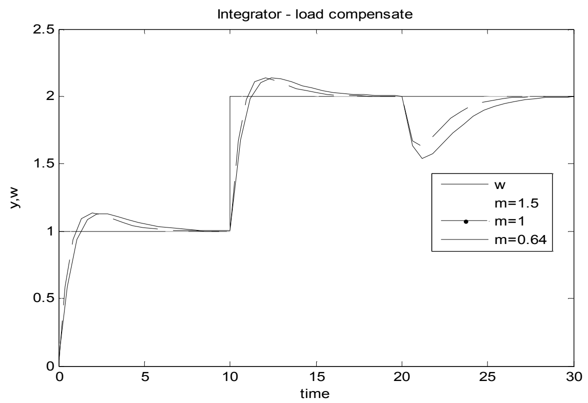

Fig. 4 Simple integrator with PI controller

The resulting feedback controller $C(s)=\dfrac{Q}{P}$ has no more of the PI or PID structure but it takes the form:

$$\frac{Q}{P}=\frac{q_2 s^2+q_1 s+q_0}{s^2+\omega^2} \qquad (25)$$

where

$$q_2=\frac{m}{b_0}-t_1; q_1=\frac{2m^2}{b_0}-t_0; q_0=\frac{m^3}{b_0} \qquad (26)$$

The control responses for three different values of m are depicted in Fig.5.

The feedforward part of the 2DOF structure for integrator (18) is given by (10) with a particular solution

$$\frac{s}{s+m}Z+\frac{b_0}{s+m}R=1 \qquad (27)$$

$$r_0=\frac{m}{b_0} \qquad (28)$$

and the feedforward transfer function of the control structure is then obtained

$$\frac{R}{P}=\frac{r_0(s+m)^2}{s^2+\omega^2} \qquad (29)$$

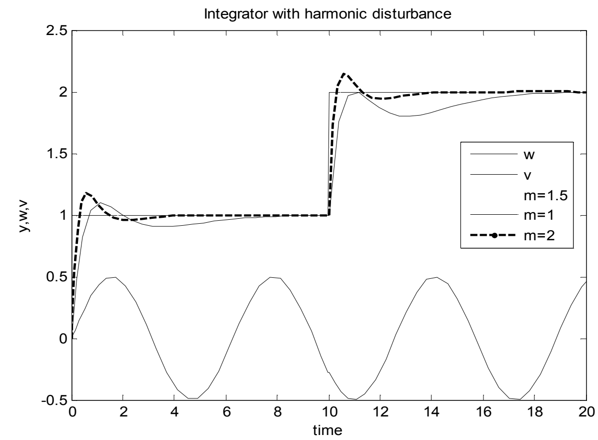

Fig. 5 Integrator with harmonic disturbance compensation

A second set of controllers for unstable systems can be derived for system governed by the transfer function:

$$G(s)=\frac{b_0}{s-a_0} \qquad (30)$$

with $a_0>0$. The stabilization feedback equation (8) takes the form

$$\frac{s-a_0}{s+m}p_0+\frac{b_0}{s+m}q_0=1 \qquad (31)$$

with all parameterization solutions

$$P=1+\frac{b_0}{s+m}T;\ Q=\frac{m+a_0}{b_0}-\frac{s-a_0}{s+m}T \qquad (32)$$

In this case (for the stepwise reference) the divisibility condition $F_w \backslash AP$ is not generically fulfilled and it is achieved for $T=t_0=-\dfrac{p_0 m}{b_0}$. The final feedback part is again in the form of PI controllers:

$$\frac{Q}{P}=\frac{q_1 s+q_0}{s} \qquad (33)$$

where

$$q_1=\frac{2m+a_0}{b_0};\ q_0=\frac{m^2}{b_0} \qquad (34)$$

Simulations for three values *m (0.6, 1.0, 2.5)* for the particular case $b_0=2$, $a_0=0.5$ are shown in Fig.6.

Two remarkable facts can be seen in Fig.4, Fig.6. The first one is that increasing value of the tuning parameter *m* lowers overshoot of the control response. The second one is that the divisibility condition enables to compensate the stepwise load disturbance which is injected in the time *t=20*.

Another question is a total rejection of overshoot. It can be achieved by utilizing of control structure 2DOF and equation (10). The control responses for *m=1.0* is depicted in Fig.7. Generally, the 2DOF

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structure always reduces overshoots after step changes of input signals (reference, load disturbance).

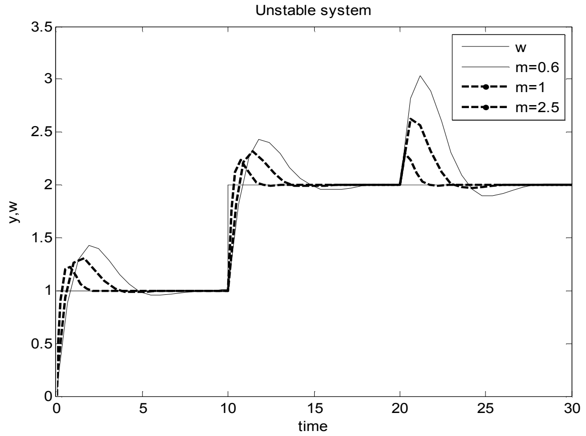

Fig. 6 Unstable system (30) with 2DOF control structure

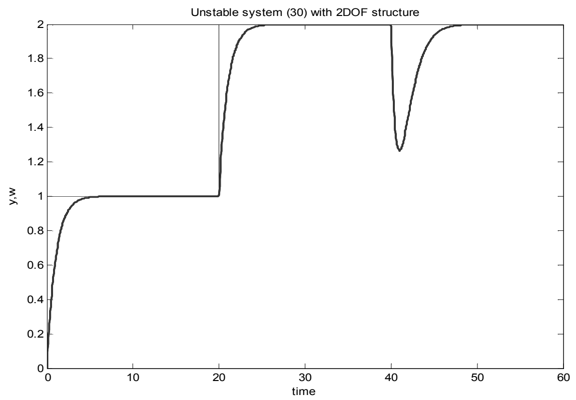

Fig. 7 Unstable system (30) with 2DOF control structure

The third class of controllers is derived for a frequent case of unstable systems with the integrator in the form

$$G(s) = \frac{b_0}{s(s-a_0)} \qquad (35)$$

The divisibility condition for a step-wise reference with $F_w = \dfrac{s}{s+m}$ is fulfilled, so the stabilizing equation (8) also ensures asymptotic tracking. This equation in this case takes the form

$$\frac{s(s-a_0)}{(s+m)^2}\frac{p_1 s + p_0}{(s+m)} + \frac{b_0}{(s+m)^2}\frac{q_1 s + q_0}{(s+m)} = 1 \qquad (36)$$

It is easy to express parameters p_i, q_i and the particular controller has the transfer function

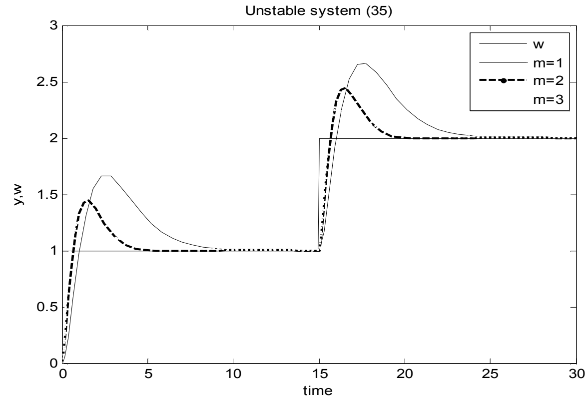

Fig. 8 Unstable system (30)

$$\frac{Q}{P} = \frac{q_1 s + q_0}{p_1 s + p_0} \qquad (37)$$

where

$$p_1 = 1; \ \ p_0 = 3m + a_0;$$
$$q_1 = \frac{3m^2}{b_0} + a_0(3m + a_0); \ \ q_0 = \frac{m^3}{b_0} \qquad (38)$$

Simulations for the case $b_0=1$ and $a_0=0.5$ and three parameters are shown in Fig.8.

6 Conclusions

The task of simultaneous regulation and disturbance attenuation for a class of unstable systems is considered. A controller design methodology is based on fractional representation in the ring of proper and stable rational functions. Resulting control laws in 1 DOF structure give a class of PI, PID controllers. Itch is important from application point of view. More complex structure 2 DOF gives more sophisticated controllers which have no more the PID structure but the benefit is in control response. The proposed methodology brings a scalar parameter m>0 which enables to tune and influence the robustness and control behaviour. The tuning parameter can be chosen arbitrarily or it can be a result of some optimization or calculation. Also problems of disturbance attenuation are analysed. The proposed results were verified in the Matlab + Simulink environment.

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