Non-Newtonian Fluid Flow down an Inclined Plane

GABRIELLA BOGNÁR¹, IMRE GOMBKÖTŐ², KRISZTIÁN HRICZÓ¹

¹Department of Analysis,
²Institute of Raw Material Preparation and Environmental Process Engineering
University of Miskolc
3515 Miskolc-Egyetemváros
HUNGARY
matvbg@uni-miskolc.hu, ejtimreg@uni-miskolc.hu, krisztian.hriczo@gmail.com

Abstract: - The velocity distributions on an inclined plane are examined in the transport of non-Newtonian fluids. This process is modelled by boundary layer flows. To the equations of continuity and motion boundary conditions are considered on the plane and on the surface of the transported material. We examine the velocity distribution in case of different material properties, constant plane speed and different inclination angle.

Key-Words: - Bulk material, boundary layer, power-law, non-Newtonian fluid

1 Introduction
Investigation of the properties of flow down an inclined plane is a subject of great theoretical and practical importance and has attracted the attention of many researchers ([1]-[3]). We consider a fluid constantly poured on the inclined plane from above. The fluid forms a steady stream moving downwards under the action of the gravity. Such an example is a river flow. This phenomenon also occurs in case of conveyor belts and in the lubrication theory.

A continuum description of granular flows would be of considerable help in predicting natural geophysical hazards or in designing industrial processes ([3], [5], [6], [8]-[11]). The constitutive equations for granular flows, which govern how the material moves under shear, are still a matter of debate. These materials can behave like a solid or like a liquid. The main characteristics of granular liquids are complex dependence on shear rate when flowing. In this sense, granular materials show similarities with classical non-Newtonian fluids.

Here we propose power-law relation between the shear stress and the shear rate for sand-water mixture, and determine experimentally the rheological parameters for different volumetric concentrations. We also apply in our investigations the results obtained for the rheological parameters in case of bentonite mud.

The stationary solution for the Navier-Stokes equation for this problem can be found analytically. We shall derive this solution for both Newtonian and non-Newtonian fluids. The effects of rheological parameters are exhibited on the velocity distributions in the boundary layer.

2 Model Description
In this paper, the material is assumed to be incompressible and approximated as an homogeneous fluid with constant density.

The governing equations for steady, fully developed flow of a non-Newtonian fluid down an inclined plane under gravity are ([1], [3], [4]-[12]):

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  
(1)

\[
u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial^2 \tau_{xy}}{\partial y^2} + g \sin \alpha,
\]  
(2)

where \(x\) and \(y\) are coordinates along and normal to the plate, respectively, \(u\) and \(v\) are the velocity
components in $x$ and $y$ direction, respectively (see Fig. 1), $\rho$ is the density of fluid, $g$ is the acceleration due to gravity. $\alpha$ is the angle of the plane to the horizontal.

Consider a uniform flow of a non-Newtonian power-law fluid past a moving plane with

$$\tau_{xy} = \gamma \left[ \frac{\partial u}{\partial y} \right]^{\alpha-1} \frac{\partial u}{\partial y},$$

where $\gamma$ is a consistency index for non-Newtonian viscosity and $n$ is called power-law index, that is $n < 1$ for pseudoplastic, $n = 1$ for Newtonian, and $n > 1$ for dilatant fluids. The value of the power exponent $n$ in (3) is $n \approx 0.3$ for mud flow [8], $n > 1$ for sand-water mixture, and the value of $n$ is approximately 2 for dry granular material.

The boundary conditions are the following when the conveyor belt is moving down with constant velocity $U$:

$$u \bigg|_{y=0} = U, \quad \text{(no-slip boundary condition)}$$

$$v \bigg|_{y=0} = 0,$$

$$u \bigg|_{y=h} = 0. \quad \text{(free-surface boundary condition)}$$

(4)

Here $h$ denotes the height.

### 2.1 Newtonian fluid

For a flow of Newtonian fluid $n=1$ in the expression (3) and $\gamma$ denotes the dynamic viscosity. Steady, fully developed, laminar, incompressible flow of a Newtonian fluid down an inclined plane (see Fig.1) under gravity the Navier-Stokes equations reduces to

$$\gamma \frac{\partial^2 u}{\partial y^2} + \rho g \sin \alpha = 0.$$  

(5)

In case of steady, fully developed flow there is no change in time and in the flow direction, moreover, the flow occurs with no pressure gradient. From equation (5) one can get

$$u_{yy} = -\frac{\rho g}{\gamma} \sin \alpha,$$  

(6)

and $u$ can be obtained as:

$$u(y) = -\frac{\rho g}{\gamma} \sin \alpha \frac{y^2}{2} + Ay + B. \quad \text{(7)}$$

Taking into considerations the boundary conditions, constants $A$ and $B$ can be determined. Applying $u \bigg|_{y=h} = 0$ we get

$$A = \frac{\rho g}{\gamma} h \sin \alpha.$$

Condition $u \bigg|_{y=0} = U$ yields $B = U$. Then the solution is

$$u(y) = -\frac{\rho g}{\gamma} \sin \alpha \left( hy - \frac{y^2}{2} \right) + U, \quad \text{(8)}$$

which describes the velocity distribution in the boundary layer when the plane is moving with constant speed $U$.

Using (8) one can obtain the expression for the volumetric flow rate of thickness $h$. The volumetric flow rate through one unit width fluid along the $z$-direction is given by

$$Q = \int_0^h u \, dy = \frac{\rho g \sin \alpha h^3}{\mu} + U h.$$  

### 2.2 Non-Newtonian fluid

Here we apply the Ostwald-de Waele power law for non-Newtonian fluid down an inclined plane with angle $\alpha$. Equation (2) with (3) reduces

$$\left( \gamma u_{yy} \right) + \rho g \sin \alpha = 0$$  

(9)

where $\gamma$ and $n$ are parameters. The flow occurs with no pressure gradient and we apply the same boundary conditions

$$u \bigg|_{y=h} = 0 \quad \text{es} \quad u \bigg|_{y=0} = U.$$  

Integrating from equation (9)

$$\left( \gamma u_{yy} \right) = -\rho g \sin \alpha,$$  

one gets
\[ u_y^n = -\frac{\rho g}{\gamma} y \sin \alpha + A. \]  \hspace{1cm} (10)  

Applying condition \( u_h \mid_{y=h} = 0 \)
\[ u_y(y) = \left( \frac{\rho g}{\gamma} \sin \alpha \right)^{\frac{1}{n}} (h - y)^{\frac{1}{n}}, \]

and
\[ u(y) = -\left( \frac{\rho g}{\gamma} \sin \alpha \right)^{\frac{1}{n}} \frac{n}{n+1} (h - y)^{\frac{n}{n+1}} + B. \]  \hspace{1cm} (11)  

Constant \( B \) is evaluated from condition \( u \mid_{y=0} = U : \)
\[ B = U + \left( \frac{\rho g}{\gamma} \sin \alpha \right)^{\frac{1}{n}} \frac{n}{n+1} h^{\frac{n}{n+1}}, \]

that means
\[ u(y) = U + \left( \frac{\rho g}{\gamma} \sin \alpha \right)^{\frac{1}{n}} \frac{n}{n+1} \left( h^{\frac{n}{n+1}} - (h - y)^{\frac{n}{n+1}} \right) \]  \hspace{1cm} (12)  

The volumetric flow rate from the integral
\[ Q = \int_0^h u \, dy \]  

for the non-Newtonian case is obtained by
\[ \int_0^h \left[ U + \left( \frac{\rho g}{\gamma} \sin \alpha \right)^{\frac{1}{n}} \frac{n}{n+1} \left( h^{\frac{n}{n+1}} - (h - y)^{\frac{n}{n+1}} \right) \right] \, dy \]

as
\[ Q = Uh + \left( \frac{\rho g}{\gamma} \sin \alpha \right)^{\frac{1}{n}} \frac{n}{2n+1} h^{\frac{2n+1}{n}}. \]

### 3 Determination of Rheological Parameters for Sand-Water Mixture

Sand is generally used in the building industry, glass production or in metallurgy processes where sand is used for mould. As a bulk material it is transported in dry or wet form through pipeline, flowing properties of both cases are essential to determine for design purposes. During mineral processing operations, \( \text{SiO}_2 \) is often part of the tailing material as well, where determination of rheology of the processed slurry is also important.

![Figure 2. Volumetric concentration 20%](image-url)

Here we shall determine the rheological parameters for some sand-water mixtures.

One of the most widely used devices for measuring rheology of fluids is rotational viscometer. Rotational viscometers are having a cylindrical container in which the fluid is filled, and a rotor which submerge into the fluid. The geometry of the tank and the body is make very narrow ring like space, filled with the fluid. While the rotor rotating at different speed, the torque can be measured caused by the friction between the fluid and the rotor, share diagram can be determined. Measuring rheology of water / solid mixtures has its limitations in rotational viscometer, since large particles are settling down rapidly causing the concentration distribution of the mixture become inhomogeneous. This is the reason, why rotational viscometers can be used measuring rheology of slurries only made of very fine particles.
Our investigation was carried out with fine glass sand powder. The maximum particle size of the sample was 72 micrometers. For the tests, glass sand/water mixtures were mixed at different volumetric concentrations (20, 25, 30 and 40% by volume) and inserted into the cylindrical tank of the ANTON PAAR type rotational viscometer.

During each measurement, 30 measurement points were taken between 100...1000 1/s shear rate, while shear stresses were measured accordingly. Data were analysed using Goldensoftware Grapher software. The results of the measurements can be seen in Fig.2-5.

From these figures we can see that the power law model (3) fits the measured data. Table 1 exhibits the values of the consistency constant $\gamma$, the power exponent $n$ and the density $\rho$ for different volumetric concentrations of sand-water mixtures.

### Table 1. Parameter values of mixtures

<table>
<thead>
<tr>
<th>Volumetric concentration of sand/water mixtures $c$</th>
<th>$\gamma$</th>
<th>$n$</th>
<th>$\rho$ [kg/m$^3$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 %</td>
<td>0.000313</td>
<td>1.475</td>
<td>1340</td>
</tr>
<tr>
<td>25 %</td>
<td>0.000538</td>
<td>1.444</td>
<td>1425</td>
</tr>
<tr>
<td>30 %</td>
<td>0.001388</td>
<td>1.360</td>
<td>1510</td>
</tr>
<tr>
<td>40 %</td>
<td>0.026902</td>
<td>1.211</td>
<td>1680</td>
</tr>
</tbody>
</table>

## 4 The Influence of Parameters

### 4.1 Sand-water mixture

Here we examine the effect of the volumetric concentration $c$ of sand-water mixtures on the velocity distribution. We perform numerical simulations with MAPLE12 and exhibit the velocity profiles in Fig.6-7. The figures exhibit that the maximum velocity increases as the volumetric concentration increases. We can observe the effect of angle $\alpha$: the maximum value of the velocity increases as $\alpha$ increases.
Bentonite mud

In [7] water based mud at different mud flow rates were examined by Jiao and Sharma. It was observed that the thickness of the mud cake was a sensitive function of the mud rheology and the mud shear rate.

A commercial Wyoming bentonite was used to prepare different type of mud. The fresh water mud was prepared by adding 40 grams of the bentonite to 1 liter of water. It was mixed with a blender and then aged for 20 hours. 2% NaCl or 2% NaCl with 3% lignosulfonate (thinner) were added to get either flocculated or dispersed mud. It was shown in [7] that the power law rheological model fits the obtained date best. Table 2 contains the rheological properties of the three types of mud.

<table>
<thead>
<tr>
<th>Mud</th>
<th>$\gamma$</th>
<th>$n$</th>
<th>$\rho$ [kg/m$^3$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fresh water mud</td>
<td>0.319</td>
<td>0.8</td>
<td>1070</td>
</tr>
<tr>
<td>Dispersed mud</td>
<td>0.313</td>
<td>0.7</td>
<td>1070</td>
</tr>
<tr>
<td>Flocculated mud</td>
<td>0.235</td>
<td>1.7</td>
<td>1070</td>
</tr>
</tbody>
</table>

Table 2. Parameter values for mud [7]

We perform numerical simulations with MAPLE12 to observe the influences of the parameters. First, fix $\phi = 15^\circ$ and $\gamma = 0.7 \, Pa \cdot s^{-n}$.

In Fig.8., it can be seen that the maximum values of the velocity decrease as $n$ increases. Next fix $\alpha = 15^\circ$ and $n = 0.4$, and we investigate the influence of $\gamma$. It is presented that the velocity maximum decreases as $\gamma$ increases (see Fig.9).

Then we examine the influence of $\alpha$ when $n = 0.4$ and $\gamma = 0.7 \, Pa \cdot s^{-n}$. For different values of angle $\alpha$ we see that the maximum value of the velocity increases as $\alpha$ increases (see Fig. 10).
5 Conclusion
We presented the mathematical model for boundary layer flow of an incompressible homogeneous fluid with constant density on an inclined plane moving with constant velocity. The velocity distribution in the material is determined for steady, fully developed, laminar flow of non-Newtonian fluids down an inclined moving plane. The Ostwald-de Waele power law model is applied. Computations were carried out for sand-water slurry with different volumetric concentration and bentonite mud. The rheological parameters were measured by ANTON PAAR type rotational viscometer for the sand-water mixtures and we applied the results for bentonite mud (see paper [7]). The effect of the consistency parameter, the power law exponent, the volumetric concentration of the sand-water mixture and the angle $\alpha$ are observed and exhibited in Fig.6-10.

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References: