Computing Efficient Solutions of Nonconvex Multi-Objective Problems via Scalarization

REFAIL KASIMBEYLI
Izmir University of Economics
Department of Industrial Systems Engineering
Sakarya Caddesi 156, 35330 Izmir
TURKEY
refail.kasimbeyli@ieu.edu.tr

Abstract: This paper presents a new method for scalarization of nonlinear multi-objective optimization problems. We introduce a special class of monotonically increasing sublinear scalarizing functions and show that the scalar optimization problem constructed by using these functions, enables to compute complete set of weakly efficient, efficient, and properly efficient solutions of multi-objective optimization problems without convexity and boundedness conditions.

Key–Words: Cone separation theorem, Sublinear scalarizing functions, Conic scalarization method, Multi-objective optimization, Proper efficiency

1 Introduction

In general, scalarization means the replacement of a multi-objective optimization problem by a suitable scalar optimization problem which is an optimization problem with a real valued objective functional. Since the scalar optimization theory is widely developed, scalarization turns out to be of great importance for the multi-objective optimization theory.

One way of constructing a single-objective optimization problem is through scalarizing functions involving possibly some parameters or additional constraints. Multi-objective optimization methods utilize different scalarizing functions in different ways.

In a large variety of methods, norms have usually been used to identify the minimal (or non-dominated) elements that are closest to some (reference) point. In particular, the family of $l_p$ norms has been studied extensively by many researchers, including Yu [15], Zeleny [16], Steuer [13], and Lewandowski and Wierzbicki [10]. Gearhart [5] studied a family of norms including the $l_p$ norms. The $l_\infty$ norm and the augmented $l_\infty$ norm turned out to be very useful in generating nondominated solutions of general continuous and discrete multiple criteria programs and led to the well-known weighted (augmented) Tchebycheff scalarization and its variations.

To characterize the minimal and properly minimal elements of nonconvex sets in normed spaces, Jahn used the Minkowski functional and proved the existence of an additional norm defined on the ordering cone which is strongly monotonically increasing on this cone [7, 8]. Then he showed that the minimal and properly minimal elements (in the sense of Borwein [2]) of a cone-bounded set can be characterized by minimizing this norm on the set. The scalarization results using monotone norms were also given by Salukvadze [12], Rolewicz [11], Yu [15], Gearhart [5], and others.

By summarizing the properties of different scalarization techniques, the following questions related to the quality or performance of the technique can be drawn:

- Does this method always yield minimal (or efficient) elements?
- If so, can all minimal (or efficient) elements be detected in this way?
- Will the preference and reference point information of the decision maker be taken into consideration by this technique?

The simple investigation of different techniques may show that not all methods can positively answer all the questions raised here.

The conic scalarization method presented in this paper is shown to answer positively all these questions.

The history of development of this method goes back to the paper [4], where Gasimov introduced a class of monotonically increasing sublinear functions on partially ordered real normed spaces and showed without any convexity and boundedness assumptions...
that support points of a set obtained by these functions are properly minimal in the sense of Benson [1]. The question of “can every properly minimal point of a set be calculated in a similar way” was answered only in the case when the space is partially ordered by a certain Bishop–Phelps cone. Since then, different theoretical and practical applications by using the suggested class of sublinear functions have been realized [6, 3, 9, 14]. The theoretical fundamentals of the conic scalarization method in general form was firstly given in [9].

In this paper, a full characterization of the conic scalarization method is presented. We show that the scalar optimization problem constructed by using the monotonically increasing sublinear scalarizing functions, enables to characterize complete set of weakly efficient, efficient, and properly efficient solutions of multi-objective problems without convexity and boundedness conditions.

The rest of the paper is organized as follows. Section 2 presents characterization theorem for minimal, weakly minimal and properly minimal elements of nonconvex sets. The description of the conic scalarization method is given in Section 3. Illustrative examples explaining advantages of the presented approach are presented in Section 4. Finally, Section 5 draws some conclusions from the paper.

2 Characterization of Minimal Elements

In this section we introduce a new kind of properly minimal elements by using the class of monotonically increasing sublinear functions. We define properly minimal elements as support points of a set with respect to the given set of functions, and present a theorem which characterizes minimal, weakly minimal and properly minimal elements of a set in terms of optimal solutions of the scalar optimization problem where no convexity and boundedness assumptions are required.

Definition 1 Let \( Y \) be a nonempty subset of \( \mathbb{R}^n \). An element \( \vec{y} \in Y \) is called a properly minimal element of \( Y \), if there exists a pair

\[
(w, \alpha) \in C^\# \quad \text{such that} \quad w\vec{y} + \alpha\|\vec{y}\|_1 \leq wy + \alpha\|y\|_1 \quad \text{for all} \quad y \in Y.
\]

Theorem 2 Let \( Y \subset \mathbb{R}^n \) be a given nonempty set and denote \( C = \mathbb{R}^+_n \). Let

\[
(w, \alpha) \in C^\alpha = \{((w_1, \ldots, w_n), \alpha) : 0 \leq \alpha \leq w_i, \quad w_i > 0, \ i = 1, \ldots, n\},
\]

and let \( \text{Sol}(SP) \) be the set of optimal solutions of the scalar optimization problem (SP):

\[
\min_{y \in Y} \{wy + \alpha\|y\|_1\}. \quad (1)
\]

Suppose that \( \text{Sol}(SP) \neq \emptyset \) for a given pair \((w, \alpha) \in C^\alpha\). Then the following hold.

(i) If

\[
(w, \alpha) \in C^w = \{((w_1, \ldots, w_n), \alpha) : 0 \leq \alpha \leq w_i, \quad w_i > 0, \ i = 1, \ldots, n \ \text{and there exists} \ k \in \{1, \ldots, n\} \ \text{such that} \ w_k > \alpha\},
\]

then every element of \( \text{Sol}(SP) \) is a weakly minimal element of \( Y \).

(ii) If \( \text{Sol}(SP) \) consists of a single element, then this element is a minimal element of \( Y \).

(iii) Every element of \( \text{Sol}(SP) \) is a properly minimal element of \( Y \) if

\[
(w, \alpha) \in C^\# = \{((w_1, \ldots, w_n), \alpha) : 0 \leq \alpha < w_i, \quad i = 1, \ldots, n\}.
\]

Conversely, if \( \vec{y} \) is a properly minimal element of \( Y \), then there exists a pair \((w, \alpha) \in C^\#\) and a point \( r \in \mathbb{R}^n \) such that \( \vec{y} \) is an optimal solution of the following scalar optimization problem:

\[
\min_{y \in Y} \{w(y - r) + \alpha\|y - r\|_1\}. \quad (2)
\]

Proof: (i) Let \((w, \alpha) \in C^w\). Assume, to the contrary, that there exists an element \( \vec{y} \in \text{Sol}(SP) \) which is not a weakly minimal solution of \( Y \). Then there exists \( y \in Y \setminus \{\vec{y}\} \) such that \( y \in \{\vec{y}\} - \text{int}(C) \). Since \((w, \alpha) \in C^w\), the function \( g_{(w, \alpha)}(y) = wy + \alpha\|y\|_1 \) is strictly monotonically increasing and therefore we have \( g_{(w, \alpha)}(y) < g_{(w, \alpha)}(\vec{y}) \), which contradicts to the hypothesis that \( \vec{y} \in \text{Sol}(SP) \).

(ii) The proof of this part is similar to that of part (i).

(iii) The result is obvious.
3 Conic Scalarization Method

Consider a multi-objective optimization problem (MOP):

$$\min_{x \in X} [f_1(x), \ldots, f_n(x)],$$

where $X$ is a nonempty set of feasible solutions and $f_i : X \to \mathbb{R}, i = 1, \ldots, n$ are real-valued functions. Let $f(x) = (f_1(x), \ldots, f_n(x))$ for every $x \in X$ and let $Y := f(X)$.

Let

$$C^a = \{(w_1, \ldots, w_n) : 0 \leq \alpha \leq w_i, w_i > 0, i = 1, \ldots, n\},$$

$$C^a_w = \{(w_1, \ldots, w_n, \alpha) \in C^a : \text{there exists } k \in \{1, \ldots, n\} \text{ such that } w_k > \alpha\},$$

$$C^a# = \{(w_1, \ldots, w_n, \alpha) : 0 \leq \alpha < w_i, i = 1, \ldots, n\}.$$

Choose preference parameters:

- Weight vector $w = (w_1, \ldots, w_n)$, where $w_i > 0$ denotes the preference degree of $i$th objective function for decision maker.
- Reference point $r = (r_1, \ldots, r_n)$. Such a point may be identified by a decision maker in cases when she or he desires to calculate minimal elements that are close to some point. The conic scalarization method does not impose any restrictions on the ways for determining reference points. These points can be chosen arbitrarily.

Choose an augmentation parameter $\alpha$ such that $(w, \alpha) \in C^a$ (or $(w, \alpha) \in C^a_w$, or $(w, \alpha) \in C^a#$).

The scalar problem for the given parameters $(w, \alpha)$ and $r$ is:

$$\text{(SP1)} \quad \min_{x \in X} \sum_{i=1}^{n} w_i(f_i(x) - r_i) + \alpha \sum_{i=1}^{n} |f_i(x) - r_i|$$

It is clear that in the case $\alpha = 0$ the objective function of the scalar optimization problem (SP1) becomes an objective function of the well-known weighted sum scalarization method. The minimization of such an objective function over feasible set enables to obtain some efficient solution $x$ of the problem (if the corresponding scalar problem has a solution), for which the minimal vector $f(x)$ is just a supporting point of the objective space with respect to the hyperplane

$$H(w) = \{y : wy = \beta\},$$

where $\beta = wf(x)$. It is obvious that minimal points which are not supporting points of the objective space with respect to some hyperplane, cannot be detected by this way. By augmenting the linear part in (SP1) with the norm term using a positive augmentation parameter $\alpha$, the hyperplane $H(w)$ becomes a conic surface defined by the cone

$$S(w, \alpha) = \{y \in \mathbb{R}^n : wy + \alpha \|y\| \leq 0\}.$$

This cone becomes smaller as $\alpha$ increases, and a smallest cone (corresponding to the given set of weights) is obtained when $\alpha$ equals its maximum allowable value $\min\{w_1, \ldots, w_n\}$ for the given set of weights. Since all these cones contain the ordering cone $-\mathbb{R}_+^n$, the presented scalarization method enables one to calculate minimal elements which are “supporting” elements of $Y$ with respect these conic surfaces. Therefore we arrive at the following conclusion. By solving problem (SP1) for “all” possible values of the augmentation parameter $\alpha$ between 0 and $\min\{w_1, \ldots, w_n\}$, one can calculate all the efficient solutions corresponding to the decision maker’s preferences (the weighting vector $w = (w_1, \ldots, w_n)$ and the reference point $r$).

4 Examples

In this section we present some demonstrative examples. Examples 1 and 2 demonstrate the strength of the conic scalarization method by emphasizing its ability to find all efficient solutions corresponding to decision maker’s preferences (objective functions weights and reference points). These special examples demonstrate that the conic scalarization method can also be used in the cases, when the problem is not convex and is not bounded and there does not exist neither an ideal point nor a nadir point.

Example 1. Consider the two-objective problem with $f_1(x_1, x_2) = x_1$, $f_2(x_1, x_2) = x_2$, where the set of feasible solutions is defined in the following form.
Then, the objective space can be defined in an analogous way by setting $Y_1 = \{(y_1, y_2) : y_1 \geq -2, y_2 \geq 2\}, Y_2 = \{(y_1, y_2) : y_1 \geq 1, y_2 \geq 1\}, Y_3 = \{(y_1, y_2) : y_1 \geq 2, y_2 \geq -1\}, Y_4 = \{(y_1, y_2) : y_1 \geq 3\},$ and then

$$Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4.$$ 

Points $(-2, 2), (1, 1)$ and $(2, -1)$ are minimal points of $Y$ which are not properly minimal, and all the boundary points of $Y$ are weakly minimal points. Computational results obtained by solving scalar problem (SP1) for different sets of preference parameters $w$ and $r$ and augmentation parameter $\alpha$ are presented in Table 1.

First and second rows of Table 1 depict efficient solutions obtained for the same $(w, \alpha) = ((2, 1), 1)$, and different reference points $(3, 1)$ and $(1, -1)$, respectively. Note that pair $(2, 1)$ belongs to $C^w\#$, reference point $(3, 1)$ is in the feasible region, and other reference point $(1, -1)$ does not belong to the feasible region of the problem. These results demonstrate that the conic scalarization method does not impose special requirements on conditions for selecting reference points.

Rows 3 to 6 of Table 1 again demonstrate efficient solutions obtained for same weighting and augmentation parameters $(w, \alpha) = ((3, 2), 2) \in C^w\#$ and different reference points from and outside of the feasible region. These rows nicely demonstrate the balance between weights and reference points provided by the conic scalarization method. For example, the reason for obtaining the same efficient solution for reference points $(1, 1)$ and $(3, 1)$, is that, these points are closer to efficient point $(1, 1)$ than to point $(2, -1)$. Reference point $(1, -1)$ is closer to efficient point $(2, -1)$, but the method again yields the efficient point $(1, 1)$. The reason is probably that, the weight of the first objective is sufficiently greater than the weight of the second objective, and this reference point is not too far to the the efficient solution $(1, 1)$ whose first coordinate is closer to the corresponding coordinate of the reference point. Since the ratio between the two weights is not sufficiently large, and reference point $(1, -2)$ is sufficiently far to efficient point $(1, 1)$, the set of preference parameters given on the eighth row of Table 1 yields the efficient point $(2, -1)$, which is more closer to the reference point than point $(1, 1)$.

Note that for every $(w, \alpha) \in C^w\#$ the objective function of scalar problem (SP1) becomes unbounded from below. This is a nice interpretation of the fact that, the problem in this example has no properly efficient solutions, and by Theorem 2 (iii) for every pair $(w, \alpha) \in C^w\#$, any solution of scalar problem (SP1) must be properly efficient.

And finally, note that any attempt to solve the scalar problem for pairs $(w, \alpha)$ with $w_i > 0, i = 1, 2$, and $\alpha = 0$ leads to the unbounded from below objective function. The only pair of parameters $(w_1, w_2)$ for weighted sum scalarization method (that is for the case $\alpha = 0$) is $(w_1, w_2) = (1, 0)$, for which the solution set gives efficient solutions $\{(-2, y_2) : y_2 \geq 2\}$. □

Example 2. Consider the two-objective problem with $f_1(x_1, x_2) = x_1$, $f_2(x_1, x_2) = x_2$, where the set of feasible solutions is defined in the following form. $X_1 = \{(x_1, x_2) : x_2 \geq -(1/3)x_1 + 11/3\}$, $X_2 = \{(x_1, x_2) : x_2 \geq -x_1 + 3\}$, $X_3 = \{(x_1, x_2) : x_2 \geq -2x_1 + 4\}$, and $X = X_1 \cup X_2 \cup X_3$.

Then, the objective space can be defined in an analogous way by setting $Y_1 = \{(y_1, y_2) : y_2 \geq -(1/3)y_1 + 11/3\}$, $Y_2 = \{(y_1, y_2) : y_2 \geq -y_1 + 3\}$, $Y_3 = \{(y_1, y_2) : y_2 \geq -2y_1 + 4\}$, and then

$$Y = Y_1 \cup Y_2 \cup Y_3.$$ 

Note that all boundary points of the set $Y$ are properly minimal. Computational results obtained for different sets of preference parameters and the augmentation values by solving the problem (SP1) for this example, are presented in Table 2.
The result obtained for the scalarizing parameters \((w, \alpha, r) = ((5, 4), 2, (0, 0))\) presented in the first row of this table shows that this set of parameters does not yield an optimal solution to the problem (SP1). The reason is that, the set of parameters \((w, \alpha) = ((5, 4), 2)\) leads to the cone which is too “large”, and therefore the corresponding scalar problem becomes unbounded from below (like to the case with \(\alpha = 0\)). By increasing the value of the augmentation coefficient \(\alpha\), the same set of weights yields different efficient solutions for different reference points (see rows 2 to 5 of Table 2). Note that these reference points can be chosen arbitrarily. In this example, points \(r = (0, 0), r = (0, 2), \) and \(r = (-1, 3)\) are not in the feasible set, point \(r = (2, 2)\) is in the interior of the feasible set. In all cases the minimal points which are (in some sense) close to the reference points are obtained.

An interesting case has been obtained for reference point \(r = (2, 2)\). The set of properly efficient solutions corresponding to the parameter set \((w, \alpha, r) = ((5, 4), 3, (2, 2))\) coincides with the whole segment of properly efficient solutions connecting points \((1, 2)\) and \((2, 0)\). This is the case when all points of this segment are supporting points of the set \(Y\) with respect to the cone representing the sublevel set of the objective function of the problem (SP1), shifted to the reference point \(r = (2, 2)\).

The rows 6 to 9 of Table 2 represent different set of efficient solutions corresponding to the same set of weights for different reference points.

And finally, rows 9 and 10 demonstrate that different efficient solutions can be obtained for the same augmentation parameter and the same reference point, when the weights of the objectives are slightly changed.

\[\square\]

5 Conclusion

In this paper, the conic scalarization method for multi-objective optimization problems is presented. The following features of this method can be emphasized:

- This method always yields efficient, weakly efficient, or properly efficient solutions if the corresponding scalar problem has a finite solution. By choosing a suitable scalarizing parameter set consisting of a weighting vector, an augmentation parameter, and a reference point, the decision maker may guarantee a most preferred efficient or properly efficient element.

- All the efficient solutions can be detected in this way.

- The preference and reference point information of decision maker is taken into consideration by this technique.

- The presented method does not need the boundedness and convexity assumptions on the problem under consideration. In addition, the ideal and nadir points are not needed for application of the conic scalarization method.

Although the different augmentations in different scalarization methods have earlier been considered, the main difference in the formulation of a class of scalarizing functions presented in this paper is that the scalarizing parameters consisting of a weighting vector and an augmentation coefficient are chosen not randomly but from an augmented dual cone providing a special relation between these parameters.

New features of the nonlinear cone separation theorem given in [9] are also developed in this paper. A new notion of a separable cone has been suggested, which simplifies the formulation of the separation property. Additionally, the separability properties of \(\mathbb{R}^n_+\) with respect to different norms are also discussed and several types of augmented dual cones have been calculated explicitly.

References:


