Analysis of a simple quasipolynomial of degree one

LIBOR PEKAŘ, ROMAN PROKOP
Department of Automation and Control, Faculty of Applied Informatics
Tomas Bata University in Zlín
nám. T. G. Masaryka 5555, 76001 Zlín
CZECH REPUBLIC
pekar@fai.utb.cz, prokop@fai.utb.cz

Abstract: - Systems with internal delays in feedback loops, so called time-delay or anisochronic ones, constitute a widespread family of industrial plants. Their linear representation in the form of the Laplace transform yields transfer functions expressed as ratios of quasipolynomials, instead of polynomials, with delay (exponential) terms in denominators. In this paper, detailed pole location analysis of a quasipolynomial of degree one is presented. This quasipolynomial is capable to express the dynamics of conventional high order systems, even without internal delays, as a model transfer function denominator. The presented analysis represents also a powerful tool for controller tuning in pole-placement control algorithms for delayed systems.

Key-Words: - Delayed systems, anisochronic systems, stability analysis, quasipolynomials

1 Introduction

In feedback control systems, delay as a generic part of many processes is a phenomenon which unambiguously deteriorates the quality of a control performance. Modern control theory has been dealing with this problem since its nascence – the well known Smith predictor has been known for longer than five decades [1]. Linear time delay systems in technological and other processes are usually assumed to contain delay elements in input-output relations only, which results in shifted arguments on the right-hand side of differential equations. All the system dynamics is hence traditionally modeled by point accumulations in the form of a set of ordinary differential equations (ODEs). The Laplace transform then yields a transfer function expressed by a serial combination of a delayless term and a delay exponential element. However, this conception is somewhat restrictive in effort to fit the real plant dynamics since inner feedbacks are of time-distributed or delayed nature.

Anisochronic models, on the other hand, offer a more universal dynamics description applying both integrators and delay elements on the left-hand side of a differential equation, either in a lumped or distributed form. Using some techniques [2], one can reduce possible integrals to a combination of shifted-argument output or state variables elements, which finally gives a transfer function as a ratio of so called quasipolynomials [3] with an infinite number of poles.

These models are applied in processes with energy or mass transportation phenomena, e.g. in chemical processes [4], in heat exchange networks [5], [6], in internal combustion engines with catalytic converter [7], in models of mass flow in sugar factory [8], in metallurgic processes [9], etc.

A large number of conference and journal papers were dedicated to stability analysis of systems with delay elements on the left-hand side of a (functional) differential equation (see e.g. [10]-[12]); nevertheless, general approaches lacking detailed analysis of a particular denominator quasipolynomial prevail. This contribution, contrariwise, offers the deep analysis of a simple quasipolynomial which, as a model transfer function denominator, is convenient to represent the dynamics of many anisochronic systems as well as high order systems even delayless [13]-[14]. Moreover, we present stability properties depended on a real non-delay parameters instead that on a delay value. The information about poles locations can serve engineers to decide quickly about dominant poles location or to place closed-loop poles when using anisochronic controllers.

2 Anisochronic System Model

Anisochronic models in general can be described by state and output functional differential equations in the form

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} &= \sum_{i=1}^{N_H} \mathbf{H}_i \frac{\mathrm{d}\mathbf{x}(t-\eta_i)}{\mathrm{d}t} \\
&+ \mathbf{A}_0\mathbf{x}(t) + \sum_{i=1}^{N_A} \mathbf{A}_i \mathbf{x}(t-\eta_i) \\
&+ \mathbf{B}_0\mathbf{u}(t) + \sum_{i=1}^{N_B} \mathbf{B}_i \mathbf{u}(t-\eta_i) \\
&+ \int_0^L \mathbf{A}(\tau)\mathbf{x}(t-\tau)d\tau + \int_0^L \mathbf{B}(\tau)\mathbf{u}(t-\tau)d\tau \\
\mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)
\end{aligned}
\tag{1}
$$

where $\mathbf{x} \in \mathbb{R}^n$ is a vector of state variables, $\mathbf{u} \in \mathbb{R}^m$ stands for a vector of inputs, $\mathbf{y} \in \mathbb{R}^l$ represents a vector of outputs, $\mathbf{A}_i$, $\mathbf{A}(\tau)$, $\mathbf{B}_i$, $\mathbf{B}(\tau)$, $\mathbf{C}$, $\mathbf{H}_i$ are matrices of compatible dimensions, $0 \le \eta_i \le L$ are lumped delays and convolution integrals express distributed delays. If $\mathbf{H}_i \neq \mathbf{0}$ for any i = 1,2,...N_H, model (1) is called *neutral*; on the other hand, if $\mathbf{H}_i = \mathbf{0}$ for every i = 1,2,...N_H, so-called *retarded* model is obtained. It should be noted that the state of model (1) is given not only by a vector of state variables in the current time, but also by a segment of the last model history of state and input variables

$$
\mathbf{x}(t+\tau),\ \mathbf{u}(t+\tau),\ \tau \in [-L, 0] \tag{2}
$$

Integrals in (1) can be rewritten into sums using the Laplace transform, which is suitable for model implementation in computers and simulations. The transform correspondence is the following

$$
\begin{aligned}
\mathcal{L}\left\{\int_0^L \mathbf{A}(\tau)\mathbf{x}(t-\tau)\mathrm{d}\tau\right\} &= \mathbf{X}(s)\int_0^L \mathbf{A}(\tau)\exp(-s\tau)\mathrm{d}\tau \\
\mathcal{L}\left\{\int_0^L \mathbf{B}(\tau)\mathbf{u}(t-\tau)\mathrm{d}\tau\right\} &= \mathbf{U}(s)\int_0^L \mathbf{B}(\tau)\exp(-s\tau)\mathrm{d}\tau
\end{aligned}
\tag{3}
$$

where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform operation. Subsequent utilization of the reverse Laplace transform yields the state equation in the form

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{z}(t)}{\mathrm{d}t} &= \sum_{i=1}^{N_H} \widetilde{\mathbf{H}}_i \frac{\mathrm{d}\mathbf{z}(t-\eta_i)}{\mathrm{d}t} + \widetilde{\mathbf{A}}_0\mathbf{z}(t) + \sum_{i=1}^{N_A+1} \widetilde{\mathbf{A}}_i \mathbf{z}(t-\eta_i) \\
&+ \widetilde{\mathbf{B}}_0\mathbf{z}(t) + \sum_{i=1}^{N_B+1} \widetilde{\mathbf{B}}_i \mathbf{z}(t-\eta_i) \\
\mathbf{z}(t) &= \left[\mathbf{x}(t) \quad \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}\right]^{\mathrm{T}}
\end{aligned}
\tag{4}
$$

where $\eta_{N_A+1} = \eta_{N_B+1} = L$. Or, alternatively, one can utilize a numerical approximation of convolution integrals, which, however, can destroy system stability in some cases, see e.g. [10] and references herein.

Notice that authors' interest is in retarded models and systems due to their higher practical usability [5]-[8]. Considering a model of retarded type and zero initial conditions, the following input-output description and the transfer matrix using the Laplace transform from (4) is obtained

$$
\begin{aligned}
\mathbf{Y}(s) &= \mathbf{G}(s)\mathbf{U}(s) \\
&= \frac{\mathbf{C}\mathrm{adj}\left[s\mathbf{I} - \widetilde{\mathbf{A}}_0 - \sum_{i=1}^{N_A+1} \widetilde{\mathbf{A}}_i \exp(-s\eta_i)\right]}{\det\left[s\mathbf{I} - \widetilde{\mathbf{A}}_0 - \sum_{i=1}^{N_A+1} \widetilde{\mathbf{A}}_i \exp(-s\eta_i)\right]} \\
&\left[\widetilde{\mathbf{B}}_0 + \sum_{i=1}^{N_B+1} \widetilde{\mathbf{B}}_i \exp(-s\eta_i)\right]\mathbf{U}(s)
\end{aligned}
\tag{5}
$$

The main advantage of the anisochronic system description in the form of the transfer function is in its practical usability when system analysis and control design.

All transfer functions in $\mathbf{G}(s)$ have identical denominator in the form

$$
\begin{aligned}
m(s) &= \det\left[s\mathbf{I} - \widetilde{\mathbf{A}}_0 - \sum_{i=1}^{N_A+1} \widetilde{\mathbf{A}}_i \exp(-s\eta_i)\right] \\
&= s^n + \sum_{i=0}^{n-1}\sum_{j=1}^{h_i} m_{ij} s^i \exp(-s\eta_{ij})
\end{aligned}
\tag{6}
$$

where

$$
h_i \le \binom{N_A + n - i}{n - i} \tag{7}
$$

which arises from the calculation of all permutations in the determinant; since, the upper bound of h_i equals the number of all combinations with repetitions of N_A+1 elements choose (n-i).

Thenceforward, a simple-input simple-output system is considered, which gives rise to the transfer function (5) in the form of a ratio of quasipolynomials instead of transfer function matrix.

ISBN: 978-1-61804-004-6

3 System Stability

Formula (6) expresses the *characteristic quasipolynomial* of retarded type of system (1) and it determines the system poles, σ_i, by solution of $m(s)=0$, and thus decides about the system stability. Due to the transcendental character of model (1) caused by exponential terms, the number of poles is infinite in general and anisochronic models are regarded as infinite-dimensional. The role of zeros is the same as for delay-free systems and the number of zeros depends on the structure of numerators in (5).

The system stability is formulated in the same way as for systems void of state delays, hence, it is determined by system poles. An anisochronic system is stable iff

$$\bar{\sigma} := \sup\{\operatorname{Re}(\sigma_i) : m(\sigma_i) = 0\} < 0 \quad (8)$$

i.e. all system poles are located in the open left half complex plane, see, e.g. [15], [16].

Both types of systems, retarded and neutral ones, embody diverse spectral properties w.r.t. poles locations and their changes depended on changes of quasipolynomials parameters. Whereas retarded systems always own finite number of unstable poles, unstable neutral systems have infinite number of these poles which constitute vertically bounded strips [15]. Another important feature is that poles locations of the retarded type is continuously depended on delays η_i, while small changes in delays for neutral systems can cause abrupt changes in the spectrum.

4 Main Result

The main goal of this contribution is to study spectral properties of a simple retarded quasipolynomial given by

$$m(s) = s + q\exp(-\tau s) \quad (9)$$

where $s \in □$, $q \in □$, $\tau \in □^{+}$, with respect to the non-delay real parameter q while τ is fixed. It was demonstrated [14], [17] that models of dynamics described by this quasipolynomial as a transfer function denominator, can be successfully used for the description of a real plant dynamics of conventional (delay-free) high order systems. Stability properties of quasipolynomial (9) have been already studied in [18]. In [17] the dynamics of (9) using a root mapping algorithm was investigated; however, a deeper analysis was not made.

Note that all the following lemmas and theorems refer to quasipolynomial (9). Proofs are omitted due the limited space and they will be the aim of a future paper.

Lemma 1. Quasipolynomial (9) has a root (a real or a complex conjugate pair) on the (Hurwitz) stability border (i.e. on the imaginary axis) iff

$$q = 0\,,\ q = (-1)^k (2k+1)\frac{\pi}{2\tau},\ k = 0,1,2,... \quad (10)$$

Moreover it holds that

$$\omega = \pm|q| \quad (11)$$

where ω is the position of the root on the imaginary axis. ■

Lemma 1 gives no information about other roots positions so that we can not decide about the rest of the quasipolynomial spectrum. The following lemmas clarify positions of other roots of (9) when (10) holds. To prove the lemmas, a significant theorem formulated e.g. in [19] has to be presented.

Theorem 1 (Root continuity). Let $f(s)$ and the sequence $\{f_n(s)\}_{n\geq 1}$ be analytic function on an (open) domain □ ⊆ □. Suppose that $\{f_n(s)\}_{n\geq 1}$ converges uniformly to $f(s)$ on the disc $D = \{s : |s - \sigma_0| \leq r\} \subseteq □$ for some $r > 0$ and that on this disc σ_0 is the only zero of $f(s)$ with multiplicity *k*. Then there exists a natural number *N* such that $\forall n \geq N$, $f(s)$ has exactly *k* zeros $\sigma_{n,1}$, ..., $\sigma_{n,k}$ in *D* and $\lim_{n\to\infty} \sigma_{n,j} = \sigma_0$, $\forall j \in \{1,...,k\}$. ■

Theorem 1 implies another important fact that retarded quasipolynomial $m(s)$ has roots continuous w.r.t. changes of *q*. That is, a limit sequence of quasipolynomials (with infinitesimal changes of *q*) results in a corresponding limit sequence of roots of $m(s)$.

Lemma 2. For *q* = 0, there is no root in the open right half complex plane. ■

Further (in Proposition 5) we will show that there are infinity many roots with a real part $\alpha = -\infty$ for $q = 0$.

Lemma 3 (Roots shift tendency). Define two sets Σ_1, Σ_2 of *q* as

$$\Sigma_1 := \left\{ q : q = (4k+1)\frac{\pi}{2\tau},\ k = 0,1,2,.. \right\}$$

$$\Sigma_2 := \left\{ \begin{matrix} q : q = 0 \text{ or } q = -(4k+3)\frac{\pi}{2\tau}, \\ k = 0,1,2,.. \end{matrix} \right\} \quad (12)$$

and two corresponding spectra of $m(s)$

$$\Theta_1 := \left\{ \sigma \in \mathbb{R} : m(\sigma)\big|_{q\in\Sigma_1} = 0 \right\}$$

$$\Theta_2 := \left\{ \sigma \in \mathbb{R} : m(\sigma)\big|_{q\in\Sigma_2} = 0 \right\} \quad (13)$$

Then the following statements hold:

1) If $q = r + \Delta$ where $r \in \Sigma_1$ and Δ is an arbitrarily small positive real number, then the spectrum of $m(s)$ have one root (real or complex conjugate pair) in the open *right* half complex plane more in comparison with Θ_1.

2) If $q = r + \Delta$ where $r \in \Sigma_2$ and Δ is arbitrarily small positive real number, then the spectrum of $m(s)$ have one root (real or complex conjugate pair) in the open *left* half complex plane more in comparison with Θ_2. ■

Lemma 3 (roots shift tendency) is expressed graphically in Fig. 1.

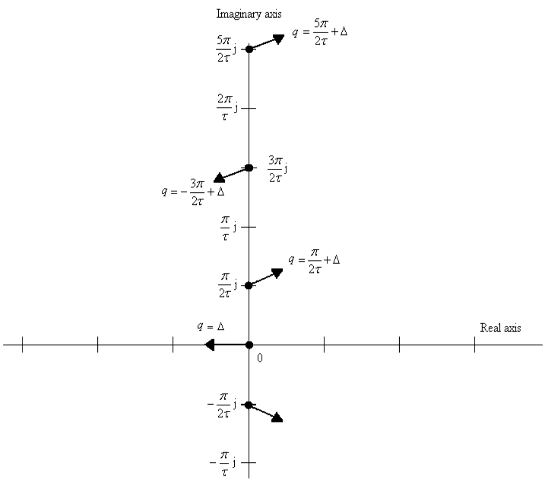

Fig. 1. Root shifting tendency for $q \in \Sigma_1, \Sigma_2$ on the imaginary axis

Theorem 2. **(**Quasipolynomial stability**)**. Quasipolynomial (9) has all roots in the open left half complex plane iff

$$q \in \left(0, \frac{\pi}{2\tau}\right) \quad (14)$$

■

The result of Theorem 2 is already presented in [18].

Proposition 1. There exists a double real root $\sigma = -\frac{1}{\tau}$ in the spectrum of $m(s)$ iff $q = \frac{1}{\tau \exp(1)}$. ■

Let us now derive and display a figure that clarifies the further statements. The inevitable relation between real and imaginary parts of the roots is

$$-\alpha = \omega \cot(\tau\omega) \quad (15)$$

Denote a real part of a root as

$$\alpha = -\frac{k_0}{\tau},\ k_0 \in \mathbb{R} \quad (16)$$

which is a multiple of the “critical” root from Proposition 1. Hence

$$\frac{1}{k_0}(\tau\omega) = \tan(\tau\omega) \quad (17)$$

Take the substitution

$$\xi = \tau\omega \quad (18)$$

Final relation

$$\frac{1}{k_0}\xi = \tan(\xi) \quad (19)$$

has a nice graphical interpretation, see Fig. 2. The figure indicates the imaginary parts of roots depending on τ, k_0. For example, if $k_0 = 1$, i.e. $\alpha = -\frac{1}{\tau}$, there is no intersection near the zero point. If one moves the real part of the root to the right, i.e. $\alpha = -\frac{1-\delta}{\tau},\ \delta > 0$, an intersection near the zero point appears, which means that the real root has become a complex conjugate pair. The following three propositions formalize i.a. this fact.

Proposition 2. If $q = \dfrac{1}{\tau \exp(1)}$, there is no root (real or complex conjugate) with $\alpha \in \left(-\dfrac{1}{\tau}, 0\right)$, i.e. with $k_0 \in (0,1)$. ■

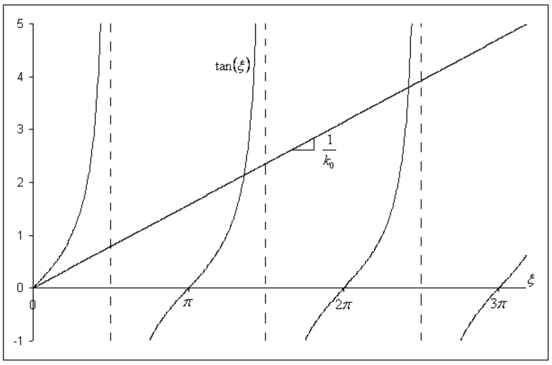

Fig. 2. Graph of functions $\dfrac{1}{k_0}\xi$ and $\tan(\xi)$

Proposition 3. If $q = \dfrac{1}{\tau \exp(1)} + \Delta$, for arbitrarily small $\Delta > 0$, the double real root $\sigma = -\dfrac{1}{\tau}$ bifurcates into a complex conjugate pair of roots $\sigma = \alpha \pm \mathrm{j}\omega$ with $\alpha < -\dfrac{1}{\tau}$. ■

Now we use Fig. 2 which clearly indicates whether there is a complex conjugate root with ω near zero, then $0 < k_0 < 1$, i.e. $-\dfrac{1}{\tau} < \alpha < 0$.

Proposition 4. If $q = \dfrac{1}{\tau \exp(1)} - \Delta$, for arbitrarily small $\Delta > 0$, then the double real root $\sigma = -\dfrac{1}{\tau}$ becomes two (different) real roots σ_1, σ_2 with $\sigma_1 < -\dfrac{1}{\tau}$ and $\sigma_2 > -\dfrac{1}{\tau}$, respectively. ■

Proposition 5. Define two sets $\Sigma_{-\infty,1}, \Sigma_{-\infty,2}$ of ω as

$$\Sigma_{-\infty,1} := \left\{\omega : \omega = \lim_{x \to \pi^+} (2k+1)\frac{x}{\tau},\ k = 0,1,2,...\right\}$$
$$\Sigma_{-\infty,2} := \left\{\omega : \omega = 0 \text{ or } \omega = \lim_{x \to \pi^+} 2k\frac{x}{\tau},\ k = 1,2,...\right\} \quad (20)$$

If there exists a root (or a complex conjugate pair of roots) of quasipolynomial (9), $\sigma = \alpha \pm \mathrm{j}\omega$, with $\alpha \to -\infty$, then the imaginary part of the root lies either in the set $\Sigma_{-\infty,1}$ or in the set $\Sigma_{-\infty,2}$.

Moreover, if $\omega \in \Sigma_{-\infty,1}$, then $q = 0^-$, i.e. it asymptotically moves to zero from the left. If $\omega \in \Sigma_{-\infty,2}$, then $q = 0^+$, i.e. it asymptotically moves to zero from the right. ■

According to Preposition 5, it is obvious that if q reaches zero from the left, there exist roots of (9) with real parts in negative infinity and imaginary parts from $\Sigma_{-\infty,1}$, and if q approaches zero from the right, there are roots again in negative infinity, the imaginary parts of which lie in $\Sigma_{-\infty,2}$. This fact explains i.a. the position where it moves the real root which appears by splitting the double real root $\sigma = -\dfrac{1}{\tau}$ when $q = \dfrac{1}{\tau \exp(1)} - \Delta$ (i.e. σ_1 from Proposition 4).

Proposition 6. Define two sets $\Sigma_{\infty,1}, \Sigma_{\infty,2}$ of ω as

$$\Sigma_{\infty,1} := \left\{\omega : \omega = \lim_{x \to \pi^-} (2k+1)\frac{x}{\tau},\ k = 0,1,2,..\right\}$$
$$\Sigma_{\infty,2} := \left\{\omega : \omega = 0 \text{ or } \omega = \lim_{x \to \pi^-} 2k\frac{x}{\tau},\ k = 1,2,..\right\} \quad (21)$$

If there exists a root (or a complex conjugate pair of roots) of quasipolynomial (9), $\sigma = \alpha \pm \mathrm{j}\omega$, with $\alpha \to \infty$, the imaginary part of the root is either in the set $\Sigma_{\infty,1}$ or in the set $\Sigma_{\infty,2}$.

Moreover, if $\omega \in \Sigma_{\infty,1}$, then $q \to \infty$. If $\omega \in \Sigma_{\infty,2}$, then $q \to -\infty$. ■

Proposition 6 gives rise to the fact that for $q \to \infty$, roots approach infinity in the real axis and their imaginary parts are from $\Sigma_{\infty,1}$. Finally, when $q \to -\infty$, real parts of roots go to infinity; however, their imaginary parts are from $\Sigma_{\infty,2}$.

We can thus imagine the existence of tangential “strips” of roots running from the positive to negative infinity and vice-versa, depending on the range of values on the imaginary axis.

5 Conclusion

Stability analysis of the first order retarded quasipolynomial by analytical means has been presented. The studied quasipolynomial can represent the denominator of an internally delayed (anisochronic) system transfer function or the characteristic quasipolynomial of the closed loop when control systems with time delays. Anisochronic models proved to be a suitable form for description of dynamic properties of high order even undelayed systems as well.

Information about the spectrum features derived in this contribution can afford engineers a potential tool for internally delayed systems analysis and also in an effort to place dominant closed loop poles of a feedback control system. In contrast to other general stability criteria, this particular result can be utilizable promptly without additional work-intensive calculations.

Acknowledgement

The work was performed with financial support of research project MSM7088352102. This support is very gratefully acknowledged.

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