Stabilizing a reaction-diffusion system with nonlocal reaction term

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Abstract: We consider a two-component reaction-diffusion system with a nonlocal reaction term. A necessary condition and a sufficient condition for the internal stabilizability to zero of one of the two components of the solution while preserving the nonnegativity of both components have been established in [6]. In case of stabilizability, a feedback stabilizing control of harvesting type has been indicated. The rate of stabilization (for the indicated feedback control) is given by the principal eigenvalue of a certain non-selfadjoint operator. A large principal eigenvalue leads to a fast stabilization. The first main goal of the present paper is to approximate this principal eigenvalue. This is done in two steps: first we investigate the large-time behavior of the solution to a logistic population dynamics with migration, and next we derive as a consequence a method to approximate the principal eigenvalue. The other main goal is to derive a conceptual iterative algorithm to improve the position of the support of the control in order to get a faster stabilization. Our results apply to prey-predator systems.

Key–Words: Reaction-diffusion system, Prey-predator system, Nonlocal reaction term, Stabilization, Principal eigenvalue, Feedback control, Feedback stabilization, Numerical iterative algorithm

1 Introduction

In the present study we consider a two-component reaction-diffusion system. Our motivating problem is a prey-predator system wherein the two species are living on the spatial domain $\Omega$. Here, prey are native species distributed over the habitat $\Omega$ while predators are alien species feeding upon prey. The biomass of captured and eaten prey at $x$ in $\Omega$ is spatially distributed over the whole range $\Omega$ occupied by the predator species. This produces local / non local interspecific interactions between the two species at the predator level, that is, the functional response to predation is local and supported in $\Omega$ while the numerical response to predation is non local and distributed over $\Omega$, see also [16] and [17]. This is again a departure from most standard predator-prey models, see [13] and [21].

In the present paper we are interested in the large time stabilization of one of the solution components toward 0 while preventing the second component going to 0. This is achieved via internal controls distributed on a small subdomain, $\omega$, of $\Omega$ preserving the nonnegativity of both components. For our prey-predator system this means stabilizing the alien predator species to 0 and preventing prey dying out.

Let us now introduce the mathematical problem we shall work with. When both populations are present predation occurs on $\Omega$; Let $h(\cdot,t)$ be the spatial density at time $t$ of a prey species distributed over a spatial domain $\Omega$ in $\mathbb{R}^N$, $N = 2$ or 3, and assume its spatio-temporal dynamics is governed by a basic logisitic model, see [13] and [21]; let us denote by $f(h,p)$ a suitable functional response to predation, see [13] and [21].

$$\partial_t h - d_1 \Delta h = rh - k_0 h^2 - f(h,p)p, \quad x \in \Omega, \ t > 0.$$  

(1.1)

wherein $r > 0$ is the natural growth rate and $k_0 > 0$ is a density dependent effect on mortality due to intraspecific competition within prey. Let $p(\cdot,t)$ be the spatial density at time $t$ of a predator species distributed over the spatial domain $\Omega$ in $\mathbb{R}^N$; in absence of the prey – assumed to be its main resource – the predator population will decay at an exponential rate $\alpha > 0$. The prey captured and eaten at time $t > 0$ and location $x' \in \Omega$ are transformed into biomass via a conversion factor $\delta > 0$ yielding a numerical response to predation $\delta f(h(x',t),p(x',t))p(x',t)$. We assume that this quantity is distributed over $\Omega$ via a generic nonnegative kernel $\ell(x,x')$ so that $\delta f(h(x',t),p(x',t))p(x',t)$ is the biomass distributed at $x \in \Omega$ resulting from predation at $x' \in \Omega$. Biomass conservation implies a consistency condition must hold: $\int_{\Omega} \ell(x,x')dx = 1$ for each $x' \in \Omega$. 

ISBN: 978-1-61804-011-4
In this setting the predator dynamics reads
\[
\partial_t p - d_2 \Delta p = -ap + \delta \int_\Omega \ell(x, x') f(h, p)(x', t)p(x', t) \, dx',
\]
x \in \Omega, t > 0. \quad (1.2)

In applications we have in mind the functional response to predation may take several standard parametric forms, such as Lotka-Volterra, \( f(h, p) = ph \), Holling type \( \theta + 1 \), \( f(h, p) = \frac{\rho h^\theta}{1 + qh^\theta} \) (\( \theta \in \mathbb{N} \)), or Beddington-De Angelis, \( f(h, p) = \frac{\rho h}{1 + qh + sp} \) (with \( \rho, q, s > 0 \)), see [13], [14] and [21] and cf. assumption (H2) below.

We choose no-flux boundary conditions corresponding to isolated populations:
\[
\partial_n h(x, t) = \partial_n p(x, t) = 0, \quad x \in \partial \Omega, t > 0,
\]
where \( \partial_n g(x) \) denotes the normal derivative of \( g \) at \( x \in \partial \Omega \). Last nonnegative and bounded initial conditions are prescribed at time \( t = 0 \):
\[
h(x, 0) = h_0(x), \quad p(x, 0) = p_0(x), \quad x \in \Omega.
\]
Then (1.1)-(1.4) is the model for our prey-predator system.

A global existence result of componentwise nonnegative strong solutions to (1.1)-(1.4) is derived in [6]. Going back to control problems two strategies are devised and investigated in [6].

In the present paper we refer only to the so-called \( p \)-stabilizability. Our strategy refers to harvesting/culling predators in an open subdomain, \( \omega \subset \Omega \), by introducing a control, \( u \in \omega \). Let \( \chi_\omega \) be the characteristic function of \( \omega \). Equation (1.2) is modified into
\[
\partial_t p - d_2 \Delta p = -ap + \delta \int_\Omega \ell(x, x') f(h, p)(x', t)p(x', t) \, dx' + \chi_\omega(x) u(x, t),
\]
for \( x \in \Omega \) and \( t > 0 \). The question to address reads: “is it possible to find a control \( u \in L^\infty_0(\omega \times [0, \infty)) \) such that the solution \( (h, p) \) of (1.1)-(1.5) remains componentwise nonnegative and satisfies \( p(\cdot, t) \to 0 \) in \( L^\infty(\Omega) \), as \( t \to +\infty \) ?”

Here are the hypotheses we are going to use throughout this paper:

(H1) \( \omega \) and \( \Omega \) are nonempty bounded domains in \( \mathbb{R}^N \), \( N \geq 2 \), with smooth boundaries \( \partial \omega \) and \( \partial \Omega \), respectively, so that locally each \( \omega, \Omega \) lies on one side of \( \partial \omega \) and \( \partial \Omega \), respectively, and \( \Omega \setminus \overline{\omega} \) is a domain;

(H2) All coefficients: \( d_1, r, k_0, d_2, a \) and \( \delta \) are positive constants.
Next \( \ell : \Omega \times \Omega \to [0, +\infty) \) is a measurable and bounded function satisfying
\[
\int_\Omega \ell(x, x') \, dx = 1 \quad \text{a.e. } x' \in \Omega.
\]

Last \( f : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) is locally Lipschitz continuous, \( h \mapsto f(h, p) \) being nondecreasing on \( [0, +\infty) \) for any \( p \geq 0 \), \( p \mapsto f(h, p) \) being nonincreasing on \( [0, +\infty) \) for any \( h \geq 0 \); \( f(0, p) = 0 \), \( \forall p \in [0, +\infty) \) and \( f(h, 0) > 0, \forall h \in (0, +\infty) \);

(H3) \( h_0, p_0 \in L^\infty(\Omega) \) are nonnegative;
\[
\|h_0\|_{L^\infty(\Omega)} > 0, \quad \|p_0\|_{L^\infty(\Omega)} > 0.
\]

Definition 1.1. The predator population is \( p \)-zero stabilizable if for any \( (h_0, p_0) \) satisfying (H3), there exists a control \( u \in L^\infty_0(\overline{\omega} \times [0, +\infty)) \) such that the solution \( (h, p) \) to (1.1)-(1.5)-(1.3)-(1.4) satisfies
\[
h(x, t) \geq 0, \quad p(x, t) \geq 0 \quad \text{a.e. } x \in \Omega, \forall t \geq 0,
\]
and
\[
\lim_{t \to +\infty} p(t) = 0 \quad \text{in } L^\infty(\Omega).
\]

Hence “\( p \)-zero stabilizable” means that the zero stabilizability holds for controls acting only on the predator population.

The study of the \( p \)-zero stabilizability is related to the magnitude of the principal eigenvalue \( \lambda_1^\omega \) for the following problem:
\[
\begin{aligned}
&\partial_t \varphi - d_2 \Delta \varphi + a \varphi(x) + \delta f(K, 0) \int_{\Omega \setminus \overline{\omega}} \ell(x, x') \varphi(x') \, dx' = \lambda \varphi(x), & x \in \Omega \setminus \overline{\omega}, \\
&\partial_\nu \varphi(x) = 0, & x \in \partial \omega, \\
&\partial_\nu \varphi(x) = 0, & x \in \partial \Omega,
\end{aligned}
\]
where \( K = \frac{r}{k_0} \), as well as to the magnitude of the principal eigenvalue \( \lambda_1^\omega \gamma \) to the next problem:
\[
\begin{aligned}
&\partial_t \varphi - d_2 \Delta \varphi + a \varphi(x) - \delta f(K, 0) \int_{\Omega \setminus \overline{\omega}} \ell(x, x') \varphi(x') \, dx' + \chi_\omega(x) \gamma \varphi(x) = \lambda \varphi(x), & x \in \Omega, \\
&\partial_\nu \varphi(x) = 0, & x \in \partial \omega, \\
&\partial_\nu \varphi(x) = 0, & x \in \partial \Omega.
\end{aligned}
\]

The existence and properties of both \( \lambda_1^\omega \) and \( \lambda_1^\omega \gamma \) follow via Krein-Rutman’s theorem ([15]) - see [5], [6].

The following stabilization result has been established in [6]- Theorem 2:
Theorem 1.1. If the predator population is $p$-zero stabilizable then $\lambda^p_1 > 0$.

Conversely, when $\lambda^p_1 > 0$ the predator population is $p$-zero stabilizable and for $\gamma$ large enough the feedback control $u := -\gamma p$ realizes (1.6) and (1.7), where $(h, p)$ is the solution to (1.1)-(1.5)-(1.3)-(1.4), corresponding to $u := -\gamma p$.

Remark 1.2. (see [6]). In case of $p$-zero stabilizability, if we take the feedback control $u := -\gamma p$ (with $\gamma$ a large enough nonnegative constant; $\gamma$ is a harvesting rate), then the solution $(h, p)$ to (1.1)-(1.5)-(1.3)-(1.4) has both components nonnegative and satisfies

$$p(t) \to 0 \text{ in } L^\infty(\Omega),$$

as $t \to +\infty$, at the rate of $e^{-\lambda^p_1 t}$.

Moreover, for the feedback stabilizing control $u := -\gamma p$ we get that

$$h(t) \to K \text{ in } L^\infty(\Omega),$$

as $t \to +\infty$.

So, the importance of approximating $\lambda^p_1$, and of finding a position (translation) of $\omega$ for which $\lambda^p_1$ has a large value is that it gives fast stabilizing rate.

Remark 1.2. It has been proved in the same paper [6] (also by using Krein-Rutman’s theorem) that

(i) the mapping $\gamma \mapsto \lambda^p_1$ is increasing on $[0, +\infty)$, and $\lambda^p_1 \to \lambda^p_2$, as $\gamma \to +\infty$;

(ii) for any stabilizing control $u$, the component $p$ of the solution to (1.1)-(1.5)-(1.3)-(1.4) converges to zero slower than $e^{-\lambda^p_1 t}$.

These show that the use of the feedback control $u := -\gamma p$ (with large $\gamma$) is an excellent strategy.

Here is the plan of the paper. In Section 2 we shall investigate the large-time behavior for a logistic population dynamics with migration. The result established here is important by itself. However, as a consequence we shall derive in the next section that

$$\lim_{T \to +\infty} \int_\Omega y^p(x, T) dx = \zeta - \lambda^\omega_1,$$

where $y^p$ is the solution to

$$\begin{cases}
\partial_t y - d_2 \Delta y + ay \\
- \delta f(K, 0) \int_\Omega \ell(x, x') y(x', t) dx' \\
+ \gamma \omega - \zeta y \\
+ (f_1 y(x, t) dx) y = 0, & x \in \Omega, \ t > 0 \\
\partial_y y(x, t) = 0, & x \in \partial \Omega, \ t > 0 \\
y(x, 0) = 1, & x \in \Omega,
\end{cases}$$

(1.8)

where $\zeta > \lambda^\omega_1$, and this leads to a method to approximate $\lambda^\omega_1$.

For a large $T > 0$,

$$J^\omega = \int_\Omega y^\omega(x, T) dx,$$

gives an approximation of $\zeta - \lambda^\omega_1$. So, instead of finding a position of $\omega$ which provides a large value for $\lambda^\omega_1$, we could treat the approximating problem of finding the position of $\omega$ which provides a small value for $J^\omega$. In Section 4 we shall calculate the derivative with respect to translations of $\omega \mapsto J^\omega$. This will allow to derive a conceptual iterative algorithm to improve at each step the position (by translation) of $\omega$ in order to get a smaller value for $J^\omega$.

For basic properties of other reaction-diffusion models with nonlocal terms we refer the reader to [8], [9], while for stabilization results for reaction-diffusion systems in biology see [1], [2], [4], [5].

For basic results concerning general reaction-diffusion systems we recommend the monograph [23], and for general results concerning the stabilization of PDE we refer to [20].

2 Large-time behavior for a logistic population dynamics with migration

Consider a biological population that is free to move in the habitat $\Omega$, and denote by $y(x, t)$ the population density at position $x$ and moment $t$, and by $d$ the diffusion coefficient.

The population dynamics is described by the following equation on $\Omega \times (0, +\infty)$:

$$\partial_t y - d \Delta y - \tilde{a}(x) y + b(x) y - \int_{\Omega} \tilde{k}(x, x') b(x') y(x', t) dx' + \mathcal{M}(\int_{\Omega} y(x, t) dx) y = 0,$$

(2.1)

where $\tilde{a}(x)$ is the natural increasing rate at position $x$, and $\mathcal{M}(\int_{\Omega} y(x, t) dx)$ is an additional mortality rate due to overpopulation (this rate depends on the total population at moment $t$); $\mathcal{M}(\int_{\Omega} y(x, t) dx) y(x, t)$ is the logistic term;

$b(x) \geq 0$ denotes the rate of population at position $x$ that leaves this position and migrates to other locations;

$k(x, x') \geq 0$ is the rate of the population that leaves position $x'$ and migrates to position $x$. It is obvious that $\tilde{k}$ satisfies

$$\int_{\Omega} \tilde{k}(x, x') dx = 1 \quad \text{a.e. } x' \in \Omega$$
(due to the fact that the migration does not modify the total population).

If we denote by \( a(x) = -\bar{a}(x) + b(x) \), and by
\( k(x, x') = \tilde{k}(x, x') \bar{b}(x') \), then (2.1) may be rewritten as
\[
\partial_t y - d\Delta y + a(x)y - \int_{\Omega} k(x, x') y(x', t) dx' = 0, \quad x \in \Omega, \ t > 0
\]
\[
\partial_y y(x, t) + \alpha y(x, t) = 0, \quad x \in \partial \Omega, \ t > 0
\]
\[
y(x, 0) = y_0(x), \quad x \in \Omega.
\]

The solution \( y \) exists, is unique, and nonnegative (see [5], [6]).

**Lemma 2.1.** (i) If \( \lambda > 0 \), then
\[
\lim_{t \to +\infty} \tilde{y}(t) = 0 \quad \text{in} \quad L^\infty(\Omega);
\]
(ii) If \( \lambda = 0 \), then
\[
\lim_{t \to +\infty} \tilde{y}(t) = c_\varphi \quad \text{in} \quad L^\infty(\Omega),
\]
where \( \varphi \) is the positive eigenfunction of (2.5), corresponding to \( \lambda = \lambda_1 \), with the norm \( \| \varphi \|_{L^2(\Omega)} = 1 \).

Here \( c = \left( \int_{\Omega} \varphi(x) \varphi^*_1(x) dx \right)^{-1} \int_{\Omega} y_0(x) \varphi^*_1(x) dx \), and \( \varphi^*_1 \) is the positive eigenfunction of (2.6), corresponding to \( \lambda = \lambda_1 \), with the norm \( \| \varphi^*_1 \|_{L^2(\Omega)} = 1 \);
(iii) If \( \lambda < 0 \), and \( y_0 \neq 0 \) \(L^\infty(\Omega)\), then
\[
\lim_{t \to +\infty} \int_{\Omega} \tilde{y}(x, t) = +\infty.
\]

**Remark 2.1.** \( \varphi_1, \varphi^*_1 \in L^\infty(\Omega) \) and satisfy
\[
a_0 \leq \varphi_1(x), \varphi^*_1(x) \quad \text{a.e.} \ x \in \Omega,
\]
where \( a_0 \) is a positive constant (see [5]).

**Theorem 2.2.** If \( \lambda_1 < 0 \) and \( y_0 \neq 0 \) \(L^\infty(\Omega)\), then there exists \( c \in (0, +\infty) \) such that
\[
\lim_{t \to +\infty} y(t) = c \varphi_1 \quad \text{in} \quad L^\infty(\Omega).
\]

**Remark 2.2.** \( c \varphi_1 \) in Theorem 2.2 is the unique positive steady state to (2.2).

The analysis of this problem is related to the magnitude of the principal eigenvalue for

||-d\Delta \varphi + a(x, t) \varphi \||
- \int_{\Omega} k(x, x') \varphi(x', t) dx' = \lambda \varphi, \quad x \in \Omega, \ t \geq 0
\]
\[
\partial_t \varphi - d\Delta \varphi + a(x, t) \varphi \varphi(x, t) + \alpha \varphi(x, t) = 0, \quad x \in \partial \Omega, \ t \geq 0
\]
\[
\varphi(x, t + T) = \varphi(x, t), \quad x \in \Omega, \ t \geq 0.
\]
3 Approximating the eigenvalue $\lambda_{1\gamma}^\omega$

Let us remind that $\lambda_{1\gamma}^\omega$ is the principal eigenvalue of

$$
\begin{cases}
-d_2 \Delta \varphi + a \varphi - \delta \int_\Omega \ell(x, x') f(K, 0) \varphi(x') dx' \\
+ \gamma \chi_\omega \varphi = \lambda \varphi,
\end{cases}
\quad x \in \Omega;
\quad \partial_\nu \varphi(x) = 0, \quad x \in \partial\Omega.
$$

Let $d := d_2, a(x) := a + \alpha \chi_\omega(x) - \zeta, k(x, x') := \delta \ell(x, x') f(K, 0), \mathcal{M}(r) := r, \alpha := 0, y_0(x) := 1$, where $\zeta \in \mathbb{R}$ is such that $\zeta > \lambda_{1\gamma}^\omega$.

In this particular case (2.2)-(2.4) becomes (1.8).

**Theorem 3.1.** The solution $y^\omega$ to (1.8) satisfies

$$
\lim_{t \to +\infty} \int_\Omega y^\omega(x, t) dx = \zeta - \lambda_{1\gamma}^\omega.
$$

**Remark 3.1.** Theorem 3.1 gives an approximation method for $\lambda_{1\gamma}^\omega$: for $T > 0$ large, we have that

$$
\zeta - \int_\Omega y^\omega(x, T) dx
$$

approximates $\lambda_{1\gamma}^\omega$.

If we replace the initial condition in (1.8) by the general one (2.4), where $y_0 \in L^\infty(\Omega), y_0(x) \geq 0$ a.e. $x \in \Omega, y_0 \not\equiv 0_{L^\infty(\Omega)}$, then the solution $y$ to this new problem has the same limit as the solution of (1.8), as $t \to +\infty$, and satisfies the conclusion of Theorem 3.1.

4 The derivative of $J^\omega$ with respect to translations

The main result in Section 3 shows that instead of finding a position for $\omega$ which gives a large value for $\lambda_{1\gamma}^\omega$, it would be very good to investigate the problem of finding a position for $\omega$ which gives a small value for

$$
J^\omega = \int_\Omega y^\omega(x, T) dx,
$$

where $y^\omega$ is the solution to (1.8). This is of course an approximating problem.

Let $\omega^* \subseteq \omega$ be a nonempty open subset of $\omega$, with a smooth boundary and such that $\omega^* \subseteq \Omega$ and $\Omega \setminus \omega^*$ is a domain. Consider $\mathcal{O}$ the set of all translations $\omega$ of $\omega^*$, satisfying $\omega \subseteq \omega$.

For any $\omega \in \mathcal{O}$ and $V \in \mathbb{R}^N$ we define the derivative

$$
dJ^\omega(V) = \lim_{\varepsilon \to 0} \frac{J^{\omega + \varepsilon V} - J^\omega}{\varepsilon}.
$$

For basic results and methods in the optimal shape design theory we refer to [19].

**Theorem 4.1.** For any $\omega \in \mathcal{O}$ and $V \in \mathbb{R}^N$ we have that

$$
dJ^\omega(V) = \gamma \int_0^T \int_{\partial\omega} \bar{y}^\omega(x, t)p^\omega(x, t)\nu(x) \cdot V d\sigma dt,
$$

where $\bar{p}$ is the solution to the adjoint problem

$$
\begin{cases}
\partial_t \bar{p} + d_2 \Delta \bar{p} - a \bar{p} \\
+ \delta \int_\Omega \ell(x', x') f(K, 0) p(x', t) dx' \\
- \gamma \chi_\omega \bar{p} + \zeta \bar{p} - (\int_\Omega y^\omega(x, t) dx) \bar{p}
\end{cases}
\quad \begin{array}{ll}
x \in \Omega, t > 0 \\
\partial_\nu p(x, t) = 0, \quad x \in \partial\Omega, t > 0 \\
p(x, T) = 1, \quad x \in \Omega.
\end{array}
$$

Here $\nu(x)$ is the normal inward versor at $x \in \partial\omega$ (inward with respect to $\omega$).

For the construction of the adjoint problems in optimal control theory we refer to [11].

**Remark 4.1.** Based on Theorem 4.1 we derive the following conceptual iterative algorithm to improve the position (translation) of $\omega \in \mathcal{O}$ at each iteration (in order to obtain a smaller value for $J^\omega$). It is a descent method (e.g. [7], Section 2.3) for the minimization of $J^\omega$ with respect to $\omega \in \mathcal{O}$.

**Step 0:** Choose $\omega^{(0)}$ the initial position of $\omega$; choose $T > 0$;

set $J^{(0)} := 0, k := 0$;

**Step 1:** compute $y^{(k+1)}$ the solution of Problem (1.8) for $t \in [0, T]$,

$$
\begin{array}{c}
\text{corresponding to } \omega := \omega^{(k)}; \\
\text{compute } J^{(k+1)} = \int_\Omega y^{(k+1)}(x, T) dx;
\end{array}
$$

**Step 2:** if $|J^{(k+1)} - J^{(k)}| < \varepsilon_1$,

then stop ($\omega^{(k)}$ is the support of the control)

else go to Step 3

**Step 3:** compute $p^{(k+1)}$ the solution of Problem (4.1) corresponding to

$$
y^{(k)} := y^{(k+1)};
$$

**Step 4:** compute

$$
\begin{array}{ll}
V := - \int_0^T \int_{\partial\omega(x)} y^{(k+1)}(x, t)p^{(k+1)}(x, t)\nu(x) d\sigma dt; \\
\text{if } |V| < \varepsilon_2,
\end{array}
$$

then stop ($\omega^{(k)}$ is the support of the control)

else go to Step 5
Step 5: compute the new position of $\omega$

$$
\omega^{(k+1)} := \rho V + \omega^{(k)}; \\
k := k + 1;
$$
go to Step 1.

In Step 5, $\rho > 0$ is a given parameter, and $\varepsilon_1 > 0$ in Step 2 and $\varepsilon_2 > 0$ in Step 4 are prescribed convergence parameters.

Remark 4.2. Another interesting problem is that when

$$
\omega = \bigcup_{j=1}^{N} \omega_j,
$$
where $\{\omega_j\}_{j=1}^{N}$ are mutually disjoint domains. The goal is to find the best position for each $\omega_j$. A similar approach can be used.

Acknowledgements: The work of S. Aniţa was supported by the CNCSIS (Romanian National Research Council) grant ID 381/2007: “Optimal control and stabilization of the continuous models of population dynamics”.

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