On Oscillations Damping and Control for a Class of Affine Systems

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Abstract: - The paper considers a class of continuous affine systems with oscillating behaviour due to the presence of pairs of complex eigenvalues in the spectrum of the system matrix. The control problem consists of damping the oscillations and tracking a piecewise constant reference signal. A control solution is proposed, based on pole placement combined with the internal model principle. Sufficient conditions for the controller existence are deduced and some issues concerning fixed and variable step simulations approaches are discussed. The results can be extended to piecewise affine hybrid systems, composed of a set of piecewise affine systems and a switching strategy based on a state space partition.

Key-Words: - affine systems, controllability, internal model, tracking, fixed step discretization, simulation.

1 Introduction and motivation

There is an increasing interest, in recent years, in the study of hybrid systems, describing the interaction between time-driven and event-based dynamics. Piecewise affine systems represent a class of hybrid systems frequently encountered as modelling approximations of a large class of nonlinearities [1] arising both in control engineering [2], [3] and biological modelling [4], [5]. The control of oscillations is an important objective for hybrid piecewise affine systems, generally associated with driving quality and comfort in the evolution of automotive systems [6].

In order to analyze and control complex piecewise affine nonlinear dynamics, a prior study of affine systems is relevant for the description of the local behaviour of the hybrid system. This paper proposes a control approach for a single affine system, ensuring oscillations damping and the tracking of a piecewise constant reference input. The two objectives are reached, under certain assumptions, using a classic pole placement procedure combined with the internal model principle.

The paper is structured as follows. Section 2 contains a description of the affine model and the formulation of the control problem, followed by the presentation of a control system structure, as a solution to a related control problem, whose existence implies stronger conditions for affine the process. Section 3 presents some simulation issues for a simple example, followed by the formulation of concluding remarks.

2 A Control Problem for an Affine System

This section introduces a control problem for a single continuous affine system, which represents a starting point for developing a control approach for a piecewise affine hybrid system.

2.1 The affine model and the control problem

Consider the continuous-time affine system
\[ \dot{x} = Ax + bu + f, \quad y = c^T x, \]
with \( x \in \mathbb{R}^n \) the continuous state, \( u \) a scalar-valued input signal, \( y \) the scalar output and \( A, b(\neq 0) \) and \( f(\neq 0) \) real matrices of appropriate dimensions, respectively. Denote \( \Lambda(A) \) the set of eigenvalues of \( A \).

Assumption 1 The matrix \( A \) has at least one pair of complex eigenvalues, i.e. \( \exists (\alpha \pm j\beta) \in \Lambda(A) \).

Assumption 2 The pair \( (A,b) \) in (1) is controllable. The pair \( (c^T, A) \) in (1) is observable.
Consider also a class of piecewise-constant exogenous signals \( r : [0, T] \rightarrow \mathbb{R} \),

\[
  r(t) = \begin{cases} 
  r_i, & 0 \leq t < t_i \\
  \vdots \\
  r_i, & t_{i-1} \leq t < t_i \\
  \vdots \\
  r_s, & t_{s-1} \leq t \leq T
  \end{cases}
\]  

(2)

where \( r_i \in \mathbb{R}, \ i = 1 \ldots s \), and the time intervals between any two consecutive switching moments \( t_i, \ i = 1 \ldots s - 1 \), may not be equal, respectively.

\textbf{Problem 1} Given the affine system (1) satisfying Assumption 1 and Assumption 2 and an exogenous signal \( r \) (2), find a control law \( u(x,r) \) s.t. in closed loop: (i) the system is stable and the oscillations of the output \( y \) are damped and (ii) the output \( y \) tracks the reference signal (2).

In view of Assumption 2, a simple theoretical approach for the oscillations damping problem is the feedback pole placement. However, the second part of the control objective, i.e. the reference tracking, cannot be reached only by a feedback law ensuring an adequate pole placement, or by a static controller.

More detailed, consider \( k_o^T \in \mathbb{R}^{\text{lex}} \) a feedback matrix s.t. for the closed loop matrix \( A_o = A + bk_o^T \) the property

\[
  \forall \lambda \in \Lambda_o \overset{\text{def}}{=} \Lambda(A + bk_o^T) \Rightarrow \lambda < 0,
\]  

(3)

holds. Denote \( x(s) = L[x(t)] \), \( y(s) = L[y(t)] \) the Laplace transformed of the state and output in (1), respectively. Then, for zero initial conditions, the state evolution of (1) with the feedback law \( u = k_o^T x + \varphi \varepsilon \), where \( \varepsilon = r - y \), is given by

\[
  x(s) = (sl - A_o)^{-1} \left[ b \varphi \varepsilon(s) + f \cdot \frac{1}{s} \right],
\]  

(4)

and, for \( r(s) = r_i / s \), the Laplace transformed output is

\[
  y(s) = \frac{1}{s} \cdot c^T (sl - A_o)^{-1} \left[ b \cdot \varphi r_i + f \right],
\]  

(5)

which yields to a nonzero stationary error \( e(\infty) = \lim_{s \to 0} s[r(s) - y(s)] \neq 0 \). Hence the pole placement strategy has to be enriched with a dynamic control strategy dedicated to the tracking problem.

\textbf{Remark 1} Note that condition (3) is stronger than condition (i) in Problem 1. In order to achieve the first part of the control objective, Assumption 2 can be relaxed: the oscillations can still be damped in closed loop if not all eigenvalues of \((A,b)\) are controllable, but the uncontrollable eigenvalues are real and stable. In this case, the uncontrollable “part” produces neither oscillations nor instability and, after a structural decomposition, the feedback matrix \( k_o^T \) - with appropriate reduced dimensions – can be applied only to the controllable part, assumed to generate output oscillations.

\subsection*{2.2 The control structure and the extended system}

Consider the control structure depicted in Fig.1, with the state equations

\[
  \dot{x} = Ax + bu + f \]

(5)

\[
  \dot{z} = -c^T x + r
\]

~

Fig. 1. The control structure for the affine system in Problem 2.

Based on the expression (2) of the reference signal, composed of delayed step signals, the second equation in (5) defines, for the controller dynamics, the \textit{internal model} of the step signal – with \( L[1(t)] = 1/s \) – as an integrator of the error \( \varepsilon = r - y \), with \( y = c^T x \).

Define the extended affine system of order \((n+1)\)

\[
  \dot{x}_e = A_e x_e + b_e u + f_e + g_e r,
\]

(6)

where \( x_e = [x^T \ z]^T \) is the extended state, \( u \) is the control input, \( r \) is the reference signal to be tracked and the system matrices are

\[
  A_e = \begin{bmatrix} A & \begin{bmatrix} 0_{m \times n} \end{bmatrix} \\ -c^T & 0 \end{bmatrix}, \ b_e = \begin{bmatrix} b \\ 0 \end{bmatrix}, \ f_e = \begin{bmatrix} f \\ 0 \end{bmatrix}, \ g_e = \begin{bmatrix} 0_{m \times d} \\ -c \end{bmatrix}.
\]  

(7)
Consider the following problem associated to Problem 1.

**Problem 2** Given (i) the system (6)-(7), with \((A,b,c^T)\) satisfying Assumption 2 and

\[
\det \begin{bmatrix} A & b \\ -c^T & 0 \end{bmatrix} \neq 0
\]  

and (ii) a set \(\Lambda_e\) of \((n+1)\) strictly negative real numbers, find a control law

\[
u_k^e \in \Lambda_e
\]  
s.t. \(\Lambda(A_e + bk_e^T) = \Lambda_e\).

In order to show that solving Problem 2 drives to a solution to Problem 1, two aspects have to be proved: firstly, the fact that the pole placement for \((A_e,b_e,0)\) is possible, provided that the first part of Assumption 2 is true and, secondly, the fact that the control system (6)-(7) with the control law (8) can track a reference signal from the class (2), i.e. that no steady state error occurs.

The answer to the first sub-problem is given by the next result.

**Lemma 1** If the pair \((A,b)\), with \(A \in \mathbb{R}^{n \times n}\) and \(b \in \mathbb{R}^n\), is controllable and the triplet \((A,b,c^T)\) satisfies (8), then the pair \((A_e,b_e)\), defined in (7), is also controllable.

**Proof:** Controllability is equivalent, for single input single output systems, to the nonsingularity of the controllability matrix. Hence, the controllability matrix of the pair \((A,b)\), defined by

\[
R = \begin{bmatrix} b \\ Ab \\ \vdots \\ A^{n-1}b \end{bmatrix}
\]  
is non-singular. The controllability matrix of the pair \((A_e,b_e)\) is defined by

\[
R_e = \begin{bmatrix} b \\ Ab \\ A^2b \\ \vdots \\ A^{n}b \\ 0 \\ -c^Tb \\ -c^TAb \\ \vdots \\ -c^T A^{n-1}b \end{bmatrix}
\]

The above matrix can be written in the form

\[
R_e = \begin{bmatrix} b \\ AR \\ 0 \\ A^2R \\ \vdots \\ A^{n-1}R \\ 0 \\ -c^T \\ -c^TAb \\ \vdots \\ -c^T A^{n-1}b \end{bmatrix}
\]

which, in view of (8) and of \(\det R \neq 0\), results also non-singular.

In order to show that the extended system (6) with a control law (9) can track an exogenous reference signal from the class (2), consider a step signal \(r(t) = r_1 \cdot 1(t)\) and compute the steady state error in Fig.1.

Denote \(y(s) = L\{y(t)\}, u_1(s) = L\{u_1(t)\}\), \(\varepsilon(s) = L\{\varepsilon(t)\}\) and \(r(s) = L\{r(t)\}\) the Laplace transformed of the corresponding time signals in Fig.1, respectively. Also, note that the spectrum \(\Lambda(A_e) = \Lambda(A+bk_e^T) \subset \Lambda_e\) is stable. Then, for zero initial conditions \(x_e(0) = 0\), the Laplace transformed of the affine system state in Fig. 1 is

\[
x(s) = (sI - A_e)^{-1} \left[ bu_1(s) + f \cdot \frac{1}{s} \right],
\]

with

\[
u_1(s) = L\{u_1(t)\} = \frac{s}{s} \cdot \varepsilon(s) = \frac{s}{s} \cdot [r(s) - y(s)].
\]

After some computation,

\[
y(s) = \frac{c^T (sI - A_e)^{-1}b \cdot \varphi(s) + f}{s + c^T (sI - A_e)^{-1}b \cdot \varphi}.
\]

Then, for \(r(t) = r_1 \cdot 1(t)\) with \(r(s) = r_1/s\),

\[
y(\infty) = \lim_{s \to 0} sy(s) = r_1.
\]

and the steady state error is

\[
\varepsilon(\infty) = \lim_{s \to 0} s\varepsilon(s) = r_1 \cdot [1 - 1] = 0.
\]

**Remark 2** A computation of the stationary state value, similar to the one presented in (4) and (5), can be performed for the affine extended system (6), resulting that \(c^T x_e(\infty) = y(\infty) = r_1\) and \(z(\infty) \neq 0\).

In conclusion, a solution to Problem 2 is a solution to Problem 1 or, solving Problem 2 is a sufficient condition for solving Problem 1.

Summing up, the procedure for solving Problem 2 has, as input data, the \(n\)-th order model (1), the reference signal (2) and a desired closed loop real stable spectrum \(\Lambda_e = \{\lambda_i \mid \lambda_i < 0\}_{i=2}^{n+1}\) and outputs the feedback matrix \(k_e^T = [k_e^T \varphi] \) in (9). If the conditions in Lemma 1 are satisfied, then the matrices (7) of the extended system are built and for the pair \((A_e,b_e)\) a classic pole-placement procedure is applied, s.t. \(\Lambda(A_e + bk_e^T) = \Lambda_e\).
Remark 3 If \( (A,b) \) in (1) is not controllable, but the uncontrollable eigenvalues of \( A \) are real and stable then, when solving Problem 2, Remark 1 can be taken into account. Also, if the condition (8) is not satisfied, for example due to the sensors placement (modelled by the output matrix \( c^T \)), then a reconfiguration of the sensors system can be tried.

3 A Simulation Example

Consider the second order affine system (1), with

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & -2\xi \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ 10 \end{bmatrix}
\]  
\( c^T = [0 \ 1], \ 0 < \xi < 1. \)  

Note that \( A \) is already stable.

The output of the system defined by (1) and (18) has to become stable and aperiodic and to track the signal \( r: [0, T] \rightarrow \mathbb{R} \),

\[
r(t) = \begin{cases} r_1, & 0 \leq t < t_1 \\ r_2, & t_1 \leq t < t_2 \\ r_3, & t_2 \leq t \leq T \end{cases}
\]  

3.1 The continuous-time solution

The matrices defined in (18) satisfy condition (8). The matrices of the extended system (6) are

\[
A_e = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2\xi & 1 \\ -1 & 0 & 0 \end{bmatrix}, \quad b_e = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},
\]

\[
f_e = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}, \quad g_e = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]  

and the desired closed loop spectrum is chosen

\[
A_e = \{-1, -2, -10\}.
\]  

Using the MATLAB function place, the resulting extended feedback matrix is, for \( \xi = 0.1 \)

\[
k_e^T = [k_o^T \ 1 \ 0] = [-31 \ -12.8 \ 20].
\]  

3.2 The simulation model

The general approach for the study of continuous systems with switched inputs is to build simulation models based on the fixed step time discretization of the continuous models, in order to avoid possible Zeno behaviour [7]. In this regard, an already classic example in the literature of automotive control is the mixed logical dynamical (MLD) approach discussed in [8].

Following this philosophy, denote \( h > 0 \) the sampling step and \( k = 0, 1, \ldots \) the discrete time variable. The discrete-time model of (1) is

\[
x_d(t+1) = A_d x_d + b_d u_d + f_d, \quad y_d = c_d^T x_d,
\]  

where, introducing the matrices in (18),

\[
A_d = \exp(A h), \quad b_d = \int_0^h \exp(A \theta) d \theta \cdot b,
\]

\[
f_d = f \cdot h, \quad c_d^T = c^T,
\]  

and, for any integer time \( k > 0 \), the sampled signals are

\[
x_d(k) = x(kh), \quad u_d(k) = u(kh),
\]  

Similarly, consider the extended system (6) with the matrices (20) and the control law \( u_e = k_e^T x_e \) specified in (22). The closed loop system is

\[
\dot{x}_e = (A_e + b_k k_e^T) x_e + f_e + g_e r.
\]  

Denote \( A_{oe} = A_e + b_k k_e^T \). The simulation model of (26) is the discrete time affine system

\[
x_{ed}(t+1) = A_{oed} x_{ed} + f_{ed} + g_{ed} r_d,
\]

\[
y_d = c_{ed}^T x_{ed},
\]

where, using the matrices given in (20),

\[
A_{oed} = \exp(A_{oe} h), \quad f_{ed} = f_e \cdot h,
\]

\[
g_{ed} = g_e \cdot h, \quad c_{ed}^T = c_e^T,
\]  

and, for any integer time \( k > 0 \), the sampled signals are

\[
x_{ed}(k) = x_e(kh), \quad r_d(k) = r(kh).
\]  

The following simulation values have been used:

\[
T = 30, \quad h = 0.001, \quad r_1 = 0, \quad r_2 = 2, \quad r_3 = 1,
\]

\[
t_1 = 5, \quad t_2 = 15, \quad \xi = 0.1
\]  

The open loop evolution of affine system (23) with the discrete input signal associated to (19) is
depicted in Fig.2, showing the oscillating behaviour and suggesting an asymptotic nonzero error.

![Simulation of the discrete affine system (23) with sampled piecewise constant input (19) and parameters (29).](image)

The closed loop evolution of the system (26) with the same discrete input signal associated to (19) is depicted in Fig.3. The oscillation on the first time interval [0, 5] is due to $r_1 = 0$. For the rest of the simulation time interval, the output has an aperiodic behaviour.

![Simulation of the discrete control system (26) with sampled piecewise constant input (19) and parameters (29).](image)

In what concerns the simulation, for arbitrary initial state conditions, of the free continuous affine systems, variable step methods can be successfully applied. Fig.4 (up) represents the phase portrait of the free discrete affine system (23), and the trajectory evolves towards the stationary point of the continuous affine stable system (1), (18), given by

$$x(\infty) = \lim_{s \to 0} s x(s) = (-A)^{-1} f = \begin{bmatrix} 0.2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

On the other side, the phase portrait depicted in Fig.4 (down), associated to the first two components of the free extended state vector in (26) - the controlled affine process -, with matrices given in (20) and (22), is obtained by using the MATLAB routine ode45, based on the Runge-Kutta method. However, the simulation results of the discrete affine systems and of the corresponding continuous time model, simulated with variable step integration methods, may slightly differ, due to the approximation introduced by the sampling procedure.
4 Conclusion

The study of affine systems, as subsystems of piecewise affine hybrid systems, is important for understanding the overall complex system behaviour. Also, the presence of the affine term imposes specific control approaches, which slightly differ from the classic linear control strategies. The control problem considered in this paper has an objective frequently encountered in the automotive systems literature: oscillations damping and reference tracking.

The proposed control approach is based on the controllability of the classic linear system associated to the affine system and implies a combination of a classic pole placement procedure – driving to a closed loop stable and aperiodic behaviour – and of the internal model principle. The two strategies are unified by introducing the extended affine system model, which aggregates the state equations of the affine process and of the internal model, respectively. Then, provided that an additional rank condition is fulfilled – related to the controllability of the extended system – it is proved that an extended pole placement procedure can be applied and implemented into the control system. A control problem implying sufficient conditions for the solution of the original control problem is introduced and solved by the design of a feedback pole placement matrix.

The simulation experiments are based on fixed step discretized models associated to the continuous time original affine systems. However, special care has to be taken when interpreting the results, due to the approximation introduced by the sampling procedure.

A future research direction is the study of implementing the controller using estimators. Also, alternatively to the pole placement approach, the feedback matrices in the controller can be synthesized as solutions of optimal problems associated to the classic linear part of the affine system –simulations have shown that an appropriate choice of the matrices in the integral criterion may drive to damped oscillations. Finally, the proposed control approach can be extended to piecewise affine hybrid systems.

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