Polynomial Approach to Robust Control of Unstable Processes with Application to a Magnetic System

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Abstract: - This paper exploits a relatively simple framework for robust control of unstable single input – single output processes. A linear model-based polynomial approach to control system design is utilized together with robust tuning of some of the closed-loop poles. The methodology is illustrated on the task of magnetic system stabilization with presence of both output and input disturbances.

Key-Words: - Unstable systems, Polynomial methods, Robust control, Single-input/single-output systems, Sensitivity functions, Spectral factorization, Magnetic levitation system.

1 Introduction

A large number of technological processes, such as various types of reactors, combustion systems, crystallizers, distillation columns, etc. possess instable behaviour [2], [12]. All these system need proper control since controlling unstable systems can be a real hazard [16]. In such cases the control designer has to understand fundamental limitations that stem from the process instability [11], [14].

There are many sources devoted to the area of unstable systems control, often covering also the case of delayed and non-minimum-phase systems, e.g. [13], [10], [9], [4], [3]. In this work, the control system design is based on the algebraic approach using polynomials, e.g. [8], [7], [1]. The advantage of this approach is in its systematic and a relatively simple way of designing controllers – it provides both controller structure as well as its parameters and it allows imposing further control requirements simply. A suitable controller is then found as a solution of Diophantine equations.

This paper is structured as follows: control system structure and requirements are stated first, followed by general solution using the polynomial approach. Further the system of magnetic levitation is introduced and described in detail [5]. Next section is focused on the controller design and fine-tuning of its parameters in order to provide robust - safe control. This is done by optimization of some of the closed-loop poles with the help of sensitivity functions and spectral factorization technique [6]. Control results are presented and discussed at the end of this article.

2 Methodology

In this work the classical control set-up of Fig. 1 is considered where G denotes a plant to be controlled by a controller C and the signals w, e, u, y stand for the reference, control error, control input and controlled variable respectively. Signals v_u and v_y represent general disturbances.

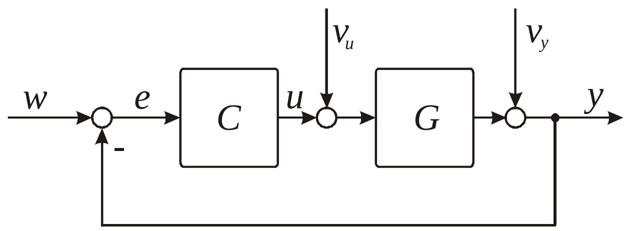

Fig. 1 Control system configuration

Let us assume that the process can be described by a linear time-invariant continuous-time model given by a transfer function

$$G(s)=\frac{b(s)}{a(s)} \tag{1}$$

where $b(s)$, $a(s)$ are coprime polynomials in the complex Laplace variable "s" satisfying the condition:

$$\deg a(s) \succ \deg b(s). \tag{2}$$

Further, the controller C can be also described by a transfer function (3) with $q(s)$, $p(s)$ coprime polynomials satisfying (4).

$$C(s)=\frac{q(s)}{p(s)} \tag{3}$$

$$\deg p(s)\geq \deg q(s) \tag{4}$$

Requirements for the control system are formulated as stability, asymptotic tracking of the reference signal, disturbances attenuation and inner properness. Besides these the system should be robust in order to cope with the real plant (not only with the adopted linear model) and possible disturbances. This is especially important in this case when dealing with unstable systems.

From the scheme of Fig. 1 and assuming (1), (3) it is easy to derive following relationships between the controlled variable y ($Y(s)$ in the complex domain) and input signals w, v_u and v_y ($W(s)$, $V_u(s)$ and $V_y(s)$ similarly); the argument "s" is in these formulas omitted somewhere to keep them more compact and readable):

$$\begin{aligned}
&Y(s)=\frac{G\cdot C}{1+G\cdot C}\cdot W(s)+\frac{G}{1+G\cdot C}\cdot V_u(s)+\\
&+\frac{1}{1+G\cdot C}\cdot V_y(s),\\
&Y(s)=\frac{b\cdot q}{a\cdot p+b\cdot q}\cdot W(s)+\frac{b\cdot p}{a\cdot p+b\cdot q}\cdot V_u(s)+\\
&+\frac{a\cdot p}{a\cdot p+b\cdot q}\cdot V_y(s),\\
&Y(s)=\frac{b\cdot q}{d}\cdot W(s)+\frac{b\cdot p}{d}\cdot V_u(s)+\frac{a\cdot p}{d}\cdot V_y(s),\\
&Y(s)=T\cdot W(s)+S_u\cdot V_u(s)+S\cdot V_y(s).
\end{aligned} \tag{5}$$

Here, the symbol d defines a characteristic polynomial of the closed-loop given as:

$$a\cdot p+b\cdot q=d\,. \tag{6}$$

Symbols S, T, S_u denote important transfer functions of the loop known as the sensitivity function, complementary sensitivity function, and input sensitivity function respectively. The sensitivity functions S and S_u are further used to make the designed control system robust.

Similarly, it is straightforward to derive formula (7) for the control error.

$$E(s)=\frac{p}{d}\left[a\cdot W(s)-b\cdot V_u(s)-a\cdot V_y(s)\right]. \tag{7}$$

2.1 Control System Stability

From (5) it is clear that the control system of Fig. 1. will be stable if the characteristic polynomial $d(s)$ given by (6) is stable. This Diophantine equation, after a proper choice of the stable polynomial $d(s)$, is used to compute unknown controller polynomials $q(s)$, $p(s)$. Sometimes it is useful to require also so called *strong stability* which guarantees also stability of the designed controller, i.e. stability of the polynomial $p(s)$ in (3). As control of unstable systems is generally more dangerous and the suggested design methodology relies on the approximate linear model of the originally nonlinear plant only, the strong stability condition is also considered in this work for safety reasons.

2.2 Asymptotic Tracking of the Reference Signal and Disturbances Attenuation

Let us assume that the reference signal $w(t)$ is a step function, defined in the complex domain as:

$$W(s)=\frac{w_0}{s}, \tag{8}$$

and, further suppose that disturbances $v_u(t)$, $v_y(t)$ can be also approximated by step-functions:

$$V_u(s)=\frac{v_{u0}}{s},\; V_y(s)=\frac{v_{y0}}{s}. \tag{9}$$

Then substituting (8)-(9) into (7) yields:

$$E(s)=\frac{p}{d}\left(a\cdot\frac{w_0}{s}-b\cdot\frac{v_{u0}}{s}-a\cdot\frac{v_{y0}}{s}\right), \tag{10}$$

which shows that in order to guarantee zero-control error in the steady-state, the denominator polynomial of the controller $p(s)$ needs to be divisible by the "s"-term. This will be fulfilled for this polynomial in the form:

$$p(s)=s\cdot\tilde{p}(s). \tag{11}$$

Then the controller (3) can be written as (12)

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$$C(s) = \frac{q(s)}{s \cdot \tilde{p}(s)}, \tag{12}$$

and the Diophantine equation (6) defining stability will be:

$$a \cdot s \cdot \tilde{p} + b \cdot q = d \,. \tag{13}$$

2.3 Control System Inner Properness

Inner properness of the control system is satisfied if all its transfer functions are proper. With regard to the conditions (2) and (4) and taking into account solvability of (6) it is possible to derive following formulae for degrees of the unknown polynomials:

$$\deg q(s) = \deg a(s)\,, \deg \tilde{p}(s) \geq \deg a(s) - 1, \tag{14}$$
$$\deg d(s) = 2 \cdot \deg a(s)\,.$$

2.4 Robust Setting of the Designed Loop

In order to cope with external disturbances and with the fact that only an approximate model of a generally nonlinear unstable plant is used for the control system design, the closed loop is designed to be robust. This is done with the help of the sensitivity functions S and S_u from (5). The sensitivity function S is defined as

$$S(s) = \frac{Y(s)}{V_y(s)} = \frac{1}{1 + G(s) \cdot C(s)} = \frac{a(s) \cdot p(s)}{d(s)} \tag{15}$$

and it describes the impact of output disturbance v_y on the process output y; moreover, it gives the relative sensitivity of the closed-loop transfer function $T(s)$ to the relative plant model error. The peak gain of its frequency response given by the infinity norm H_∞ is a good measure of the loop robustness, e.g. [15].

The input sensitivity function $S_u(s)$ describes the impact of the input (load) disturbance on the process output and it is given as:

$$S_u(s) = \frac{Y(s)}{V_u(s)} = \frac{G(s)}{1 + G(s) \cdot C(s)} = \frac{b(s) \cdot p(s)}{d(s)}\,. \tag{16}$$

In this work it is suggested to use both sensitivity functions and their H_∞ norms to tune some of the closed-loop poles in order to make the designed control system more robust, i.e. safer. The procedure is shown further on the presented example of the magnetic system stabilization.

3 Magnetic Levitation System

The magnetic system CE 152 presented in Fig. 2 below is a laboratory-scale model designed for studying system dynamics and experimenting with control algorithms. It demonstrates control problems associated with nonlinear unstable systems mainly.

Fig. 2 The CE 152 magnetic levitation apparatus

The system consists of a coil levitating a steel ball in the magnetic field. A basic control task is to control the ball position. Description of the system can be found in [5] and the references cited therein.

A mathematical model of this system can be derived in the following form [5]:

$$\frac{m_k}{k_{AD}k_x}\ddot{y} - \frac{k_{fv}}{k_{AD}k_x}\dot{y} = \frac{k_{DA}^2 k_i^2 u^2 k_c}{\left(\frac{y - k_{AD}y_0}{k_{AD}k_x} - x_0\right)^2} - m_k g\,, \tag{17}$$

where y denotes the controlled variable - ball position and u is the control input proportional to the voltage from the D/A converter. Other symbols are clearly defined in Table 1.

Table 1 Parameters of the magnetic system

Symbol	Description	Value and unit
k_{AD}	A/D converter gain	0.2 MU [a]/V
k_{DA}	D/A converter gain	20 V/MU [a]
k_{fv}	Damping constant	$0.02\ \mathrm{N \cdot s/m}$
k_x	Position sensor gain	821 V/m
k_i	Power amplifier gain	0.3 A/V
k_c	Coil constant	$1.769\text{x}10^{-6}\mathrm{N \cdot m^2/A^2}$
m_k	Ball mass	8.27×10^{-3} kg
x_0	Coil offset	7.6×10^{-3} m
g	Gravity constant	$9.81\ \mathrm{m/s^2}$
y_0	Position sensor offset	0.0183 V
$y(t)$	Ball position	MU [a]
$u(t)$	Input signal	MU [a]

[a] Voltage converted to 0-1 machine unit (MU).

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For the purpose of controller design the nonlinear model (17) can be linearized to the form of a second-order proportional system (18) [5]:

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_0}{s^2 + a_1 \cdot s + a_0}. \quad (18)$$

Then, for the ball levitating in the middle of the space the transfer functions is [5]:

$$G(s) = \frac{18400}{s^2 - 2.418 \cdot s - 3998}. \quad (19)$$

It is easy to check that the system has one stable and one unstable pole located on the real axis, nearly symmetrically with respect to the origin.

4 Control System Design

The task is to design a control system for the described magnetic system. It is based on the nominal linear model (19) but it must fulfil the given requirements stated in the section 2 of this paper also for different operating points.

Assuming the transfer function of the controlled system (18)-(19), the degrees of the unknown polynomials will according to (14) be: $\deg q(s) = 2$, $\deg \tilde{p}(s) \geq 1$ and $\deg d(s) = 4$. Therefore the simplest controller structure according to (12) will be:

$$C(s) = \frac{q(s)}{s \cdot \tilde{p}(s)} = \frac{q_2 \cdot s^2 + q_1 \cdot s + q_0}{s \cdot (\tilde{p}_1 \cdot s + \tilde{p}_0)}. \quad (20)$$

Its coefficients are obtained by a solution of the Diophantine equation (13) for some stable characteristic polynomial $d(s)$. Therefore, the next task is to choose this polynomial. Here it is suggested to have it in this form:

$$d(s) = n(s)(\mathrm{s}+\alpha)^2, \quad (21)$$

where $\alpha \succ 0$ is a free tuning constant and $n(s)$ is a stable polynomial computed from the denominator polynomial of the controlled system $a(s)$ using the spectral factorization technique [6]:

$$a^*(s)a(s) = n^*(s)n(s). \quad (22)$$

This choice of the characteristic polynomial will not only guarantee stability of the resultant control system but also connection to the original process behaviour and it will leave space enough for further possible tuning. Solving (22) yields:

$$n(s) = s^2 + n_1 \cdot s + n_0 = s^2 + 126.483 \cdot s + 3998 . (23)$$

It is easy to check that whereas the original polynomial $a(s)$ has poles $p_1 = 64.5$ and $p_2 = -62.0$, i.e. the first one is unstable, the result of the factorization $n(s)$ (23) provides both stable poles $p_1 = -64.5$ and $p_2 = -62.0$. Now the characteristic polynomial (21) can be rewritten as:

$$d(s) = (s^2 + 126.483 \cdot s + 3998) \cdot (s + \alpha)^2, \quad (24)$$

where the only free parameter $\alpha \succ 0$ can be used for further tuning. In this work this is done using the sensitivity functions of the loop $S(s)$ and $S_u(s)$ in order to make the designed control system robust, as outlined in the section 2.4 of this paper.

Dependence of the H_∞-norms of both sensitivity functions on the parameter α is presented in Fig. 3.

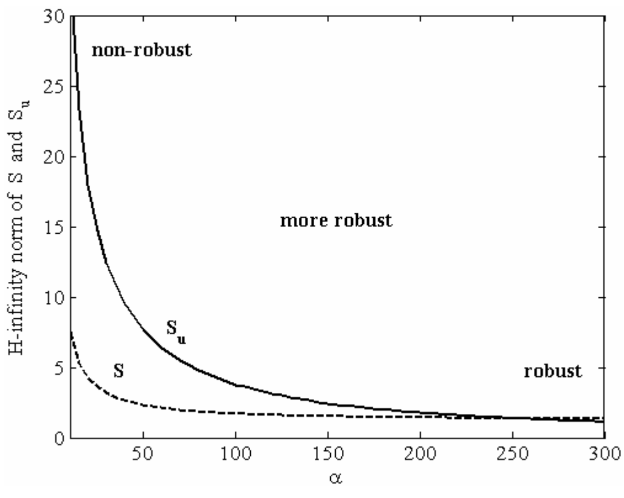

Fig. 3 H_∞-norm of sensitivity functions S, S_u with α

From the plot it is obvious that the smaller value of the constant α the more sensitive the closed-loop system is, and vice versa – the higher value of α the more robust control system (regarding the influence of both disturbances and possible changes in the process model). Based on this information the free tuning parameter α was chosen as $\alpha = 200$. This choice will provide robust control system and approximately the same sensitivity for both

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disturbances. Besides this it can be seen as a trade-off between the desired robustness of the loop and limitations on the control input (higher values of α result in more aggressive control action and consequently more overshoots and oscillations).

Then, the designed controller has this form:

$$C(s)=\frac{q(s)}{s\cdot\tilde{p}(s)}=\frac{q_2\cdot s^2+q_1\cdot s+q_0}{s\cdot\left(\tilde{p}_1\cdot s+\tilde{p}_0\right)}= \\ =\frac{3.25\cdot s^2+479\cdot s+8691}{s\cdot(s+528.9)}, \qquad (25)$$

with the coefficients computed from (13),(24) as:

$$\tilde{p}_1=1;\quad \tilde{p}_0=2\cdot\alpha+n_1-a_1;\quad q_0=\alpha^2\cdot n_0/b_0; \\ q_1=\left[\alpha^2\cdot(1+n_1)+2\cdot\alpha\cdot n_0-a_0\cdot\tilde{p}_0\right]/b_0; \qquad (26) \\ q_2=\left(2\cdot\alpha\cdot n_1+n_0-a_1\cdot\tilde{p}_0-a_0\right)/b_0.$$

It is easy to check that the strong stability condition (besides stability of the control system also stability of the controller is required – see section 2.1) will be fulfilled as the coefficient $\tilde{p}_0$ is always positive for $\alpha \succ 0$.

5 Experiments

Several experiments were performed on the magnetic system in order to test the designed control loop. First, control in different operating points were analysed for two settings of the tuning parameter α - robust one ($\alpha=200$) and, non-robust ($\alpha=50$). Some of the control responses are presented below in Fig. 4 and Fig. 5.

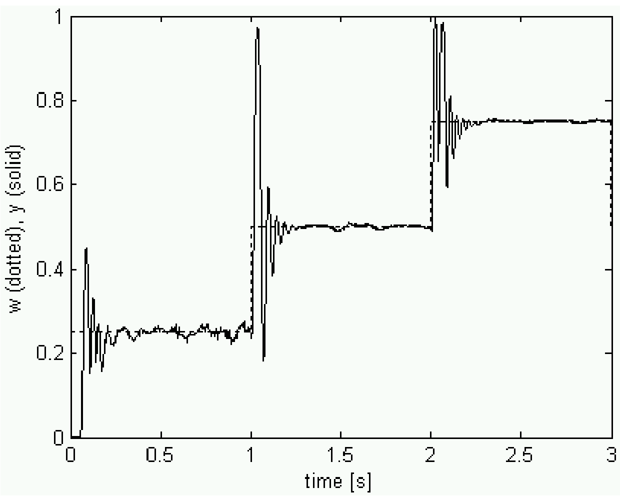

Fig. 4 Control response in different operating points: robust setting ($\alpha=200$)

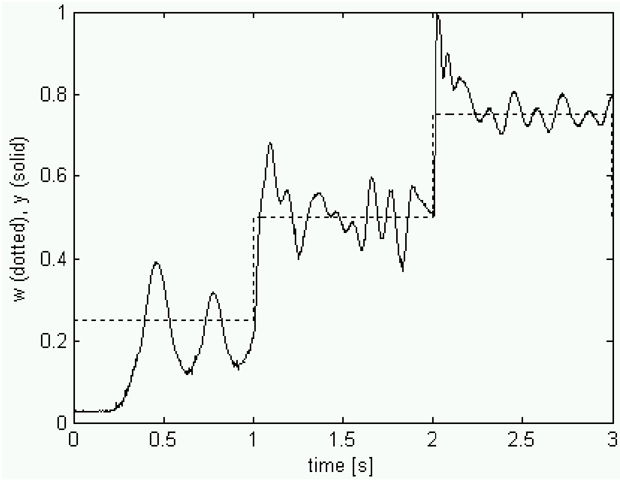

Fig. 5 Control response in different operating points: non-robust setting ($\alpha=50$)

From the graphs it is obvious that the suggested robust setting for ($\alpha=200$) provides more stable response and better tracking of the reference signal. It gives relatively big overshoots but this can be improved by e.g. different control configuration, as shown in [5].

Further attention was focused on the disturbance attenuation. During the control both disturbances (affecting control input at the time 1 sec. and controlled output at the time 2 sec.) were injected into the loop and the response was analysed. Both disturbances were step-functions as assumed in section 2.2 of this paper and their amplitude was 10% of the set-point signal. The figures below show some of the achieved responses.

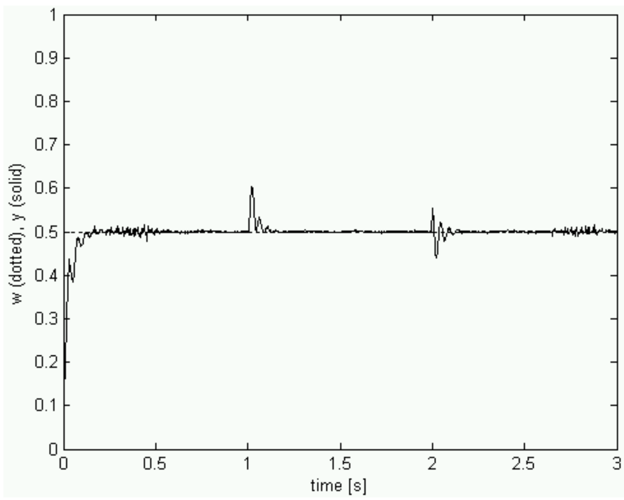

Fig. 6 Disturbance attenuation: robust setting ($\alpha=200$)

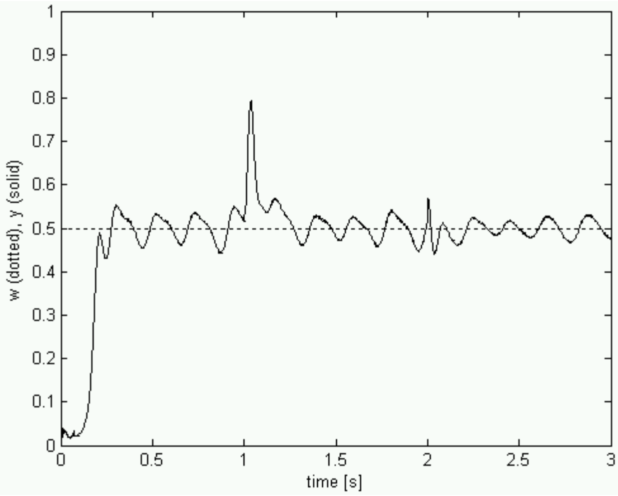

Fig. 7 Disturbance attenuation: non-robust setting ($\alpha = 50$)

As can be clearly seen from the graphs, the robust setting of the tuning parameter α provides better responses to both disturbances.

6 Conclusion

This paper presented a relatively simple framework for control of unstable single input – single output processes. The resultant controller is designed to be robust with respect to both, changes in the operating point (adopted model) and disturbances affecting manipulated or controlled variables. The presented experimental results show applicability of the approach to safer control of unstable processes. Further, it can be easily extended to cover also multi input – multi output processes.

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