Description of strange attractors using invariants of phase-plane

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Abstract: - Describing the attractors of chaotic dynamical systems has been one of the achievements of chaos theory. Attractors are parts of phase space of the dynamical system. There can be many geometrical sets that are attractors. When these sets are hard to describe, then the attractor is a strange attractor. Like snowflakes, these strange attractors come in infinite variety with no two the same. The aim of this paper is to propose a slightly different way to "see" a strange attractor. Three mean quantities are defined and the chaotic motion of the Ueda oscillator, the simplest quadratic oscillator and the Rucklidge oscillator is analyzed in order to draw the so-called invariants of phase-plane, who may be regarded as "marks" of the strange attractors.

Key-Words: - Chaos, invariant of phase-plane, strange attractor

1 Introduction

Chaos is an aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions. Thus, chaotic motion of a dynamical system implies that two trajectories that are initially arbitrarily close in plase space will diverge exponentially in time from each other. An attractor is a set of states, or points in phase-space, invariant under dynamical progression of the system, towards which neighboring trajectories of states approach asymptotically during the evolution. In the case of chaotic systems, attractors have fractal dimension and don't create a continuous or finite-dimensional curve or neither. These types of attractors have been defined as "strange".

Appearance of chaos has been identified through various tools in the past. The well-known are: the phase space method, the time series method, bifurcation diagram, the Poincare section of surface, frequency-map analysis, Lyapunov-characteristic exponents, and most recently the Fast Lyapunov indicator, the 0-1 test, the Dynamic Lyapunov exponent and the Smaller alignment index.

In this paper we propose a slightly different approach to chaotic motion of a dynamical system. Although the trajectories corresponding to two nearby initial conditions separate exponentially in time, there are a set of mean quantities, which are less sensitive to initial conditions, and can be used to "recognize" a particular set of strange attractors. These values are called *invariants of phase-plane* and will be defined in the next section.

2 Geometric Invariants. Definitions

We consider a chaotic dynamical system governed by the system of differential equations

$$\dot{x} = f(t, x, \mu) \qquad (1)$$

where t is the time variable, x a set of unknown functions (coordinates) and μ a set of control parameters. A common way to understand the solution of eq. (1) is to study the projection of a trajectory *x(t)* on phase-plane *u-v*, which corresponds for two of the coordinates of the set x

If we'll study the motion for a long period of time, the trajectory crosses the line $u = u_C$, with u_C a constant, in points $(u_C, v_1), (u_C, v_2), \ldots, (u_C, v_N)$. We define the mean value

$$v_m = \frac{1}{N}\sum_{n=1}^{N} v_n \qquad (2)$$

For a fixed set μ, it is found that v_m is approximately constant with initial conditions. Let suppose now that exists M points for which $v_n \geq v_m$, namely $(u_C, v_i), i = \overline{1, M}$, and K points for which $v_n < v_m$, namely $(u_C, v_j), j = \overline{1, K}$. In this way it was realized a division of the N points in two groups. For convenience of writing, there was a reordering of the N points in the two sub-groups.

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We define also the values

$$v_u = \frac{1}{M}\sum_{i=1}^{M} v_i \quad , \quad v_d = \frac{1}{K}\sum_{j=1}^{K} v_j \qquad (3)$$

Like v_m, both v_u and v_d are independent of initial conditions for a given set μ. Changing u_C with another value we'll obtain other mean values v_m, v_u and v_d. Continuously modifying the value u_C, on find that points (u_C, v_m), (u_C, v_u) and (u_C, v_d) are registered, each one, on a curve. The three curves are called *the invariants of phase-plane.*

3 Numerical Results

We have applied above defined invariants for the models given below:

3.1 Ueda oscillator

Chaotic oscillations of periodically driven nonlinear oscillator were studied in some details around 1980 by the Japanese researcher Ueda. Systems of such kind can be realized as mechanical and electronic devices

The equation of motion can be transformed into the system of differential equations

$$\frac{dx}{dt} = y, \frac{dy}{dt} = -x^3 - by + a\sin\omega t \qquad (4)$$

with a, b and ω real constants. It has shown that for $a = 7.5, b = 0.05$, $\omega = 1$ and initial conditions $(x_0, y_0) = (2.5, 0.0)$ the trajectory corresponds to a chaotic motion (see Fig. 1).

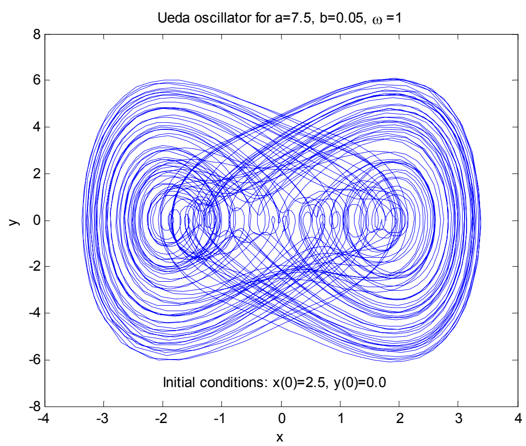

Fig.1: Ueda attractor ($a = 7.5, b = 0.05, \quad \omega = 1$)

Indeed, the Lyapunov exponents are $\lambda_1 = 0.1034, \lambda_2 = -0.1534$ (the former indicates a chaotic trajectory). If we choose the fixed value $x = 0$ and change the initial conditions so that $(x_0, y_0) \in [2.50, 2.55] \times [0.00, 0.05]$, on obtain the situation shown in Fig. 2. It is obvious that the three values y_u, y_m, y_d are sufficiently independent of initial conditions, although the trajectories corresponding to two close initial conditions are very different after a time interval not very large (see Fig. 3).

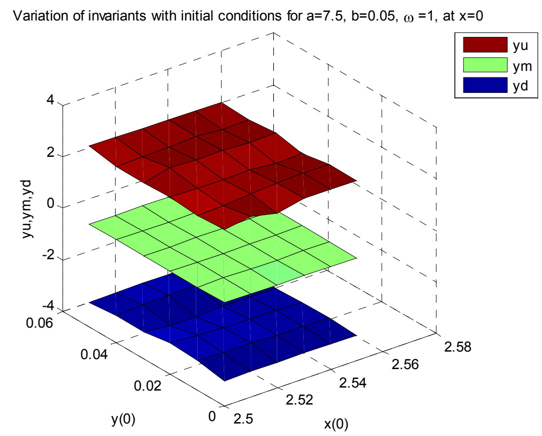

Fig.2: Dependence of y_u, y_m, y_d on initial conditions

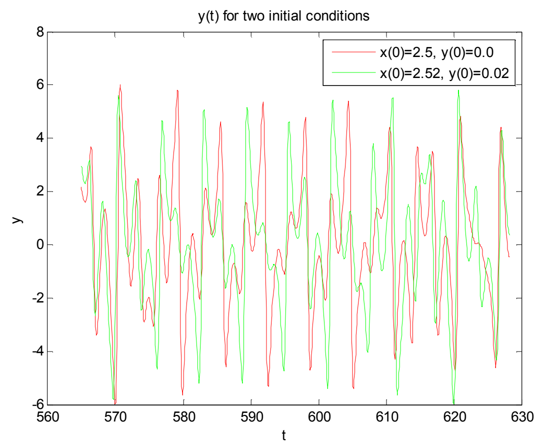

Fig.3: Time evolution of two orbits with close initial conditions

Changing x between its extreme values, on obtain the invariants shown in Fig. 4.

3.2 Simplest quadratic chaotic oscillator

Simplest quadratic oscillator is defined by the following set of differential equations

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$$\frac{dx}{dt} = y, \frac{dy}{dt} = z, \frac{dz}{dt} = -az + y^2 - x \qquad (5)$$

For $a = 2.017$ and initial conditions $(x_0, y_0, z_0) = (-0.9, 0.0, 0.5)$ the trajectory is chaotic. This can be observed through the phase plots given in Figs 5 and 6.

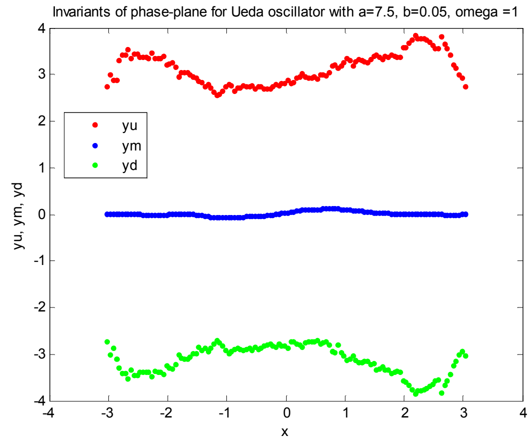

Fig. 4: Invariants of phase-plane for Ueda oscillator

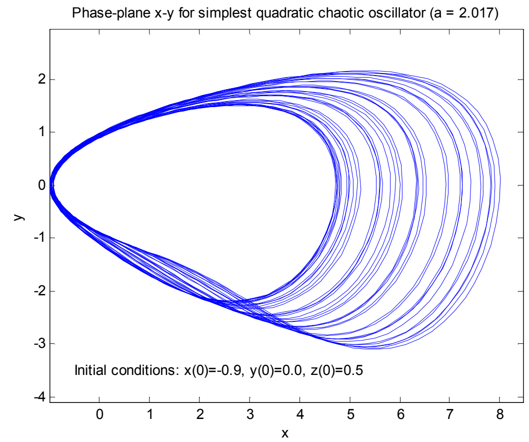

Fig.5: Phase-plane x-y for simplest quadratic chaotic oscillator ($a = 2.017$)

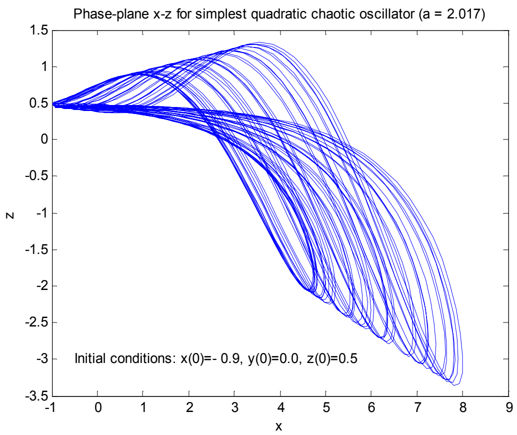

Fig.6: Phase-plane x-z for simplest quadratic chaotic oscillator ($a = 2.017$)

The Lyapunov exponents are $\lambda_1 = 0.0551$ and $\lambda_2 = -2.0721$. For this oscillator, the invariants of *x*-*y*, respectively *x*-*z*, phase planes are computed with the parameter value and initial conditions mentioned above and the results are displayed in Figs. 7 and 8.

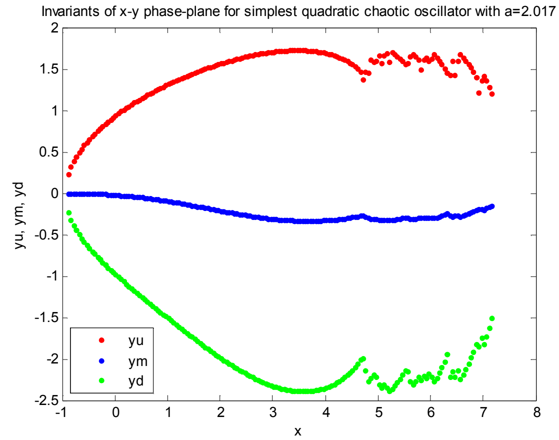

Fig.7: Invariants of phase-plane x-y for simplest quadratic chaotic oscillator ($a = 2.017$)

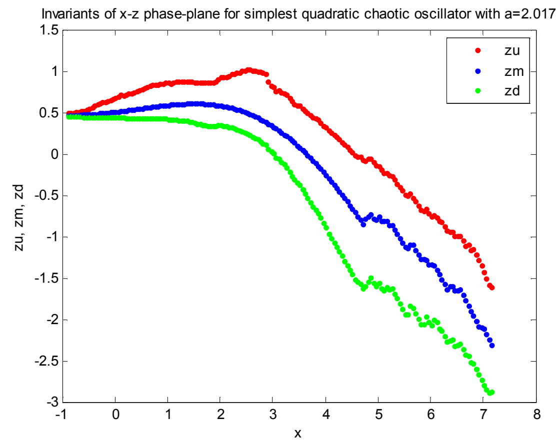

Fig. 8: Invariants of phase-plane x-z for simplest quadratic chaotic oscillator ($a = 2.017$)

3.3 Rucklidge oscillator

For this oscillator, the time evolution of an orbit is governed by the differential equations

$$\frac{dx}{dt} = -ax + by - yz$$

$$\frac{dy}{dt} = x \qquad (6)$$

$$\frac{dz}{dt} = -z + y^2$$

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For parameter values $a = 2.0, b = 6.7$ and initial conditions $(x_0, y_0, z_0) = (0.0, 0.8, 0.0)$ on obtain the chaotic attractor given in Figs. 9 and 10 (phase planes x-y and x-z).

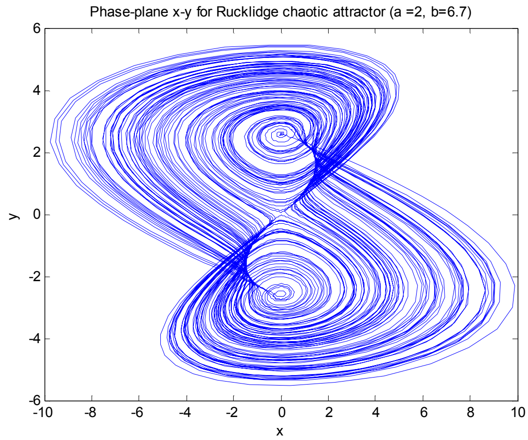

Fig.9: Phase-plane x-y for Rucklidge chaotic oscillator ($a = 2.0, b = 6.7$)

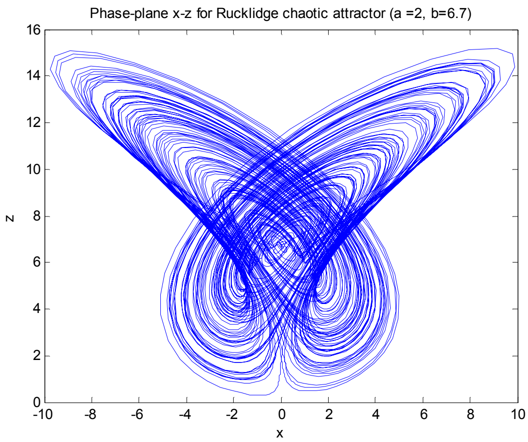

Fig.10: Phase-plane x-z for Rucklidge chaotic oscillator ($a = 2.0, b = 6.7$)

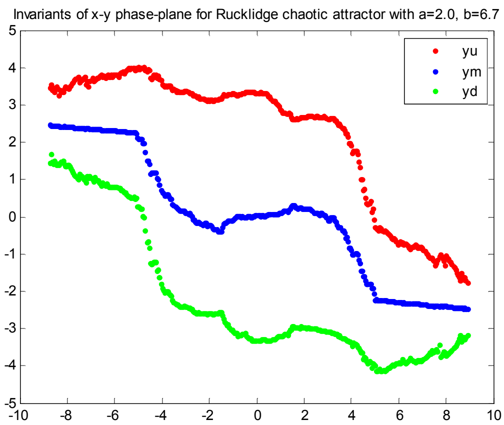

Fig. 11: Invariants of phase-plane x-y for Rucklidge chaotic oscillator ($a = 2.0, b = 6.7$)

The invariants of there phase-planes can be seen in Figs. 11 and 12.

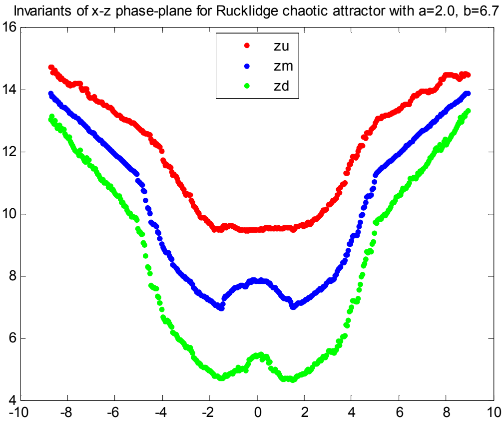

Fig. 12: Invariants of phase-plane x-z for Rucklidge chaotic oscillator ($a = 2.0, b = 6.7$)

4 Conclusion

In the present paper we have drawn the so-called invariants of phase-plane for the Ueda chaotic oscillator, simple quadratic chaotic oscillator and Rucklidge chaotic oscillator. These invariants can be seen like a “mark” of the associated strange attractors.

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