A Mathematical Model for a Willow Flute

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Abstract: - In the article we examine a mathematical model for a Norwegian flute. We find the possible frequencies of the sound produced by the flute, analyse how the pitches can be altered when changing the parameter *h* in the boundary condition at *x* = *l*, and determine the energy distribution in the sound.

Key-Words: - willow flute, harmonics, natural scale, wave equation, and energy distribution

1 Introduction

The willow flute (Norwegian: *seljefløyte*, Finnish: *pitkähuilu* or *pajupilli,* Swedish: *sälgflöjt* or *sälgpipa*) is a Scandinavian folk instrument of the recorder family existing in two forms: an end-blown flute, often called a whistling flute, and a side-blown flute.

Fig.1: A willow flute

This paper focuses on the side-blown flute (see Fig.1, from [6]) which is between 40 and 80 centimetres in length. It consists of a tube with a transverse fipple mouthpiece that is constructed by putting a wood plug with a groove in one end of the tube, and cutting a sound hole (edged opening) a short distance away from the plug (see Fig.2, from [7]). The air flow is directed through a passageway across the edge creating a sound.

As the flute has no finger holes, different pitches are produced by overblowing and by using a finger to cover, half-cover or uncover the hole at the far end of the tube. The *seljefløyte* plays tones in the harmonic series called the **natural scale**. When the end of the tube is left open, the flute produces one fundamental and its overtones, playing it with the end of the tube closed produces another harmonic scale.

2 Problem Formulation

Playing the flute causes periodic oscillations of the air pressure inside the instrument. These pressure disturbances are governed by the wave equation.

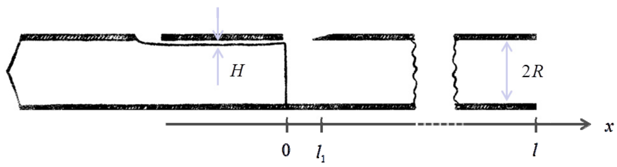

Fig.2: Definition of geometrical parameters for the flute

Assuming for simplicity that the pressure over the cross-section of the tube is constant, the willow flute can be modelled by

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2} + F(x), \; c^2 = \frac{B}{\rho}, \tag{1}$$

where $p(x,t)$ denotes the acoustic pressure, function $F(x)$ is some external source, ρ density of air ($\approx 1.2041 \; kg/m^3$). Let's suppose that there is no appreciable conduction of heat, therefore the behaviour of sound waves is adiabatic. So, B is the adiabatic bulk modulus of air, which is approximately given by $B = \gamma p_a$. Here γ is the ratio of the specific heats of air at constant pressure and at constant volume (≈ 1.4), and p_a is the ambient pressure ($\approx 101{,}325 \; Pa$).

Let's establish boundary and initial conditions. At $x = 0$ we have 3rd type boundary condition

$$p_x(0,t) - \kappa_1 p(0,t) = 0 \, . \tag{2}$$

The boundary condition at the far end of the tube depends upon whether the end is open, closed or partially open. For an open end of a tube, the total pressure at the end and atmospheric pressure must be approximately equal. In other words, the acoustic pressure p is zero:

$$p(l,t) = 0 . \quad (3)$$

At a closed end there is an antinode of pressure, as most of the sound is reflected:

$$p_x(l,t) = 0 . \quad (4)$$

In many research papers and works where wind instruments have been considered (see, e.g., [1] - [3]), only these two types of boundary conditions are used. But one should bear in mind that the player can also close the end of the tube only halfway. In that case the Robin's condition

$$p_x(l,t) + \kappa_2 p(l,t) = 0 \quad (5)$$

must be applied (see [4]). The value of κ_2 depends on how much the end is closed.

Conditions (3)-(5) can be written in the following form

$$(1-h)p_x(l,t) + hp(l,t) = 0 , \; 0 \le h \le 1 . \quad (6)$$

But the initial conditions are

$$p(x,0) = \phi(x) ,$$
$$p_t(x,0) = \psi(x) .$$

Nonhomogeneous equation (1) can be solved by finding a solution expressed in the form

$$p(x,t) = \sum_n X_n(x) T_n(t) ,$$

where $X_n(x)$, $n = 1,2,3,\dots$ are the eigenfunctions of the associated homogeneous problem, having the following expressions:

$$X_n(x) = \sin(\mu_n x + \varphi_n), \; \tan(\varphi_n) = \frac{\mu_n}{\kappa_1} ,$$

$$\|X_n\| = \frac{l}{2}\left(1 - \frac{\sin(\mu_n l)}{\mu_n l}\cos(\mu_n l + 2\varphi_n)\right).$$

The corresponding eigenvalues μ_n are roots of these transcendental equations:

end open

$$\tan(\mu_n l) = -\frac{\mu_n}{\kappa_1} ;$$

end closed

$$\cot(\mu_n l) = \frac{\mu_n}{\kappa_1} ;$$

end partially open

$$\cot(\mu_n l) = \frac{1}{\kappa_1 + \kappa_2}\left(\mu_n - \frac{\kappa_1 \kappa_2}{\mu_n}\right).$$

To determine the functions $T_n(t)$, we represent the source function and initial conditions as Fourier series. So $T_n(t)$ will satisfy the simple initial value problem

$$T_n''(t) + c^2 \mu_n^2 T_n(t) = F_n ,$$
$$T_n(0) = \phi_n ,$$
$$T_n'(0) = \psi_n .$$

Hence, the solution of problem (1) is

$$p(x,t) =$$
$$-\int_0^l \phi(\xi) \frac{\partial^2}{\partial t^2} G(x,\xi,t) d\xi - \int_0^l F(\xi) G(x,\xi,t) d\xi$$
$$-\int_0^l \psi(\xi) \frac{\partial}{\partial t} G(x,\xi,t) d\xi + \int_0^l F(\xi) G(x,\xi,0) d\xi \quad (7)$$

with Green's function

$$G(x,\xi,t) =$$
$$\sum_{n=1}^{\infty} \frac{\cos(c\mu_n t)}{c^2 \mu_n^2} \frac{X_n(x) X_n(\xi)}{\|X_n\|^2} .$$

2.1 Geometrical parameters and functions

In the paper we use the linear wave equation referring to the acoustic pressure. Actually the process is much more complex, as there are many things that should be taken into account. First of all, nonlinear equation could be used when dealing with mass-transfer. Secondly, there is a question about hydrodynamic behaviour of the jet of air. Both experimental and theoretical results suggest (see, e.g., [5]) that the airstream tends to form vortices when impinging the edged opening. In the case of one dimensional plane wave, we use a source function to describe this process. Thirdly, the mass flow in the passageway can be approximated by 3rd type boundary condition (2).

All results in the next two sections are obtained with the following parameter values:

$$l \approx 0.656 , \; l_1 = 0.01 , \; H = \frac{R}{5} ,$$

$$\kappa_1 = p_1 \frac{H^2 \pi}{R^2 \pi} , \; p_1 = 101{,}324 ,$$

and the initial conditions:

$$\phi(x) = 0 ,$$

$$\psi(x) = \begin{cases} 0 & 0 < x < l_1 \\ 1 - \dfrac{x - l_1}{\varepsilon} & l_1 \le x \le l_1 + \varepsilon \\ 0 & l_1 + \varepsilon < x < l \end{cases} , \; \varepsilon = \frac{l - l_1}{d} ,$$

where d is a positive constant,

and the source function:

$$F(x) = \frac{\kappa_1}{100} \cdot \frac{l-x}{l}.$$

3 Pitch and Frequency

One of the fundamental properties of sound is its pitch. Pitch is a subjective attribute of sound related to the frequency of a sound wave. Increasing the frequency causes a rise in pitch. But when decreasing the frequency, the pitch of the note diminishes.

The tone of the lowest frequency is known as the **fundamental** or **first harmonic**. All other possible pitches whose frequencies are integer multiples of the fundamental frequency are called the **upper harmonics** or **overtones**.

Using relation (7), we obtain the frequencies of the vibration mode of the willow pipe:

$$f_n = \frac{a\mu_n}{2\pi},\ n = 1,2,3,\ldots$$

If the parameter κ_1 is sufficiently large, the possible frequencies for the flute are close to harmonics:

$$f_n \approx \frac{an}{2l},\ n = 1,2,3,\ldots \text{ (end open)}$$

$$f_n \approx \frac{an}{4l},\ n = 1,3,5,\ldots \text{ (end closed)}$$

As μ_n is dependent on l, we can change the pitch by changing the length of the tube. If the pipe has finger holes, one can also change the effective length of the instrument by opening different holes. By altering the pressure of the air blown into the instrument, we change the value of n, that is, we jump between solutions p_n, resulting in discrete differences in pitch. The pitch can also be changed by modifying h in the boundary condition at $x = l$.

3.1 Open-end scale

For the given parameters (see Section 2) the fundamental for the open flute is Middle C or C4. The second harmonic is then an octave above the fundamental; the third harmonic is G5, and so on.

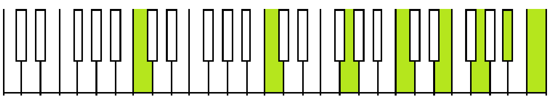

Fig.3: Notes of open-end C scale

One can notice (see Table 1) that some of these frequencies differ from the frequencies of the notes on a piano in twelve-tone equal temperament (12-TET) with the 49th key, the fifth A (called A4), defined as 440 Hz. Many people should be capable of detecting the difference if it is as little as 2 Hz.

Note	Piano	Willow flute
C4	261.63	261.63
C5	523.25	523.27
G5	783.99	784.90
C6	1046.50	1046.53
E6	1318.51	1308.16
G6	1567.98	1569.80
B♭6	1864.66	1831.43
C7	2093.00	2093.06

Table 1: Frequencies (Hz) of willow flute's open-end C scale compared with frequencies of 12-TET scale

The frequency of the m^{th} key in 12-TET can be found from the equation

$$f_m = 440 \cdot 2^{\frac{m-49}{12}}.$$

3.2 Closed-end scale

Closing the end drops the fundamental an octave below the pitch of the pipe open at the end. The next possible note G4 has approximately three times the frequency of the fundamental C3, the next one, E5, five times, and so on. This means that only the odd harmonics are present.

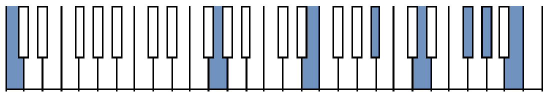

Fig.4: Notes of closed-end C scale

Table 2 shows that the willow flute's pitches do not quite conform to those produced by the piano.

Note	Piano	Willow flute
C3	130.81	130.82
G4	392.00	392.45
E5	659.26	654.08
B♭5	932.33	915.72
D6	1174.66	1177.35
F♯6	1479.98	1438.98
G♯6	1661.22	1700.61
B6	1975.53	1962.25

Table 2: Frequencies (Hz) of willow flute's closed-end C scale compared with frequencies of 12-TET scale

3.3 Intermediate scale

We can get further effects, when covering the end hole only halfway (this implies that we should change the parameter h in the boundary condition

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(6)). In this case the flute produces an intermediate set of pitches that fall in between those produced with the end closed and those when the end is open.

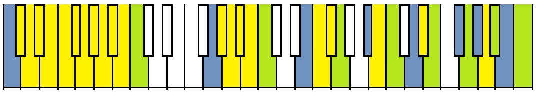

Fig.5: Approximate location of the willow flute's pitches on a piano. C scale

Pitches produced with end closed
Pitches produced with end open
Pitches produced with end partially open

In Fig.6 you can see how the parameter h affects location of pitches played by willow flute when leaving the end partially open.

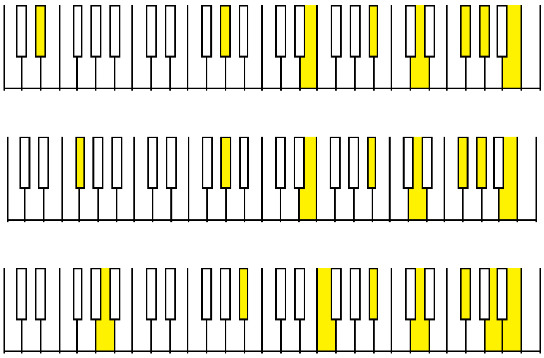

Fig.6: Approximate location of pitches when $h = 0.5$; $h = 0.7$; $h = 0.9$

4 Energy of Tones

As the sound of the willow flute playing is relatively harmonic, the energy of the sound is concentrated at certain frequencies of vibration. The more dominant the certain pitch, the greater the concentration of energy at that frequency. A person playing this particular instrument can get different pitches through manipulation of the supplied air, which is, changing initial conditions. If the initial conditions are defined as in Section 2, you can modify the distribution of energy between the fundamental and its overtones by changing ε.

You must blow as softly as possible to produce the fundamental, as it is hard to get. Blow a little harder and you get the first overtone and so on. The harder you blow the higher harmonics you get to be dominant (see Fig.7 for open-end flute).

Formula for calculating the energy distribution is

$$E(t) = \frac{1}{2}\int_0^l \rho p_t^2(x,t)dx - \int_0^t d\iota \int_0^l B p_x^2(x,\iota) p_t(x,\iota)dx .$$

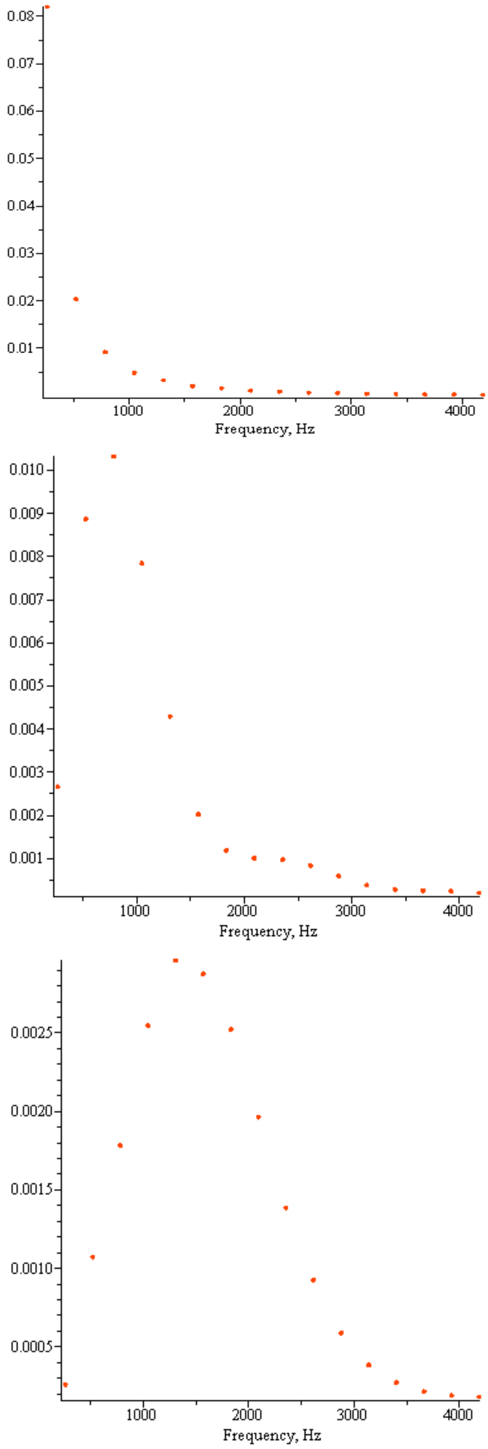

Fig.7: Energy distribution (J) across the frequencies when $\varepsilon = l$; $\varepsilon = l/3$; $\varepsilon = l/6$

5 Conclusion

We have examined how different boundary conditions affect the frequencies of pitches produced by the willow flute. And we have also determined the distribution of energy of tones.

Although linear differential equation (1) could be considered as an adequate

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approximation when describing acoustic properties of the willow pipe, it is necessary to take into account many other things, e.g., nonlinear effects. But that is a subject for future research.

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