Study of ADZT properties for spectral analysis

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Abstract: In this paper we present a new time-frequency transform called the Approximate Discrete Zolotarev Transform (ADZT), and demonstrate its efficacy in analysing non-stationary signals. Several experiments have confirmed two main properties of ADZT: i) in contrast to the Short-Time Fourier Transform (STFT) or the Wavelet Transform (WT), the choice of window type and length are not critical; and ii) the similarity between ADZT and the Fourier transform (FT) implies a simple physical interpretation of results in contrast to Wigner-Ville time-frequency distribution (WVD) or Choi-Williams time-frequency distribution (CWD).

Key–Words: Time-frequency analysis, spectral analysis, spectrogram, Zolotarev polynomials, Zolotarev transform, non-stationary signal

1 Introduction

In this section we briefly describe selected methods for spectral analysis. Usually, STFT is used to time-frequency analysis. The main advantage of STFT is the intuitive interpretation of the results. The product of time and frequency resolution of STFT is constant for a given window size and type which limits its use for non-stationary signal analysis. Wavelet Transforms (CWT or DWT) are multi-resolution and conditionally fully reversible [8], but they have poor frequency resolution. DWT, in particular, is a well established engineering method with most applications related to signal coding and data compression [9]. The Wigner-Ville time-frequency distribution (WVD) exploits the relation between the autocorrelation function of a signal and its power spectral density [7], [10], [14] to define an instantaneous, time dependent auto-correlation function and subsequently a time dependent power spectrum of the signal. It can be proved [7] that the resultant time dependent power spectra are closely related to the signal instantaneous frequency. The main limitations of the WVD are the existence of spectral cross-terms which are caused by multiple components of observed signal. This disadvantage is partly removed by using a smoothing function (2-D low-pass filter) to suppress interferences between spectral components in Choi-Williams time-frequency distribution (CWD) [16]. The main disadvantages of WVD and CWD are their irreversibility. The frames are considered as a generalization to the regular basis used for a signal expansion [10]. The Gabor expansion [7] and the wavelet transform are two well-known special cases. Generally, the signal is decomposed using a given frame operator that is constructed in such as way as to extract the main features from the signal. The main difference between basis and frames are less rigid requirements on frames. There is a group of methods which deal with instantaneous frequency and envelope. First, the Hilbert-Huang Transform belongs to this group. The name designated by NASA, was proposed by Huang et al(1996, 1998, 1999, 2003) [11]. The Hilbert-Huang transform consists of two steps. In the first step, a signal is decomposed into a sum of sub-signals called empirical modes (EM) in the process of Empirical Mode Decomposition (EMD). The process is iterative, and its purpose is to decompose the input signal into a set of signals possessing an instantaneous frequency that is positive only [11]. In the second step, the instantaneous frequency of each empirical mode is calculated with the aid of the Hilbert transform. It is important that the process of EMD itself does not guarantee physically reasonable decomposition of the signal. However, it is often the case that the signal is reasonably decomposed [12], [13]. The main disadvantage of HHT is its irreversibility. Lately, data adaptive window length transforms were introduced.
These transforms suppose the signal model of complex exponential in noise and use the optimal window length depending on the estimated instantaneous frequency (e.g. Katkovnik [17]).

2 Theoretical background

2.1 Zolotarev polynomials

The Zolotarev basis consists of two symmetrical Zolotarev polynomials \( z\cos \) and \( z\sin \), which have interesting unique properties due to their selectivity and iso-extremality. These polynomials can be expressed as weighted series of Chebychev polynomials of the first and second kind [2] and they can be expressed as using cosine or sine notation, respectively

\[
z\cos (N, w_p) = \sum_{\mu=0}^{N} a_{2\mu} (w_p) \cos (2\pi \mu n),
\]

where \( N \) denotes degree of polynomials and \( w_p \) signifies its selectivity, which controls the shape of central lobe (see Fig. 1a). In view of fact that a sine is an odd function, \( z\sin \) contains two central lobes (see Fig. 1b) which control its selectivity by setting parameter \( w_p \)

\[
z\sin (N, w_p) = \sum_{\mu=1}^{N} b_{2\mu-1} (w_p) \sin (2\pi \mu n).
\]

Algorithm for computing coefficients \( a_{2\mu} \) and \( b_{2\mu} \) is given in [2]. From the spectral point of view, every \( z\cos \) and \( z\sin \) can be decomposed to a stationary \( S(k) \) and non-stationary \( N(k) \) parts

\[
S_Z(k) = v_k S(k) - (1 - v_k) N(k),
\]

where \( v_k \) is a weighting factor. The stationary part is standard cosine or sine of degree \( N \) and non-stationary part corresponds to a specific window function which shape is controlled by \( w_p \). Fig. 1 illustrates \( z\cos \) and \( z\sin \) of 12th-order along with their spectra (see Fig. 1c, d). Stationary part \( S(k) \) of \( z\cos \) contains 12th spectral coefficient only, while lower coefficient indexes creates non-stationary part \( N(k) \).

2.2 Zolotarev series

Zolotarev series can be formed in a similar way as Fourier series, because the selective basis polynomials \( z\cos \) and \( z\sin \) belong to the class of time-limited signals (periodic) with a discrete spectrum. The Fourier basis consists of the complex exponential

\[
\exp(i2\pi lt) = \cos(2\pi lt) + i \sin(2\pi lt), \quad l \in \mathbb{Z}.
\]

Then the band limited signal \( s(t) \) can be expressed as

\[
s(t) = \sum_{l=-N}^{N} S(l) \exp(i2\pi lt),
\]

where \( S(l) \) is coefficient of Fourier series, which represents the frequency spectrum of the signal

\[
S(l) = \langle \exp(-i2\pi lt), s(t) \rangle = 0, \forall |l| > N.
\]

The basis function of Zolotarev series is composed of two selective Zolotarev polynomials \( z\cos \) and \( z\sin \) introduced in the previous chapter

\[
z\exp(l, i2\pi t) = z\cos(l, 2\pi t) + i \ z\sin(l, 2\pi t)
\]

\[
= \sum_{\mu=-l}^{l} a_{2\mu} \cos(2\pi \mu t) + i \ \sum_{\mu=-l}^{l} b_{2\mu-1} \sin(2\pi \mu t)
\]

\[
= \sum_{\mu=-l}^{l} c_{2\mu}' \exp(2\pi \mu t),
\]

where \( a_{2m}', b_{2m}', c_{2m}' \) are power normalized coefficients [3]. Then the band-limited signal \( s(t) \) can be expressed as

\[
s(t) = \sum_{l=-N}^{N} S_Z(l) \exp(l, i2\pi t),
\]
where function $c_{exp}(l, i2\pi t)$ is the complementary function biorthogonal to $z_{exp}(l, i2\pi t)$ and $S_{Z}(l)$ are spectral coefficients of the Zolotarev spectrum defined

$$S_{Z}(l) = \langle z_{exp}(l, -i2\pi t), s(t) \rangle = 0, \forall |l| > N. \quad (9)$$

Using (3), (5) and (7) spectral coefficient $S_{z}(l)$ can be expressed as

$$S_{Z}(l) = \left( \sum_{n=-l}^{l} c_{2n}^{l} \exp(2\pi nt), s(t) \right) = \sum_{n=-l}^{l} c_{2n}^{l} S(n). \quad (10)$$

This equation gives one Zolotarev spectral coefficient $S_{z}(l)$ as a weighted sum of Fourier spectral coefficients.

2.3 Approximate Discrete Zolotarev Transform

Because the evaluation of the coefficients of Zolotarev polynomials $a_{2m}^{l}, b_{2m}^{l}$ is rather tedious, the Approximate Discrete Zolotarev Transform (ADZT) was suggested in [3]. ADZT is evaluated by minimizing the spectral function $|S_{z}|^2$ leading to the relation between coefficients of the Fourier and Zolotarev spectra

$$S_{Z} = Z \cdot S, \quad (11)$$

where vectors $S_{Z}$ and $S$ contain Zolotarev and Fourier spectral coefficients, respectively. Matrix $Z$ consists of coefficients of selective Zolotarev polynomials $a_{2n}^{l}$. Matrix $Z$ is not an unitary matrix, but it is completely invertible, providing complete reversibility of ADZT without any restriction [3].

The process of matrix $Z$ creation can be summarized as follows:

1. Compute spectral coefficients by Fast Fourier transform (FFT) of the analysed signal $S(k) = FFT\{s(n)\}$
2. Separate stationary part $S(k)$ and non-stationary part $\mathcal{N}(k)$ of the actual spectral coefficient $S(k)$ according to (3).
3. Compute weighting factor $v_{k}$ and weighting coefficients of Zolotarev spectrum $c_{2n}^{l}$ (10) by minimization of $|S_{z}|^2$.
4. Compose unitary complex matrix $Z$ using coefficients $c_{2n}^{l}$.

2.4 Short-Time Approximate Discrete Zolotarev Transform

The Short-Time Approximate Discrete Zolotarev Transform (STADZT) can be used for short-time spectral analysis. STADZT can be defined similarly as the Short Time Fourier Transform (STFT), but instead of the exponential function the Zolotarev basis $z_{exp}$ is used

$$S_{Z}(l, n) = \sum_{m=-\infty}^{\infty} s(m)w(m - n)z_{exp}(l, i2\pi n), \quad (12)$$

where $w(m)$ must be the final length rectangular window resulting in signal segmentation without any other weighting. Thus all parameters of STADZT are the same as the parameters of STFT, apart from the shape of the segmentation window. Experiments revealed that when some other window shape is used, the interference between the window and the ADZT time selective basis deteriorates the performance of ADZT. This property is the most significant difference between STFT and STADZT.

3 Properties of ADZT

3.1 Spectral leakage

The discrete Fourier transform (DFT) takes an important role in digital signal processing. Although DFT is a tool for processing periodic signals, it is widely used for estimating the spectra of aperiodic signals. This fact, together with the time-limitation of the signals, leads to undesirable spectral leakage. When using a rectangular window, the spectral leakage is not present only if the orthogonality of a signal segment and the DFT basis vectors is not violated. This case is illustrated in Fig. 2, where the ADZT coefficients are identically equal to the DFT spectral coefficients, suggesting the optimality of DFT for this signal.

Fig. 3 illustrates the case when the orthogonality is violated, and hence a spectral leakage effect arises in the DFT spectrum, while the ADZT spectrum is free of leakage, with the exception of the two first ADZT coefficients. In principle, these two coefficients are always equal to DFT coefficients [3].

4 Analysis of selected non-stationary signal

This section compares the STADZT time resolution with the STFT, WT, WVD and CWD time resolution for modelled signals. The main reasons for the choice of these methods are:
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The parameters of STFT and STADZT analyses are given in figure captions. There is one general difference. While the Hamming window is used for STFT, the rectangular window is used for STADZT, as explained above. Both types of analyses are chosen for the maximum segment overlap without any decimation in frequency. This overlap can be reduced in practice, of course. The wavelet transform is performed with the Morlet mother wavelet and a fixed scaling factor. We omit any optimization of WT by choosing a mother wavelet. Wigner-Ville and Choi-Williams time-frequency distribution is used according to [15].

4.1 Unit (Dirac) impulse

The first example is the spectrogram of one unit impulse (see Fig. 4a) illustrating the basic time resolution of the transforms. The spectrogram created by STFT transforms the impulse into a rectangular area with non-uniformly spread energy in the time and frequency domains. The biggest energy of the impulse is concentrated in the middle of this area (see segment 255 in Fig. 4b). This effect is caused by the Hamming window. If the rectangular window is used, the energy is spread uniformly in time and frequency. The wavelet transform is able to localize the impulse correctly at low scales, which have the highest time resolution (see Fig. 4c). The time localization of the impulse is rather poor for higher scales. The Wigner-Ville time-frequency distribution achieves good time resolution for almost the whole frequency range (see Fig. 4d). The energy of unit impulse is concentrated in two mirroring areas. It caused by cross-terms which can be suppressed by using smoothing window (see the next section). The Zologram created by STADZT reveals the impulse quite clearly (see Fig. 4e)). This is a consequence of using the selective Zolotarev basis. The time resolution improves for higher frequencies, thus the impulse is displayed as peak narrowing from the lowest frequency to the highest frequency.

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4.2 Sudden change of frequency

This example shows the ability of the three transforms to analyse a signal composed of one harmonic wave, the frequency of which is twice changed - $f_1 < f_3 < f_2$ (see Fig. 5a). The segment length for STFT is comparable with the distance between the changes. Then STFT frequency resolution is much better than in the previous example, but the time localization of the changes is not very accurate (see Fig. 5b). The scalogram is rather poor in this case. The estimation of the frequencies and the time localization are not very good. The time instants of the changes cannot be recognized. STADZT offers satisfactory time and frequency resolution. The changes of frequency are detected with high accuracy (see Fig. 5d). Experiments have confirmed that the time resolution of STADZT is almost independent of the segment length for this signal type (see Fig. 6). A long window can therefore be used to obtain excellent frequency resolution without deteriorating the time resolution.

The time localization of two frequency changes (analysing signal is same as in the previous example) is quite good in both cases WVD and CWD. The main disadvantage of WVD is difficult interpretation of resulted spectrogram (see Fig. 7b) because the spectral interferences (cross-terms) are occurred. It is caused by multiple components of signal [15]. CWD suppresses spectral interferences of WVD by using smoothing kernel (see Fig. 7c).

5 Conclusion

This paper presents a set of experimental results for non-stationary spectral signal analysis using the
new time-frequency Approximate Discrete Zolotarev Transform. This transform achieves better results than the widely-used Fourier and Wavelet transforms and it does not involve any spectral interferences which are presented in Wigner-Vile time-frequency distribution. ADZT does not need any abridging of the length of a signal segment in order to achieve better time selectivity. In fact, ADZT excludes the use of any window and results are therefore less dependent on parameter setting than the results of STFT (window type and length) or WT (mother wavelet type). Information about Zolotarev polynomials and their application can be found on the web site: amber.feld.cvut.cz/selective transforms.

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