Comparing Solutions of Hyperbolic and Parabolic Heat Conduction Equations for L-shape Samples

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Abstract: - In this paper we state initial-boundary value problems (IBVPs) for both parabolic, and hyperbolic heat conduction equations describing intensive quenching (IQ) process for a thin L-shape sample. We construct analytical solutions of inverse and direct problems in the form of 2nd kind integral equation, and compare the rate of change of the temperatures in a small neighbourhood of the initial time *t=0* after determining solutions for these problems.

Key-Words: - steel quenching, hyperbolic heat equation, parabolic heat equation, L-shape sample, direct problem, inverse problem, exact solution, integral equation

1 Introduction

As it is well known, heat conduction in a solid body can be described by the well-known Fourier equation

$$\frac{\partial T}{\partial t} = a^2 \nabla^2 T + \frac{f}{\tilde{c}\rho}, \qquad (1)$$

where T denotes the temperature of the body with thermal conductivity k, and f is the density of heat sources, t is time, $a^2 = k/\tilde{c}\rho$, $\tilde{c}$ specific heat capacity, ρ density of the body.

In a number of physical situations Eq. (1) implies arbitrarily high thermal propagation speed. One of such cases is IQ. When immersing the heated part into a quenchant, the initial speed of propagation for the heat tends to infinity but actually is finite (see [8]). So it's better to use a hyperbolic heat conduction equation, which admits a finite speed of propagation for T

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = a^2 \nabla^2 T + \frac{\tau}{\tilde{c}\rho}\frac{\partial f}{\partial t} + \frac{f}{\tilde{c}\rho}, \qquad (2)$$

with τ as a relaxation time. (More about hyperbolic heat equation can be found, e.g., in the book [10], and bibliography [6], [7].)

τ, k, $\tilde{c}$, ρ are generally dependant on T and on the material, but throughout the paper we assume that these are constants and that there are no sources of heat, so $f = 0$.

Let's image that we have an element with rectangular fins (see Fig., from [4]) that is heated and then cooled rapidly (IQ process) in a suitable fluid, e.g., water or brine. Since the figure can be divided into several symmetrical parts, we can use Eqs. (1), (2) to describe IQ process for one L-shaped part only and get the same results as if we had quenched the entire figure. In order to simplify the problem, we use non-dimensionalization (see [4]).

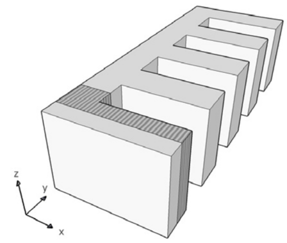

Fig.: Element with fins

2 Using Hyperbolic Heat Conduction Equation to Describe IQ Process

2.1 Mathematical Formulation of 3D Problem

In this section we are going to consider Eq. (2) for describing IQ process and solve IBVP using Green's function method (it's been discussed in [2], [3] as well). To modify the method to obtain a closed-form Green's function for so called regular non-canonical domain, we are going to represent the original domain as a finite union of canonical sub-domains with appropriate boundary conditions along the planes (or lines in 2D) connecting two neighbour domains. We may therefore suppose that

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the given L-shaped sample is made up from two rectangular cuboids (rectangles in 2D), *the base* $\{x \in [0,\delta], y \in [0,1], z \in [0,\omega]\}$ and *the foot* $\{x \in [\delta, \delta + l], y \in [0,b], z \in [0,\omega]\}$ joined along the surface $x = \delta$. By means of that we'll be able to define IQ process for each part separately.

Let's assume that $V^0(x,y,z,t)/V(x,y,z,t)$ denotes the dimensionless temperature distribution in the base/foot. Thus, in terms of the dimensionless variables the hyperbolic equations of heat conduction have the following form

$$\tau_{r,0}\frac{\partial^2 V^0}{\partial t^2} + \frac{\partial V^0}{\partial t} = a^2\left(\frac{\partial^2 V^0}{\partial x^2} + \frac{\partial^2 V^0}{\partial y^2} + \frac{\partial^2 V^0}{\partial z^2}\right) \quad (3)$$

and

$$\tau_r \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} = a^2\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right). \quad (4)$$

To state the IBVP for determining the temperature of the sample, we need to formulate both boundary and initial conditions. Along the planes $y = 0$, $y = 1$ symmetry conditions must be applied

$$\frac{\partial V^0}{\partial n} = 0, \ \frac{\partial V}{\partial n} = 0,$$

where n denotes the exterior normal to the boundary $\partial\Omega$ of the domain Ω on which the given equations are to be solved. But at the other sides ($S \subset \partial\Omega$) of the sample a heat exchange takes place with the surrounding medium, whose temperature is given by $\Theta(x,y,z,t)$,

$$k\frac{\partial V^0}{\partial n} + hV^0 = h\Theta \ \text{ on } S_1,$$

$$k\frac{\partial V}{\partial n} + hV = h\Theta \ \text{ on } S_2,$$

h heat transfer coefficient. (It is also possible to consider nonlinear boundary conditions, see [1]. In that case solutions of hyperbolic and parabolic equations differ essentially.)

As the base is in ideal thermal contact with the foot, continuity of temperature and heat flux are imposed at the interface $x = \delta$ between the adjacent parts of the sample

$$V^0\big|_{x=\delta-0} = V\big|_{x=\delta+0},$$

$$\left.\frac{\partial V^0}{\partial x}\right|_{x=\delta-0} = \left.\frac{\partial V}{\partial x}\right|_{x=\delta+0}.$$

Let us also establish the initial conditions

$$V^0\big|_{t=0} = V_0^0(x,y,z), \ V\big|_{t=0} = V_0(x,y,z), \quad (5)$$

$$\left.\frac{\partial V^0}{\partial t}\right|_{t=0} = W_0^0(x,y,z), \ \left.\frac{\partial V}{\partial t}\right|_{t=0} = W_0(x,y,z). \quad (6)$$

In IQ conditions (6) are not really known. But the initial time-rate of the temperature change should be calculated to compare it with critical cooling rate to predict heat transfer modes, as the initial cooling rate can be in different ranges (see [8]). Therefore, we can assume that the temperature distribution and the speed of temperature change are given at the end of the process

$$V^0\big|_{t=T} = V_T^0(x,y,z), \ V\big|_{t=T} = V_T(x,y,z),$$

$$\left.\frac{\partial V^0}{\partial t}\right|_{t=T} = W_T^0(x,y,z), \ \left.\frac{\partial V}{\partial t}\right|_{t=T} = W_T(x,y,z).$$

2.2 Formulation of Direct Problem in 2D

Let's suppose that the sample we have is thin in the z–direction ($\omega << 1$, $\omega << b$, $\omega << \delta$).Hence, to reduce the 3D problems (3), (4) considered before to 2D ones, we will introduce average values of all the functions used before over the interval $[0,\omega]$, for example,

$$v^0(x,y,t) = \frac{1}{\omega}\int_0^\omega V^0(x,y,z,t)dz,$$

$$v(x,y,t) = \frac{1}{\omega}\int_0^\omega V(x,y,z,t)dz,$$

and use the well-known expressions for functions $v^0(x,y,t)$ and $v(x,y,t)$

$$v^0(x,y,t) = \exp\left(-\frac{t}{2\tau_{r,0}}\right)u^0(x,y,t),$$

$$v(x,y,t) = \exp\left(-\frac{t}{2\tau_r}\right)u(x,y,t).$$

By substituting these two into 2D equations and setting dimensionless environmental temperature $\theta(x,y,t) = 0$ for $\forall(x,y) \in S$, with the exception of $x = 0$ where $\theta(0,y,t) = 1$, we arrive at (see [3])

$$\frac{\partial^2 u^0}{\partial t^2} = a_\tau^2\left(\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2}\right) - cu^0, \quad (7)$$

$$\frac{\partial^2 u}{\partial t^2} = a_\tau^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - cu, \quad (8)$$

with

$$a_\tau^2 = a^2/\tau_r, \ c = a_\tau^2\frac{2\beta}{\omega} - \frac{1}{4\tau_r^2}, \ \beta = h/k.$$

It is assumed here that we have one and the same relaxation time τ_r ($\tau_{r,0} = \tau_r$) for both parts of the sample, so continuity conditions for the temperature and the heat flux at the intersection of the base and the foot can be automatically assured.

Putting together the partial differential equations, the boundary, and the initial conditions, we have a direct problem, or an inverse problem when using final conditions.

2.3 Formulation of Inverse Problem in 2D

As it was mentioned before, the initial rate of temperature change is not known but must be determined. So we now address the inverse problem of determining that rate given the final data: the temperature distribution and the rate of change of the temperature at $t = T$. This corresponds to solving the given equations backwards. We can transform this problem into a direct problem by introducing a new time variable

$$\tilde{t} = T - t,$$

so that

$$\tilde{u}^0(x, y, \tilde{t}) = u^0(x, y, T - \tilde{t}), \tag{9}$$

$$\tilde{u}(x, y, \tilde{t}) = u(x, y, T - \tilde{t}). \tag{10}$$

In the new variables (9), (10), the wave Eqs. (7), (8) read

$$\frac{\partial^2 \tilde{u}^0}{\partial \tilde{t}^2} = a_\tau^2 \left(\frac{\partial^2 \tilde{u}^0}{\partial x^2} + \frac{\partial^2 \tilde{u}^0}{\partial y^2} \right) - c\tilde{u}^0, \tag{11}$$

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} = a_\tau^2 \left(\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} \right) - c\tilde{u}. \tag{12}$$

The function transformation (10) satisfies the boundary condition

$$\left. \left(\frac{\partial \tilde{u}^0}{\partial x} - \beta \tilde{u}^0 \right) \right|_{x=0} = -\beta \exp\left(\frac{T - \tilde{t}}{2\tau_r} \right), y \in [0,1].$$

But other required boundary conditions have the same form as those in subsection 2.1 with zero environmental temperature. Combining the continuity conditions at the right hand side border of the base yields

$$\left. \left(\frac{\partial \tilde{u}^0}{\partial x} + \beta \tilde{u}^0 \right) \right|_{x=\delta-0} = \tilde{F}^0(y, \tilde{t}),$$

$$\tilde{F}^0(y, \tilde{t}) = \begin{cases} 0, & y \in (b,1] \\ \left. \left(\dfrac{\partial \tilde{u}}{\partial x} + \beta \tilde{u} \right) \right|_{x=\delta+0}, & y \in [0,b] \end{cases}. \tag{13}$$

At the left-hand side border of the foot we get 3rd type boundary condition as well

$$\left. \left(\frac{\partial \tilde{u}}{\partial x} - \beta \tilde{u} \right) \right|_{x=\delta+0} = \tilde{F}(y, \tilde{t}),$$

$$\tilde{F}(y, \tilde{t}) = \left. \left(\frac{\partial \tilde{u}^0}{\partial x} - \beta \tilde{u}^0 \right) \right|_{x=\delta-0}, \quad y \in [0,b]. \tag{14}$$

But the initial conditions are

$$\left. \tilde{u}^0 \right|_{\tilde{t}=0} = \left. u^0 \right|_{t=T}, \quad \left. \tilde{u} \right|_{\tilde{t}=0} = \left. u \right|_{t=T}, \tag{15}$$

$$\left. \frac{\partial \tilde{u}^0}{\partial \tilde{t}} \right|_{\tilde{t}=0} = -\left. \frac{\partial u^0}{\partial t} \right|_{t=T} = -\overline{w}_T^0(x, y), \tag{16}$$

$$\left. \frac{\partial \tilde{u}}{\partial \tilde{t}} \right|_{\tilde{t}=0} = -\left. \frac{\partial u}{\partial t} \right|_{t=T} = -\overline{w}_T(x, y), \tag{17}$$

where

$$\overline{w}_T^0(x, y) = \exp\left(\frac{T}{2\tau_r} \right) \left[w_T^0(x, y) + \frac{v_T^0(x, y)}{2\tau_r} \right],$$

$$\overline{w}_T(x, y) = \exp\left(\frac{T}{2\tau_r} \right) \left[w_T(x, y) + \frac{v_T(x, y)}{2\tau_r} \right].$$

2.3.1 Solution for the Base

Under the assumption that the function $\tilde{F}^0(y, \tilde{t})$ is given, the solution of Eq. (11), satisfying appropriate boundary conditions and the initial conditions (15), (16) admits the following integral representation

$$\begin{aligned}
&\tilde{u}^0(x, y, \tilde{t}) = \\
&a_\tau^2 \int_0^{\tilde{t}} d\iota \int_0^\delta d\zeta \int_0^1 \tilde{u}^{0,1}(\zeta, \upsilon, \iota) \frac{\partial^2}{\partial y^2} \tilde{G}^0(x, y, \zeta, \upsilon, \tilde{t} - \iota) d\upsilon \\
&- \int_0^{\tilde{t}} d\iota \int_0^\delta d\zeta \int_0^1 \frac{\partial^2}{\partial \iota^2} \tilde{u}^{0,1}(\zeta, \upsilon, \iota) \tilde{G}^0(x, y, \zeta, \upsilon, \tilde{t} - \iota) d\upsilon \\
&- c \int_0^{\tilde{t}} d\iota \int_0^\delta d\zeta \int_0^1 \tilde{u}^{0,1}(\zeta, \upsilon, \iota) \tilde{G}^0(x, y, \zeta, \upsilon, \tilde{t} - \iota) d\upsilon \\
&+ \int_0^\delta d\zeta \int_0^1 u_T^0(\zeta, \upsilon) \frac{\partial}{\partial \tilde{t}} \tilde{G}^0(x, y, \zeta, \upsilon, \tilde{t}) d\upsilon \\
&- \int_0^\delta d\zeta \int_0^1 \tilde{u}^{0,1}(\zeta, \upsilon, 0) \frac{\partial}{\partial \tilde{t}} \tilde{G}^0(x, y, \zeta, \upsilon, \tilde{t}) d\upsilon \\
&- \int_0^\delta d\zeta \int_0^1 \overline{w}_T^0(\zeta, \upsilon) \tilde{G}^0(x, y, \zeta, \upsilon, \tilde{t}) d\upsilon \\
&- \int_0^\delta d\zeta \int_0^1 \frac{\partial}{\partial \tilde{t}} \tilde{u}^{0,1}(\zeta, \upsilon, 0) \tilde{G}^0(x, y, \zeta, \upsilon, \tilde{t}) d\upsilon \\
&+ \tilde{u}^{0,1}(x, y, \tilde{t}),
\end{aligned} \tag{18}$$

where

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$$\widetilde{u}^{0,1}(x,y,\widetilde{t}) = \frac{1+\beta x}{2\beta+\beta^2\delta}\widetilde{F}^0(y,\widetilde{t}) - \frac{\beta(x-\delta)-1}{2\beta+\beta^2\delta}\beta\exp\left(\frac{T-\widetilde{t}}{2\tau_r}\right),$$

with the Green's function given by (see [9])

$$\widetilde{G}^0(x,y,\zeta,\upsilon,\widetilde{t}) = = \sum_{i=1}^{\infty}\sum_{j=0}^{p_i-1}\frac{\varphi_i(x)\varphi_i(\zeta)\phi_j(y)\phi_j(\upsilon)\sinh\left(\widetilde{t}\sqrt{|f_{i,j}|}\right)}{\|\varphi_i\|^2\|\phi_j\|^2\sqrt{|f_{i,j}|}} + \sum_{i=1}^{\infty}\sum_{j=p_i}^{\infty}\frac{\varphi_i(x)\varphi_i(\zeta)\phi_j(y)\phi_j(\upsilon)\sin\left(\widetilde{t}\sqrt{f_{i,j}}\right)}{\|\varphi_i\|^2\|\phi_j\|^2\sqrt{f_{i,j}}}, \quad (19)$$

where

$$f_{i,j} = a_\tau^2(\lambda_i^2+\kappa_j^2)+c\,.$$

The natural number p_i that appears in formula (19) can be obtained from the inequalities

$$f_{i,j}<0 \text{ for } j=\overline{1,p_i-1},$$
$$f_{i,j}>0 \text{ for } j=\overline{p_i,\infty}.$$

We obtain the eigenvalues from

$$\cot(\lambda_i\delta) = \frac{\lambda_i^2-\beta^2}{2\lambda_i\beta},\ \kappa_j = j\pi\,.$$

But

$$\varphi_i(x) = \cos(\lambda_i x)+\frac{\beta}{\lambda_i}\sin(\lambda_i x),\ \phi_j(y)=\cos(\kappa_j y)$$

are called the eigenfunctions of the problem with

$$\|\varphi_i\|^2 = \frac{(\lambda_i^2+\beta^2)\delta+2\beta}{2\lambda_i^2},\ \|\phi_j\|^2 = \begin{cases}1 & j=0\\ \frac{1}{2} & j\neq 0\end{cases}$$

as their normal squares.

2.3.2 Solution for the Foot

The solution to the second problem (12) is conveniently sought in the form

$$\widetilde{u}(x,y,\widetilde{t}) = a_\tau^2\int_0^{\widetilde{t}}d\tau\int_\delta^{\delta+l}d\xi\int_0^b\widetilde{u}^1(\xi,\eta,\tau)\frac{\partial^2}{\partial y^2}\widetilde{G}(x,y,\xi,\eta,\widetilde{t}-\tau)d\eta$$
$$-\int_0^{\widetilde{t}}d\tau\int_\delta^{\delta+l}d\xi\int_0^b\frac{\partial^2}{\partial\tau^2}\widetilde{u}^1(\xi,\eta,\tau)\widetilde{G}(x,y,\xi,\eta,\widetilde{t}-\tau)d\eta$$
$$-c\int_0^{\widetilde{t}}d\tau\int_\delta^{\delta+l}d\xi\int_0^b\widetilde{u}^1(\xi,\eta,\tau)\widetilde{G}(x,y,\xi,\eta,\widetilde{t}-\tau)d\eta$$
$$+\int_\delta^{\delta+l}d\xi\int_0^b u_T(\xi,\eta)\frac{\partial}{\partial\widetilde{t}}\widetilde{G}(x,y,\xi,\eta,\widetilde{t})d\eta$$
$$-\int_\delta^{\delta+l}d\xi\int_0^b\widetilde{u}^1(\xi,\eta,0)\frac{\partial}{\partial\widetilde{t}}\widetilde{G}(x,y,\xi,\eta,\widetilde{t})d\eta$$
$$-\int_\delta^{\delta+l}d\xi\int_0^b\overline{w}_T(\xi,\eta)\widetilde{G}(x,y,\xi,\eta,\widetilde{t})d\eta$$
$$-\int_\delta^{\delta+l}d\xi\int_0^b\frac{\partial}{\partial\widetilde{t}}\widetilde{u}^1(\xi,\eta,0)\widetilde{G}(x,y,\xi,\eta,\widetilde{t})d\eta$$
$$+\widetilde{u}^1(x,y,\widetilde{t}), \quad (20)$$

$$\widetilde{u}^1(x,y,\widetilde{t}) = \frac{\beta(x-\delta-l)-1}{2\beta+\beta^2 l}\widetilde{F}(y,\widetilde{t})\,.$$

Thus, the Green's function can be written as

$$\widetilde{G}(x,y,\xi,\eta,\widetilde{t}) = = \sum_{m=1}^{\infty}\sum_{n=1}^{q_m-1}\frac{\varphi_m(x)\varphi_m(\xi)\phi_n(y)\phi_n(\eta)\sinh\left(\widetilde{t}\sqrt{|g_{m,n}|}\right)}{\|\varphi_m\|^2\|\phi_n\|^2\sqrt{|g_{m,n}|}} + \sum_{m=1}^{\infty}\sum_{n=q_m}^{\infty}\frac{\varphi_m(x)\varphi_m(\xi)\phi_n(y)\phi_n(\eta)\sin\left(\widetilde{t}\sqrt{|g_{m,n}|}\right)}{\|\varphi_m\|^2\|\phi_n\|^2\sqrt{|g_{m,n}|}},$$

and we have

$$g_{m,n} = a_\tau^2(\mu_m^2+\nu_n^2)+c\,,$$
$$g_{m,n}<0 \text{ for } n=\overline{1,q_m-1},$$
$$g_{m,n}>0 \text{ for } n=\overline{q_m,\infty}.$$

The eigenvalues may be computed by finding the roots of these transcendental equations

$$\tan(\mu_m l) = \frac{2\mu_m\beta}{\mu_m^2-\beta^2},\ \tan(\nu_n b) = \frac{\beta}{\nu_n}.$$

They have the corresponding eigenfunctions

$$\varphi_m(x) = \cos(\mu_m(x-\delta))+\frac{\beta}{\mu_m}\sin(\mu_m(x-\delta)),$$
$$\phi_n(y) = \cos(\nu_n y),$$
$$\|\varphi_m\|^2 = \frac{(\mu_m^2+\beta^2)l+2\beta}{2\mu_m^2},\ \|\phi_n\|^2 = \frac{\sin(\nu_n b)}{2\beta}+\frac{b}{2}.$$

2.3.3 Junction of Both Solutions

Upon coupling Eq. (18) with the formula (14), we have a representation for the combination of the solution for the base and its derivative at the border between both parts. In order to determine a similar representation for $\widetilde{F}(y,\widetilde{t})$, we substitute expression (20) into (13). Plugging the latter into the former, we obtain non-homogeneous Volterra-Fredholm integral equation of the 2nd kind at the interface between both parts of the L-shape sample

$$\widetilde{F}^0(y,\widetilde{t}) == \Phi^0(y,\widetilde{t}) -$$

$$-\int_0^{\tilde{t}} d\iota \int_0^b L(\widetilde{F}^0(\upsilon,\iota), y, \upsilon, \tilde{t}, \iota) K^0(y,\upsilon,\tilde{t},\iota) d\upsilon .$$

After finding solution to this integral equation, we can calculate the temperature distribution in the sample and find the rate of the temperature change.

3 Using Parabolic Heat Conduction Equation to Describe IQ Process

3.1 General Statement of 2D Problem

In this section we'll outline how to find the rate of change of the temperature when using parabolic heat conduction equation to describe the temperature in the sample (see [4], [5] for more). By using average value of the functions over the interval $[0,\omega]$ and applying appropriate boundary conditions, 3D equations in two spatial variables are transformed into

$$\frac{\partial v^0}{\partial t} = a^2\left(\frac{\partial^2 v^0}{\partial x^2} + \frac{\partial^2 v^0}{\partial y^2}\right) - \frac{2\beta}{\omega} v^0, \qquad (21)$$

$$\frac{\partial v}{\partial t} = a^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{2\beta}{\omega} v. \qquad (22)$$

Putting

$$v^0(x,y,t) = \exp\left(-\frac{2\beta}{\omega}t\right) u^0(x,y,t),$$

$$v(x,y,t) = \exp\left(-\frac{2\beta}{\omega}t\right) u(x,y,t)$$

and substituting these expressions in Eqs. (21), (22) and all the conditions, we get the following problem for the functions $u^0(x,y,t)$ and $u(x,y,t)$

$$\frac{\partial u^0}{\partial t} = a^2\left(\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^0}{\partial y^2}\right), \qquad (23)$$

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \qquad (24)$$

All the conditions are like as in the preceding section with the exception of

$$\left.\left(\frac{\partial u^0}{\partial x} - \beta u^0\right)\right|_{x=0} = -\beta \exp\left(\frac{2\beta}{\omega}t\right), y \in [0,1].$$

3.1.1 Exact Solution of 2D Problem

It is straightforward to apply the technique of Section 2 (see subsection 2.3) to derive the solutions to IBVPs (23) and (24). Thus, the solution for the base has a form

$$u^0(x,y,t) =$$

$$a^2 \int_0^t d\iota \int_0^\delta d\zeta \int_0^1 u^{0,1}(\zeta,\upsilon,\iota) \frac{\partial^2}{\partial y^2} G^0(x,y,\zeta,\upsilon,t-\iota) d\upsilon$$

$$-\int_0^t d\iota \int_0^\delta d\zeta \int_0^1 \frac{\partial}{\partial \iota} u^{0,1}(\zeta,\upsilon,\iota) G^0(x,y,\zeta,\upsilon,t-\iota) d\upsilon$$

$$+\int_0^\delta d\zeta \int_0^1 v_0^0(\zeta,\upsilon) G^0(x,y,\zeta,\upsilon,t) d\upsilon$$

$$-\int_0^\delta d\zeta \int_0^1 u^{0,1}(\zeta,\upsilon,0) G^0(x,y,\zeta,\upsilon,t) d\upsilon$$

$$+u^{0,1}(x,y,t),$$

where

$$u^{0,1}(x,y,t) = \frac{1+\beta x}{2\beta + \beta^2\delta} F^0(y,t) -$$

$$-\frac{\beta(x-\delta)-1}{2\beta+\beta^2\delta} \beta \exp\left(\frac{2\beta}{\omega}t\right).$$

We use the eigenfunctions in Section 2 to expand the Green's function

$$G^0(x,y,\zeta,\upsilon,t) =$$

$$= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{\varphi_i(x)\varphi_i(\zeta)\phi_j(y)\phi_j(\upsilon)}{\|\varphi_i\|^2 \|\phi_j\|^2} \times$$

$$\exp\left(-a^2\left(\lambda_i^2 + \kappa_j^2\right)t\right).$$

But the function satisfying the stated problem for (24) assumes the following form

$$u(x,y,t) =$$

$$a^2 \int_0^t d\tau \int_\delta^{\delta+l} d\xi \int_0^b u^1(\xi,\eta,\tau) \frac{\partial^2}{\partial y^2} G(x,y,\xi,\eta,t-\tau) d\eta$$

$$-\int_0^t d\tau \int_\delta^{\delta+l} d\xi \int_0^b \frac{\partial}{\partial \tau} u^1(\xi,\eta,\tau) G(x,y,\xi,\eta,t-\tau) d\eta$$

$$+\int_\delta^{\delta+l} d\xi \int_0^b v_0(\xi,\eta) G(x,y,\xi,\eta,t) d\eta$$

$$-\int_\delta^{\delta+l} d\xi \int_0^b u^1(\xi,\eta,0) G(x,y,\xi,\eta,t) d\eta$$

$$+u^1(x,y,t),$$

$$u^1(x,y,t) = \frac{\beta(x-\delta-l)-1}{2\beta+\beta^2 l} F(y,t).$$

Here the Green's function is defined by

$$G(x,y,\xi,\eta,t) =$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\varphi_m(x)\varphi_m(\xi)\phi_n(y)\phi_n(\eta)}{\|\varphi_m\|^2 \|\phi_n\|^2} \times$$

$$\exp\left(-a^2\left(\mu_m^2 + \nu_n^2\right)t\right)$$

Using the same method as in the preceding section we obtain Volterra-Fredholm integral

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equation and get the initial time-rate of the temperature change.

4 Comparing the Rate of the Temperature

When v_p^0, v_h^0, the solutions of parabolic and hyperbolic heat conduction equations are found, we can differentiate these expressions with respect to t and compare the rates of change of the temperatures in a small neighbourhood of the initial time $t = 0$ by setting $t = \varepsilon$

$$\left|\frac{\partial v_h^0}{\partial t} - \frac{\partial v_p^0}{\partial t}\right|_{t=\varepsilon},$$

where

$$\left.\frac{\partial v_h^0}{\partial t}\right|_{t=\varepsilon} = -\left.\frac{\partial \tilde{v}_h^0}{\partial \tilde{t}}\right|_{\tilde{t}=T-\varepsilon}$$

and

$$\frac{\partial \tilde{v}_h^0}{\partial \tilde{t}} = \exp\left(-\frac{T-\tilde{t}}{2\tau_r}\right)\left(\frac{1}{2\tau_r}\tilde{u}_h^0 + \frac{\partial \tilde{u}_h^0}{\partial \tilde{t}}\right),$$

$$\frac{\partial v_p^0}{\partial t} = \exp\left(-\frac{2\beta}{\omega}t\right)\left(-\frac{2\beta}{\omega}u_p^0 + \frac{\partial u_p^0}{\partial t}\right).$$

The same expressions can be derived for the functions v_h, v_p.

5 Conclusion

We have constructed exact solution for time inverse problem for hyperbolic heat equation for 2D L-shape sample. The solution for determination of initial heat flux is obtained in the form of Fredholm integral equation of 2nd kind with continuous kernel. We have also analytically compared the rate of change of the temperatures that are calculated from both parabolic and hyperbolic heat conduction equation.

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