The Nyquist criterion for LTI Time-Delay Systems

LIBOR PEKAŘ, RADEK MATUŠŮ, PETR DOSTÁLEK, JAN DOLINAY
Department of Automation and Control Engineering, Faculty of Applied Informatics
Tomas Bata University in Zlin
nam. T. G. Masaryka 5555, 76001 Zlin
CZECH REPUBLIC
pekar@fai.utb.cz

Abstract: - This paper extends results about stability and stabilization of a retarded quasipolynomial obtained using the Mikhaylov criterion earlier. Retarded quasipolynomials appear as numerators and denominators of linear time-invariant time-delay systems (LTI-TDS). A LTI-TDS system of retarded type (destitute of distributed delays) is said to be stable if all roots of its characteristic quasipolynomial are located in the open left-half complex plane. The contribution transforms the formulation of spectrum assignment of a characteristic quasipolynomial into the language of the Nyquist criterion for the open loop of a control system. Again, the argument principle is utilized to derive generalized Nyquist criterion for LTI-TDS. Stability measures related to the criterion are discussed with the specifications for LTI-TDS. An illustrative example is presented to illuminate the results.

Key-Words: - Time-delay systems, Nyquist criterion, stability, stabilization, quasipolynomial, gain margin

1 Introduction

Asymptotic stability, spectrum analysis and stabilization of linear time-invariant time-delay systems (LTI-TDS) have been challenging tasks in control theory during last decades. The problems are nontrivial even for these simple-modeled systems due to their infinite dimensional nature. A vast bulk of various significant results was obtained and reported; without any attempt to be exhaustive, see for instance [1] – [7].

LTI-TDS can be represented in the input-output description by the Laplace transfer function as a ratio of so-called quasipolynomials in one complex variable, instead of polynomials as for delay-free systems. Delay in the feedback can significantly deteriorate the quality of control performance, namely stability and periodicity. For lumped delays, the denominator quasipolynomial decides about the control system asymptotic stability because of the fact that its zeros are system poles with the same meaning as for polynomials; however, the spectrum is infinite due to a quasipolynomial transcendental form. Analysis of asymptotic properties of the characteristic quasipolynomial and its spectrum is one of a possible ways how to handle LTI-TDS stability.

The presented paper extends results obtained for a retarded quasipolynomial with two delays in [8] and for the control feedback with a first order LTI-TDS in [9]. The findings in these papers were obtained via the argument principle (or via the Mikhaylov stability criterion). Applying the argument principle for the control feedback along with the knowledge the open loop frequency response results in the use of the well known Nyquist criterion. The notorious precept about the number of open loop unstable poles, however, is not easy to utilize in the case of LTI-TDS due to the infinite spectrum which is of an effort to be calculated [10]-[11]. Hence, in this paper we simply derive the generalized Nyquist criterion for LTI-TDS and we turn over some stability measures, such as gain and phase margins and the minimum of the sensitivity function. There are some specific features about the measures LTI-TDS which are discussed in the paper.

An example supported by simulations in Matlab-Simulink is presented to clarify the results and statements.

2 Preliminaries

Recall basic results about the argument (increment) principle for retarded quasipolynomials and findings introduced in [8] and [9].

Theorem 1 (Argument increment principle) [12]. Consider a retarded quasipolynomial of the form

$$m(s) = s^n + \sum_{i=0}^{n-1} \sum_{j=1}^{h_i} m_{ij} s^i \exp(-s\vartheta_{ij}) \tag{1}$$

If $m(s) \neq 0$ for any imaginary $s = \mathrm{j}\omega$, $\omega \in \mathbb{R}$, function $m(s)$ has no zero in the closed right half s-plane iff the argument of $m(s)$ reaches the increment

$$\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta \arg m(s)} = \frac{n\pi}{2} \tag{2}$$

■

Definition 1 (Retarded quasipolynomial stability). Retarded quasipolynomial of the form (1) is said to be *asymptotically stable* if it has no root in the closed right half s-plane, i.e. if there is no σ such that $m(\sigma) = 0, \mathrm{Re}\{\sigma\} \geq 0$. ■

Proposition 1 (Number of unstable roots) [13]. Consider a retarded quasipolynomial (1). Then the number N_U of poles of $m(s)$ located in the closed right half s-plane (i.e. unstable ones) is

$$N_U = \frac{n}{2} - \frac{1}{\pi} \underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta \arg m(s)} \tag{3}$$

■

The following results have been derived for simple quasipolynomials with $n = 1$ and $h_0 = 1$ and $h_0 = 2$, respectively.

Theorem 2 [9]. Consider the quasipolynomial

$$m(s) = s + a\exp(-\vartheta s) + kq \tag{4}$$

where $a \neq 0 \in \mathbb{R}$; $k, \vartheta > 0 \in \mathbb{R}$ are fixed, whereas q is selectable. Then, if

$$a\vartheta \leq 1 \tag{5}$$

the quasipolynomial (4) is asymptotically stable iff

$$q > \frac{-a}{k} \tag{6}$$

Contrariwise, if

$$a\vartheta > 1 \tag{7}$$

the quasipolynomial (4) is asymptotically stable iff

$$q > \frac{-a\cos(\vartheta\omega_0)}{k} \tag{8}$$

where the *crossover frequency* ω_0 is the minimum nonzero element of the set

$$\Omega_0 := \{\omega : \omega > 0, \mathrm{Im}\{m(\mathrm{j}\omega)\} = 0\} \tag{9}$$

■

Definition 2. Consider quasipolynomial

$$m(s) = s + a\exp(-\vartheta s) + kq\exp(-\tau s) \tag{10}$$

with $a \neq 0 \in \mathbb{R}$; $k, \vartheta, \tau > 0 \in \mathbb{R}$. Here, the set of *crossover frequencies* is defined as

$$\Omega_1 := \{\omega : \omega > 0, \mathrm{Im}\{m(\mathrm{j}\omega)\} = \mathrm{Re}\{m(\mathrm{j}\omega)\} = 0\} \tag{11}$$

The *critical frequency* ω_C is defined as

$$\omega_C := \min\left\{\omega : \omega \in \Omega_1, \underset{s=\omega\mathrm{j},\omega\in[0,\omega_C)}{\Delta \arg m(s)} = 0, \underset{s=\omega\mathrm{j},\omega\in[\omega_C,\infty)}{\Delta \arg m(s)} = \frac{\pi}{2}\right\} \tag{12}$$

for a particular *critical gain* q_C given by

$$q_C = \frac{\omega_C - a\sin(\vartheta\omega_C)}{k\sin(\tau\omega_C)} \tag{13}$$

■

Remark 1 [8]. Elements $\omega_1 \in \Omega_1$ are calculated as all solutions of the transcendental equation

$$\omega_1 \cos(\tau\omega_1) = a(\sin((\vartheta - \tau)\omega_1)) \tag{14}$$

■

Theorem 3 [8]. If $\sin(\tau\omega_C) > 0$, then quasipolynomial (10) is stable iff

$$\frac{-a}{k} < q < \frac{\omega_C - a\sin(\vartheta\omega_C)}{k\sin(\tau\omega_C)} \tag{15}$$

Contrariwise, if $\sin(\tau\omega_C) < 0$, then quasipolynomial (10) is stable iff

$$q > \frac{\omega_C - a\sin(\vartheta\omega_C)}{k\sin(\tau\omega_C)} \geq \frac{-a}{k} \tag{16}$$

where ω_C is the critical frequency. ■

3 Main Result – Nyquist Criterion

In this chapter the Nyquist criterion for retarded LTI-TDS based on the argument principle is presented. As usual, the Nyquist criterion gives information about the closed-loop stability based on the knowledge of the overall phase shift (argument increment) of the open-loop transfer function around the critical point -1.

Consider a simple control system as in Fig. 1 and the corresponding closed loop transfer function

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$$G_{WY}(s)=\frac{Y(s)}{W(s)}=\frac{G_R(s)G(s)}{1+G(s)G_R(s)}=\frac{G_0(s)}{1+G_0(s)} \quad (17)$$

where $G_0(s)$ is the open-loop transfer function. Note that for LTI-TDS transfer functions are no longer rational but *meromorphic*.

Express the transfer functions as

$$G(s)=b(s)/a(s),\ G_R(s)=q(s)/p(s) \quad (18)$$

where $a(s), b(s), q(s), p(s)$ are retarded quasipolynomials and $G(s)$ is strictly proper and $G_R(s)$ is proper (the properness is defined as for delay-free systems using the highest *s*-power). Then

$$G_{WY}(s)=\frac{Y(s)}{W(s)}=\frac{\frac{q(s)b(s)}{p(s)a(s)}}{\frac{p(s)a(s)+q(s)b(s)}{p(s)a(s)}} \quad (19)$$

where the characteristic quasipolynomial $m(s)$ reads

$$m(s)=p(s)a(s)+q(s)b(s) \quad (20)$$

We can formulate and prove the following theorem.

Theorem 4 (The Nyqusit criterion for retarded quasipolynomials). Let the plant and the controller have transfer functions as in (18) and the control system be in a simple form as in Fig. 1. Let $a(s)$ and $p(s)$ have no root on the imaginary axis, i.e. $a(s)\neq 0, p(s)\neq 0$ for any imaginary $s=\mathrm{j}\omega$, $\omega\in\mathbb{R}$.

Then, if

$$\underset{s=\omega\mathrm{j},\omega\in[0,\infty)}{\Delta\arg}\ p(s)a(s)=l\pi/2 \quad (21)$$

The closed-loop system is asymptotically stable if

$$\underset{s=\omega\mathrm{j},\omega\in[0,\infty)}{\Delta\arg}\ (1+G_0(s))=(n-l)\frac{\pi}{2} \quad (22)$$

where n is the highest *s*-power in the closed-loop characteristic quasipolynomial $m(s)$ as in (20). ■

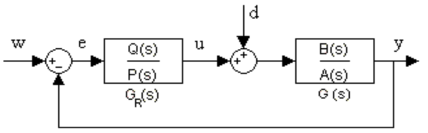

Fig. 1 Simple feedback control loop

Proof. Since the plant transfer function is strictly proper, the highest *s*-power n of $m(s)=p(s)a(s)+q(s)b(s)$ equals that of $p(s)a(s)$. If

$$\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg}\ m(s)=n\pi/2 \quad (23)$$

then the closed-loop system is asymptotically stable according to Theorem 1 (i.e. it characteristic quasipolynomial has all roots in the open left-half *s*-plane), and, simultaneously,

$$\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg}\ m(s)/(a(s)p(s))=n\pi/2-l\pi/2 \quad (24)$$

From (17) and (18) is, moreover, obvious that

$$\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg}\ m(s)/(a(s)p(s))=\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg}\ (1+G_0(s)) \quad (25)$$

and the proof is finished. □

Thus, to test the closed-loop asymptotic stability, one can figure the Nyquist plot of $G_0(s)$ and count its overall number of encirclements around the critical point -1, or more precisely, the overall phase shift of the curve.

Now, the natural question is, whether the notorious precept about the number of unstable poles of $G_0(s)$ (as for delay-free systems) can be used. The answer is the following modification of Theorem 4.

Theorem 5 (The Nyqusit criterion for retarded quasipolynomials – an alternative formulation). Let the plant and the controller have transfer functions as in (18) and the control system be in a simple form as in Fig. 1. Let $a(s)$ and $p(s)$ have no root on the imaginary axis, i.e. $a(s)\neq 0, p(s)\neq 0$ for any imaginary $s=\mathrm{j}\omega$, $\omega\in\mathbb{R}$.

Then, the closed-loop system is asymptotically stable if

$$\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg}\ (1+G_0(s))=n_U\pi \quad (26)$$

where n_U is the number of poles of $G_0(s)$ with positive real parts (i.e. unstable poles). ■

Proof. Assume results from Theorem 4 and Proposition 1. If there in no pure complex conjugate pair of poles of $G_0(s)$ (i.e. roots of $a(s)p(s)$), all unstable poles have positive real parts. The number of unstable poles is given by (3). If notations (21) and (26) are taken into account, one can write

$$n_U = \frac{n}{2} - l\frac{1}{\pi} \Rightarrow l = (n - 2n_U)\frac{\pi}{2} \tag{27}$$

Substitution (27) into (22) yields (26), finally. □

4 Stability Measures

Using the Nyquist plot, a factor of stability can be measured by the *gain margin* and the *phase margin* which characterize the "distance" of the plot from the critical point. As alternate measure, one can take the *minimum of the sensitivity function* into account. All these measures can serve not only for the stability measurement but also for the controller tuning.

The classic conception of some of the measures is deficient in many cases. For example, the gain margin means the amplification of $G_0(s)$ so that the closed loop becomes unstable (or equivalently, remains stable). Traditionally, it is supposed that the Nyquist plot crosses the negative real axis in the coordinate grater than -1; in other words, the gain margin is greater than one. However, this is not true in general. Similarly, the phase margin expresses the phase shift which makes the closed loop being on the stability border; however, there can be more than one these frequencies. Thus, we suggest these definitions of the gain margin and the phase margin:

Definition 3. Consider that the closed loop system is stable. The *lower gain margin* $A_{m,\min}$ is defined as

$$A_{m,\min} := \max_{\omega_{CP,\min} \in \Omega_{CP,\min}} \left(-1/\operatorname{Re}\{G_0(\mathrm{j}\omega_{CP,\min})\}\right) \tag{28}$$

The *upper gain margin* $A_{m,\max}$ reads

$$A_{m,\max} := \min_{\omega_{CP,\max} \in \Omega_{CP,\max}} \left(-1/\{\operatorname{Re} G_0(\mathrm{j}\omega_{CP,\max})\}\right) \tag{29}$$

where $\omega_{CP,\min}$ and $\omega_{CP,\max}$ are the appropriate *phase crossover frequencies* (through the imaginary axis) from sets

$$\begin{aligned}\Omega_{CP,\min} &:= \{\omega : \operatorname{Im}\{G_0\} = 0,\ \operatorname{Re}\{G_0(\mathrm{j}\omega)\} \in (-\infty,-1) \cup (0,\infty)\} \\ \Omega_{CP,\max} &:= \{\omega : \operatorname{Im}\{G_0\} = 0, \operatorname{Re}\{G_0(\mathrm{j}\omega)\} \in (-1,0)\}\end{aligned} \tag{30}$$

■

Definition 4. Consider that the closed loop system is stable. The *phase margin* φ_m is defined as

$$\begin{aligned}\varphi_m &:= \min_{\omega_{CA} \in \Omega_{CA}} \left(\left|\pi + \arg G_0(\mathrm{j}\omega_{CA}) \bmod (2\pi)\right|\right) \\ \Omega_{CA} &:= \{\omega : |G_0(\mathrm{j}\omega)| = 1\}\end{aligned} \tag{31}$$

where ω_{CA} is the appropriate *gain (amplitude) crossing frequency*. ■

Corollary 1. The closed loop system remains stable iff the open loop gain varies within the interval

$$(A_{m,\min}, A_{m,\max}) \tag{32}$$

and the absolute value of the open loop phase shift change is lower than φ_m. ■

Some authors [14], [15] suggest to use a single parameter $M_{\max}$, instead of both $A_{m,\cdot}$, φ_m, which is the maximum (peak) of the sensitivity function.

Definition 5. The *maximum (peak) of the sensitivity function* is defined as

$$\begin{aligned}M_{\max} &:= \sup_{\omega} |G_{WE}(\mathrm{j}\omega)| \\ &= \|G_{WE}(s)\|_\infty = \|1 - G_{WY}(s)\|_\infty\end{aligned} \tag{33}$$

■

The value of $M_{\max}$ corresponds directly with a dumping ratio of a closed-loop system; the lower $M_{\max}$ is, the more dumped the system is. Regarding the Nyquist plot the inverse value of $M_{\max}$ means the distance of a point on the Nyquist plot from the critical point -1, i.e.

$$M_{\max} = 1/\inf_{\omega} |G_0(\mathrm{j}\omega) + 1| \tag{34}$$

5 Demonstration Example

Example 1. Let the LTI-TDS plant be described by the transfer function

$$G(s) = \frac{b(s)}{a(s)} = \frac{\exp(-1.1s)}{s - 5\exp(-s)} \tag{35}$$

The controlled system is unstable which is clear from the Mikhaylov plot $a(\mathrm{j}\omega)$ displayed in Fig. 2 (a detailed zoom to the origin of the complex plane is added) since the overall phase shift (the argument change) is $-5\pi/2$, i.e. $l = -5$. In other words, the plant has three unstable poles because of Theorem 1. Consider a proportional controller $q = q_0$ and the task is to find the appropriate range of q_0 so that the closed loop is asymptotically stable.

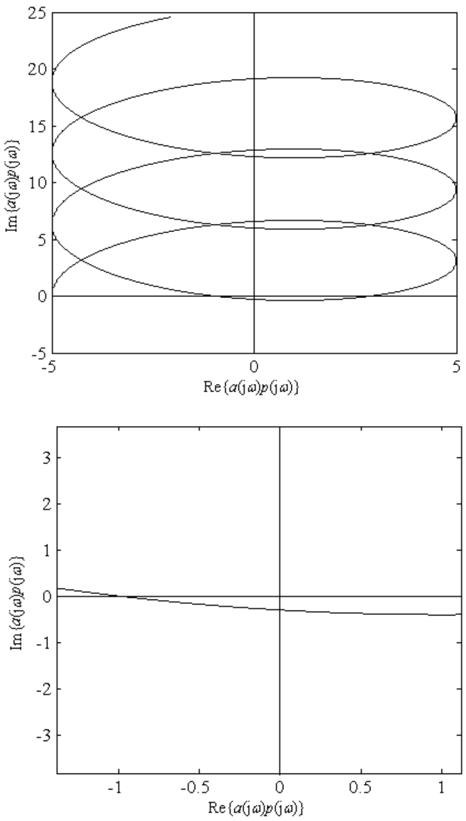

Fig. 2 Mikhaylov plot of $a(s)$ from Example 1 (a) and a detail of the vicinity of the origin (b).

Let us use the Mikhaylov criterion first. Hence, the closed-loop characteristic quasipolynomial reads

$$m(s) = s - 5\exp(-s) + q_0 \exp(-1.1s) \tag{36}$$

According to Remark 1, one can calculate the set of crossover frequencies as $\Omega_1 = \{0.953, 4.741, 6.702, 10.385, 12.498, ...\}$ and easily verify that the critical frequency satisfying definition (12) is $\omega_C = 0.953$ which gives rise to the critical gain $q_C = 5.803$. Since $\sin(0.953) > 0$, Theorem 3 results in the stabilizing interval

$$5 < q_0 < 5.803 \tag{37}$$

Because $n = 1$, the closed loop is stable according to the Nyquist criterion (see Theorem 4 and Theorem 5) if

$$\underset{s=\omega \mathrm{j}, \omega \in [0,\infty)}{\Delta \arg} \left(1 + G_0(s)\right) = 3\pi \tag{38}$$

Set e.g. $q_0 = 5.4$ and display the Nyquist plot of the open loop, see Fig. 3. Indeed, the overall phase shift around the point -1 is 3π.

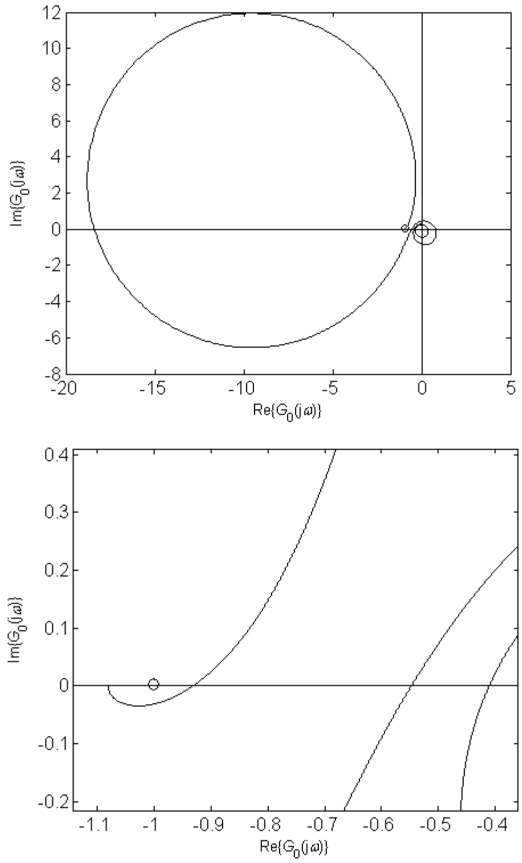

Fig. 3 Nyquist plot of $G_0(s)$ from Example 1 (a) and a detail of the vicinity of the critical point -1 (b), for $q_0 = 5.4$.

In order to measure the stability, find phase crossing frequencies defined in (30). Amazingly, it holds that

$$\Omega_1 = \Omega_{CP,\min} \cup \Omega_{CP,\max} \tag{39}$$

which also means that the phase crossing frequencies are independent of q_0. $\mathrm{Re}\{G_0(\mathrm{j}\omega_{CP,\min})\}$ are $\{-1.08, -18.362, 0.781, 0.411, ...\}$, whereas the set of $\mathrm{Re}\{G_0(\mathrm{j}\omega_{CP,\max})\}$ is $\{-0.931, -0.548, -0.408...\}$, which gives rise to $A_{m,\min} = 0.926$, $A_{m,\max} = 1.075$. This result agrees with boundaries (37).

Gain crossing frequencies can be calculated as $\Omega_{CA} = \{0.634, 3.354, 5.756\}$ yielding $\varphi_m = \min\{|-0.031|, |0.988|, |-0.595|\} = 0.031$ rad.

Obviously, the measures indicate a very low closed-loop stability safeness since the recommended values are approximately $A_m \approx 3, \varphi_m \approx \pi/3$.

4 Conclusion

The paper was aimed to present the Nyquist criterion modified to be valid also for LTI-TDS of the retarded type. It was i.a. verified that the obligatory statement about the number of open-loop unstable poles holds for these systems as well. Stability measures related to the Nyquist criterion were put more precisely for retarded LTI-TDS. This contribution extended authors´ previous results about the Mikhaylov criterion for first order LTI-TDSs, which is demonstrated on a simulation example.

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