A way to a new multi-spectral transform

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Abstract: We introduce fundamentals of a new selective transform which is purposely called the Zolotarev transform. The paper contains innovative and original methods and algorithms for multi-spectral analysis of non-stationary signals using Zolotarev polynomials.

Key–Words: Multi-spectral transforms, Zolotarev polynomials, Selective sine and selective cosine functions

1 Introduction

There are several approaches available for addressing non-stationary signals [2], including the short time Fourier Transform (STFT), the Wavelet Transform (WT), and the Hilbert-Huang Transform (HHT). A large group of non-linear multi-resolution transforms is also available. However, these approaches suffer from the difficult interpretation. Therefore, the STFT, WT, and HHT are the most frequently used tools for multi-component analysis of non-stationary signals. Several approaches for addressing non-stationary signals are solely based on statistical algorithms [3]. We consider the reversibility of the signal analysis as a fundamental criterion for discriminating these methods. The most essential reversible approach is application of the Discrete Fourier Transform (DFT) to a moving windows along the whole signal called the Short Time Fourier Transform (STFT). The main advantage of the STFT is the existence of an efficient algorithm for forward and backward numerical evaluation, the Fast Fourier Transform (FFT), and intuitive interpretation of the results. As a disadvantage we have to notice that the STFT has time resolution that significantly varies with length of the window. The Wavelet Transform (WT) was invented to improve addressing of non-stationary signals and have obtained an important place between engineering methods. The advantage of the transform is mainly good data compression but disadvantages are in their frequency resolution and in the fact that the performance depends on a proper choice of the mother wavelet which influences also reversibility of the transform. Irreversible methods are often represented by algorithms estimating instantaneous frequency (IF). As the well known representative we present the Hilbert-Huang transform (HHT). The HHT iteratively decomposes the signal into a sum of sub-signals called empirical modes (EM). The EM are signals possessing of instantaneous, positive only, frequency which do not guarantee physically reasonable decomposition. In the second step the IF of each EM is calculated by the Hilbert transform. Another algorithm of IF estimation use the Wigner distribution (WD) with varying and data-driven window length [3]. The algorithm calculates WD, the function of IF, for an increasing sequence of the window length values. The estimated IF is found as the calculated WD for the longest window satisfying estimation error boundaries. An advantage of the algorithm is well estimated IF, however the backward algorithm does not exist. Researchers have devoted enormous efforts to the WT and STFT. Numerous relevant publications confirm that this has been a field of considerable interest for decades [4], [7], [6], etc. In our approach, we propose to overcome the mentioned disadvantages by developing fundamentals for the Discrete Zolotarev Transform (DZT) and its approximation, called the Approximate Discrete Zolotarev Transform (ADZT). Both transformations are built on approximation theory as a reversible time-frequency transform exploiting advantages of the irreversible algorithms for analysis of non-stationary signals. Here we restrict our explanation to the main results only, trying to avoid a complete mathematical treatment of Zolotarev polynomials.
2 Zolotarev polynomials for a selective transform

Zolotarev polynomials of the first and second kind are related to polynomials of the form

\[ f(x) = x^n - \beta(n-2)x^{n-2} + \cdots + \beta(1)x + \beta(0), \]

which deviate least from zero in two disjoint intervals where \( \sigma \) is a given real number. We have confined our polynomials to the intervals \((-1, w_s) \cup (w_p, 1)\) by the linear transformation of \( x \)

\[ w = x \frac{1 + w_s}{2} - \frac{1 - w_s}{2}, \]

and have developed a complete analytic procedure for expansion of Zolotarev polynomials of the first kind through Chebyshev polynomials \( T_m(w) \),

\[ Z_{p,q}(w|\kappa) = \sum_{m=0}^{n} a(m) T_m(w) \]

where \( n = p+q \) denotes degree, and \( p \) and \( q \) represent number of zeroes to the right and to the left from the main lobe, resp. - Fig.1. This analytic procedure was developed in order to replace the standard parametric representation of these polynomials [8]

\[ x(u) = \frac{\text{sn}^2(u|\kappa) + \text{sn}^2(\frac{p}{n} K(\kappa)|\kappa)}{\text{sn}^2(\frac{p}{n} K(\kappa)|\kappa)} \]

\[ y(u) = \frac{(-1)^p}{2} \left[ \left( \frac{H(u - \frac{p}{n} K(\kappa))}{H(u + \frac{p}{n} K(\kappa))} \right)^{\frac{n}{2}} + \left( \frac{H(u + \frac{p}{n} K(\kappa))}{H(u - \frac{p}{n} K(\kappa))} \right)^{\frac{n}{2}} \right], \]

where \( H(u \pm \frac{p}{n} K(\kappa)) \) is the Jacobi eta function, \( \text{sn}(u|\kappa) \) is the Jacobi elliptic sine function, and \( K(\kappa) \) is the complete elliptic integral of the first kind of modulus \( \kappa \). Using this notation we can write the edges as

\[ w_p = 2\text{sn}^2(\frac{q}{n} K(\kappa)|\kappa) - 1 \]

\[ w_s = 1 - 2\text{sn}^2(\frac{p}{n} K(\kappa)|\kappa). \]

The Zolotarev polynomial of the second kind was derived [5] in the quest for the complete selective basis of a vector space suitable for non-stationary signal decomposition. In 1997, we discovered the algebraic form of Zolotarev polynomials refraining from a parametric representation, and developed an extremely efficient algorithm for evaluating them. These methods allow computation of expansion coefficients for

Zolotarev polynomials of the first and second kind \( Z_{p,q}(w|\kappa) \) in terms of power series expansion and expansion into Chebyshev polynomials. In contrast to power series representation, the Chebyshev polynomial approach leads to coefficients valued in an astonishingly small range. The algorithms are of linear complexity with respect to the polynomial order, and are robust enough to easily generate tens of thousands of degree polynomials. Generally, Zolotarev polynomials are selective, iso-extremal and we can equivalently call them selective sine and selective cosine functions. We deal especially with symmetrical specifications for which Zolotarev polynomials possess interesting unique properties. Zolotarev polynomials are composed of a weighted series of Chebyshev polynomials of the first kind, and the second kind, respectively.

2.1 Selective Zolotarev cosine

Zolotarev polynomials of the first kind \( Z_{p,q}(w|\kappa) \) are related to the symmetrical selective cosine \( zcos \) by the simplification

\[ Z_{m,m}(w|\kappa) = (-1)^m T_m \left( \frac{2u^2 - 1 - k'^2}{1 - k'^2} \right) \]

if we define

\[ k' = \frac{1 - \kappa'}{1 + \kappa'} \]

This is an even iso-extremal function with one central lobe, as depicted in Fig.2. \( zcos \) is determined by
Figure 2: Selective cosines of the 5th degree, for $k' \equiv w_p = 0, 0.1, 0.15$.

two parameters: degree $m$ and selectivity $k'$. The degree stands for a multiple of the period and the selectivity represents a new parameter defining the shape of the central lobe. The width $2k'$ and the height $y_m = Z_{m,m}(0|\kappa)$ of the lobe are interconnected by

$$y_m = \frac{1}{2} \left[ \left( \frac{1 + k'}{1 - k'} \right)^m + \left( \frac{1 - k'}{1 + k'} \right)^m \right].$$

For our purposes, there are two important points $w_p \equiv k'$ and $-w_p \equiv -k'$ where the symmetrical Zolotarev polynomial exceeds a value of ±1. As $z\cos$ is symmetrical, its selectivity can be determined by one parameter $k'$. Provided that $k' = 0$ the selective cosine is equal to a standard cosine function. By increasing the selectivity the central lobe grows and shifts zeros

$$w_p^2 = k'^2 + k^2 \cos^2 \frac{2\mu - 1}{4m} \pi, \ \mu = 1 \ldots m.$$  

to the edges of the intervals $(-1, w_s) \cup (w_p, 1)$. Symmetrical Zolotarev polynomials are expressed as a finite sum of Chebyshev polynomials.

$$Z_{m,m}(w|\kappa) = (-1)^m T_m \left( \frac{2w^2 - 1 - k'^2}{1 - k'^2} \right) = \sum_{\mu=0}^{m} a(2\mu)T_{2\mu}(w).$$  

This polynomial is of even degree $2m$. The algorithm uses even coefficients $a(2\mu)$ only, and the odd coefficients are equal to zero. Equation (11) can be written using cosine notation

$$z\cos(N, w_p) = \sum_{\mu=0}^{N} a_{2\mu}(w_p) \cos(2\pi\mu n).$$

From the spectral point of view, every $z\cos$ can be decomposed to stationary and non-stationary parts, see Fig.3. The stationary part is a standard cosine of degree $N$ - Fig.3b), and the non-stationary part resembles a specific window function - Fig.3c). This window dynamically adapts to the non-stationarity of the signal, so it is not advisable to use another weighting function.

Figure 3: Decomposition of a selective cosine (left), real parts of the Fourier spectrum (right). Selective cosine of $N = 6, w = 0.2$, its stationary part, and non-stationary part.

### 2.2 Selective Zolotarev sine

The selective Zolotarev sine is related to Zolotarev polynomials of the second kind. Since a sine is an odd function, the basis function $z\sin$ is described by two central lobes, and consequently there are left and right symmetrical crossing points $(\mp w_s, w_p)$, see Fig.4. Again, $z\sin$ is determined by the degree and selectivity. If $w_p = w_s$ is valid, $z\sin$ is equal to a standard sine. These 4 points are mutually interconnected, and therefore it is possible to choose for selectivity a single one, e.g. $w_p$. A formula for $z\sin$ evaluation (13) utilizes a sine series which has only odd coefficients $b_{2\mu-1}(w_p)$.

$$z\sin(N, w_p) = \sum_{\mu=1}^{N} b_{2\mu-1}(w_p) \sin(2\pi\mu n).$$

The spectrum of $z\sin$ can also be decomposed to stationary and non-stationary parts, see Fig.5. The $z\sin$ spectrum has the imaginary part of the Fourier spectrum. Fig.5 also shows that $z\sin$ consists of the standard sine function (see Fig.5b) and the specific
window (see Fig.5c) which can adapt to the non-stationarity of the signal.

Figure 5: Decomposition of a selective sine (left), imaginary part of the Fourier spectrum (right). Selective sine of $N = 5$, $w = 0.3538$, its stationary part, and non-stationary part.

### 3 Zolotarev series

Selective basis functions $z\cos$ and $z\sin$ with their spectrum belong to the class of time-limited (periodic) signals with a discrete spectrum. This means we can use them for forming Zolotarev series in a similar way as we form the Fourier series. It is familiar that the Fourier basis consists of the complex exponential

$$\exp(i2\pi\ell t) = \cos(2\pi\ell t) + i \sin(2\pi\ell t), \ \ell \in \mathbb{Z}. \ (14)$$

Then a band-limited signal $s(t)$ can be expressed by the coefficients of Fourier series $S(\ell)$, which represent the frequency spectrum of the signal

$$s(t) = \sum_{\ell=-N}^{N} S(\ell) \exp(i2\pi\ell t), \quad (15)$$

where

$$S(\ell) = \langle \exp(-i2\pi\ell t), s(t) \rangle = 0, \ \forall |\ell| > N. \quad (16)$$

The previous chapter introduced the relation of the $z\cos$ and $z\sin$ functions to the Zolotarev polynomials. We replace the Fourier basis (14) by these functions and obtain following form, which represents the Zolotarev basis

$$z\exp(\ell, i2\pi t) = z\cos(\ell, 2\pi t) + i z\sin(\ell, 2\pi t)$$

$$= \sum_{\mu=-\ell}^{\ell} a_{2\mu}^2 \cos(2\pi\mu t) + \sum_{\mu=-\ell}^{\ell} b_{2\mu-1} \sin(2\pi\mu t)$$

$$= \sum_{\mu=-\ell}^{\ell} c_{2\mu}^2 \exp(2\pi\mu t), \quad (17)$$

where we use normalized coefficients $a_{2\mu}^2, b_{2\mu-1}$ and $c_{2\mu}^2$, for example

$$\gamma_{r,\ell} a_{2\mu}^2 = a_{2\mu}^2 \text{ where } \gamma_{r,\ell}^2 \sum_{\mu=0}^{\ell} a_{2\mu}^2 = 1.$$

Now we can replace the Fourier basis (14) by the new Zolotarev basis (17) and express a band-limited signal $s(t)$ (15) as

$$s(t) = \sum_{\ell=-N}^{N} S_Z(\ell) z\exp(\ell, i2\pi t), \quad (18)$$

where the function $z\exp(\ell, i2\pi t)$ is biorthogonal to $z\exp(\ell, i2\pi t)$ (17) [5]

$$\{z\exp(i2\pi\ell t)\}_{\ell=-N}^{N} = \{z\exp(i2\pi\ell t)\}_{\ell=-N}^{N}^{-1} \quad (19)$$

$S_Z(\ell)$ represents spectral coefficients of the Zolotarev spectrum defined by

$$S_Z(\ell) = \langle z\exp(\ell, -i2\pi t), s(t) \rangle = 0, \ \forall |\ell| > N. \quad (20)$$
Using (15) and (17), the definition of \( S_Z(\ell) \) (20) can be rewritten as

\[
S_Z(\ell) = \sum_{n=-\ell}^{\ell} c_{2n}' \exp(2\pi nt), s(t) = \sum_{n=-\ell}^{\ell} c_{2n}' S(n),
\]

which expresses the calculation of the \( \ell \)th Zolotarev spectrum coefficient as a weighted sum of spectral coefficients up to the \( \ell \)th order with a weight defined by \( c_{2m}' \). The relation between coefficients of the Fourier spectrum and the Zolotarev spectrum can be expressed by a matrix form as

\[
S_Z = Z \cdot S,
\]

where the vector \( S \) contains coefficients of Fourier spectrum

\[
S = [S(0), \ldots, S(N), S(-N), \ldots S(-1)]^T
\]

and \( S_Z \) denotes a vector consisting of spectral coefficients of the Zolotarev spectrum

\[
S_Z = [S_Z(0), \ldots, S_Z(N), S_Z(-N), \ldots S_Z(-1)]^T,
\]

where \( N \) is the order of the respective spectrum. \( Z \) is a square complex regular matrix of the \((2N+1)\)th order which contains coefficients \( c_{2m}' \) of the Zolotarev basis (17). If we use notation of the common Fourier basis functions \( W \)

\[
W = \{ \exp(i2\pi \ell t) \}_{\ell=-N}^{N}
\]

then using (22) and (25), the band-limited signal \( s(t) \) (18) can be expressed as

\[
s(t) = (Z \cdot W(t))^{-1} \cdot S_Z
\]

4 A Way to Approximate the Discrete Zolotarev Transform

Zolotarev polynomials and their spectra belong to the class of time-limited functions with a discrete spectrum. In this treatment we deal with a vector space in which the band-limitedness implicates a basis expansion. This means that we can use these polynomials in forming the Zolotarev series. The Zolotarev polynomials are time-limited in the interval \((-1,1)\) (see Fig.1). The spectra of Zolotarev polynomials are discrete and band-limited (see Figs.3 and 5), with the bandwidth given uniquely by the polynomial order.

Both the time limitations and the band limitations enable natural discretization of the Zolotarev polynomial basis, yielding no aliasing in the spectral and time domain. The discretization in the time domain leads to the Discrete Zolotarev Transform, a discrete version of the Zolotarev series. Let \( s(t) \) be a periodic signal limited in time and frequency, and let \( \hat{s}(t) \) be its approximation. The approximation error in the time domain is given by

\[
e(t) = s(t) - \hat{s}(t).
\]

Minimizing the cost function

\[
\min \int_{T_0} (s^2(t) - 2s(t)\hat{s}(t) + \hat{s}^2(t))dt
\]

can be equivalently achieved by maximizing the term \( \int_{T_0} s(t)\hat{s}(t)dt \) that represents the correlation between the signal \( s(t) \) and its approximation \( \hat{s}(t) \). Thus we can conclude that minimization of the approximation error leads to the maximal correlation between signal \( s(t) \) and its approximation with a polynomial basis \( \{\Phi_m(t)\}_{m=0}^{N} \)

\[
\hat{s}(t) = \sum_{m=0}^{N} \hat{\Phi}_m(t),
\]

and using the Fourier expansion the basis polynomial \( \Phi_m(t) \) can be written in the form

\[
\Phi_m(t) = \sum_{k=-m}^{m} \alpha_k \exp(i2\pi kt),
\]

where \( \alpha_k \) are its spectral coefficients. If we apply the Fourier transform to (27) we get

\[
\mathcal{F}\{s(t) - \hat{s}(t)\} = \sum_{k=-N}^{N} S(k) \exp(i2\pi kt) - \sum_{m=0}^{N} \hat{\Phi}_m(t) \sum_{k=-m}^{m} \alpha_k \exp(i2\pi kt)
\]

\[
= \sum_{k=-N}^{N} \left[ S(k) - \alpha_k \sum_{m=k}^{N} \hat{\Phi}_m(t) \right] \exp(i2\pi kt)
\]

This indicates that the spectral coefficients of \( \mathcal{F}\{s(t) - \hat{s}(t)\} \) are given as

\[
E(\ell) = S(\ell) - \alpha_{\ell} \sum_{m=\ell}^{N} \hat{\Phi}_m(t).
\]

If we assume Zolotarev polynomials, the approximation error (27) can be equivalently defined in the frequency domain as

\[
S(\ell) - S_Z(\ell), \ell = 0, 1, 2, ..N - 1,
\]
where $S(\ell)$ is the spectrum of signal $s(t)$ and
\[
S_Z(\ell) = v_\ell S(\ell) - (1 - v_\ell) \mathcal{N}(\ell). \tag{33}
\]
The Zolotarev spectrum $S_Z(\ell)$ is composed of two mutually weighted parts, with weighting factor $v_\ell$, of the stationary part $S(\ell)$, and the non-stationary part $\mathcal{N}(\ell)$. Without loss of generality, it is possible to force the stationary part $S(\ell)$ of the Zolotarev transform to be equal to the Fourier spectral coefficient $S(\ell)$. The spectral cost function can then be expressed as
\[
\sum_\ell |E(\ell)|^2 = \sum_\ell |S(\ell) - S_Z(\ell)|^2 = \sum_\ell |S(\ell) - (v_\ell S(\ell) - (1 - v_\ell) \mathcal{N}(\ell))|^2 = \sum_\ell (1 - v_\ell)^2 |S(\ell) + \mathcal{N}(\ell)|^2. \tag{34}
\]

Then minimization of the spectral cost function leads to a proper selection of $v_\ell$ for each $S(\ell)$ and $\mathcal{N}(\ell)$, $\ell = 0, 1, \ldots, N - 1$, giving the optimal rate between the stationary $S(\ell)$ and non-stationary $\mathcal{N}(\ell)$ parts of the spectral line $S_Z(\ell)$. This result is the heart of the DZT computation. It can be proved that the basis, constructed of the Zolotarev polynomial of the first and second kind, forms a complete vector space with orthogonal properties related to the trigonometric functions. As the evaluation of the non-stationary part based on the elliptic functions is quite complicated, we have developed an approximation of the fast version of the DZT called the ADZT [5] which is fully reversible. This property together with the ability of suppressing a spectral leakage is illustrated in Fig.6

![Figure 6: DFT (b) and ADZT (c) amplitude spectrum of the signal $\cos(2\pi k_4 n)$, $k = 4.8$. Its reconstruction from the DFT spectrum (d) ($SNR = 312 \text{ dB}$) and from the ADZT spectrum (e) ($SNR = 313 \text{ dB}$).](image)

5 Conclusion

We have presented selective novel basis functions suitable for multi-spectral analysis of non-stationary signals. The basis functions are closely related to Zolotarev polynomials of the first and second kind. The algebraic solutions which led to recursive algorithms for these two selective Zolotarev polynomial basis functions were developed by M. Vlcek [8] and R. Spetik [5]. We have also derived the selective series as a solid theoretical background for the DZT. Time-frequency resolution of the ADZT can be continuously matched to the input signal for optimal representation. This property is the consequence of zcos and zsin features. From the first experiments we have learnt that ADZT also provides a powerful method for detecting sudden changes in non-stationary signals.

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