Pole Reconstruction of Systems from Laguerre Basis Representations

ALEXANDROS SOUMELIDIS
Systems and Control Laboratory
Kende 13–17, H-1111 Budapest
HUNGARY
soumelidis@sztaki.hu

FERENC SCHIPP
Eötvös Loránd University
Department of Numerical Analysis
Pázmány Péter 1/C, H-1117 Budapest
HUNGARY
schipp@numanal.inf.elte.hu

JÓZSEF BOKOR
Systems and Control Laboratory
Kende 13–17, H-1111 Budapest
HUNGARY
bokor@sztaki.hu

Abstract: This paper investigates an opportunity to identify poles of systems with unknown structure on the basis of discrete-time spectral signal representations generated on the basis orthogonal Laguerre systems. The convergence of the Laguerre representation coefficients is analyzed within the framework of a hyperbolic metrics generated by the Blaschke functions associated with the Laguerre system, and a conceptual base is given to solve the task of finding the poles of the system. An algorithm is proposed to find the poles, and a discussion is given upon the criteria of finding all of them. An efficient implementation of the algorithm based upon frequency domain measurement is available for practical computations.

Key–Words: Systems theory, System identification, Pole structure estimation, Frequency domain methods, Rational orthogonal bases

1 Introduction

Determining the pole locations associated to signals and systems modeling is needed in many applications and several approaches are known to cope with this problem. Detection of spectral peaks in noisy signals – a frequent task in analyzing vibrating mechanical systems, electrical rotating machines, or industrial plants (such as nuclear power plants) – is in essence a pole-identification problem. The phase angle associated with the pole can be identified as the frequency, and its absolute value – corresponding to the attenuation represented by the pole – can be connected with the height and width of a spectral peak. The spectral peak-analysis can be performed by Fourier-analysis of the signals that can efficiently be realized by using the Fast-Fourier-Transform (FFT) algorithm. However the exact pole locations cannot be determined by this way, spectral peaks cannot be separated in many cases, as well as analyzing the height and width of the peaks does not result in unique solutions for determining the poles attenuations.

For use of parametric methods in time domain one can cite variations of Prony–methods [5] and those assuming linear signal- and system-models, i.e. applying autoregressive (AR) or autoregressive moving-average (ARMA) model identifications and associated spectral analysis [4], as well as identification of matrix partial fraction models [3], and subspace approaches [6]. Disadvantage of these approaches is that associated to the parametrization problem, both the structure and the parameters have to be estimated, leading to, in many situation, unreliable results.

Another approach is the use of rational orthogonal bases [2] that needs a priori knowledge upon the pole locations. Special attention paid on the problems of pole selection and validation [1]. There exist methods to refine the pole locations starting from an approximate placement of poles [7], however the general identification problem has not been solved so far.

This paper proposes a new approach that is closely related to signal and system representations using discrete rational Laguerre–basis in \( H^2(\mathbb{D}) \). Based upon the analysis of the Laguerre–coefficients of the rational transfer function – that can be computed using frequency domain data obtained from non-uniformly distributed measurements – by applying a specific hyperbolic transform originated from the Blaschke–function inherent in the Laguerre–representation, the poles of the system can be identified. The transform is related to the Poincaré unit disc model of the hyperbolic geometry. This will allow a nice geometric interpretation of the resulting identification algorithm.

2 The discrete Laguerre–system

Definition 1 The discrete Laguerre system based upon parameter \( b (b \in \mathbb{D}) \) is defined as:

\[
\Phi_n(z) = \frac{\sqrt{1 - |b|^2}}{1 - b z} B_n^b(z) \quad (n = 0, 1, 2 \ldots),
\]
where
\[ B_0(z) = e^{i\delta} \frac{1}{1 - \overline{b}z} \]
is the Blaschke function with the arbitrary constant \( \delta \in [0, 2\pi) \).

It is well known – see [9] – that the discrete Laguerre system forms an orthonormal basis in the Hardy space \( H^2(\mathbb{D}) \). Any function \( F \in H^2(\mathbb{D}) \) can be expressed by the representation:
\[
F(z) = \sum_{n=0}^{\infty} l_n \Phi_n(z),
\]
where the coefficients \( \{l_n\} \) – the so-called Laguerre coefficients – can be computed by using the inner-product belonging to the space \( H^2(\mathbb{D}) \) as:
\[
l_n = \langle F, \Phi_n \rangle.
\]

By substituting the expression of the Blaschke function in (1) by selecting 0 for the value of the arbitrary parameter \( \delta \), the elements of the Laguerre system get the form
\[
\Phi_n(z) = \sqrt{1 - |b|^2} \frac{(z - b)^n}{(1 - \overline{b}z)^{n+1}} \quad (n = 0, 1, 2 \ldots)
\]

The representation of any function \( F \in H^2(\mathbb{D}) \) can be expressed on the basis of an orthogonal projection upon the Laguerre system components, i.e. the representation coefficients can be expressed as:
\[
l_n = \left\langle F(z), \sqrt{1 - |b|^2} \frac{(z - b)^n}{(1 - \overline{b}z)^{n+1}} \right\rangle = \frac{\sqrt{1 - |b|^2}}{2\pi} \int_{-\pi}^{\pi} F(e^{it}) \frac{(e^{-it} - \overline{b})^n}{(1 - be^{-it})^{n+1}} dt = \frac{\sqrt{1 - |b|^2}}{2\pi} \int_{-\pi}^{\pi} F(e^{it}) \frac{(1 - \overline{b}e^{it})^n}{(e^{it} - b)^{n+1}} e^{it} dt
\]

Rewriting this form into a complex contour integral upon the unit circle, i.e. by applying the substitution \( z = e^{it} \) and considering \( T := \{ z \in \mathbb{C} : |z| = 1 \} \),
\[
l_n = \frac{\sqrt{1 - |b|^2}}{2\pi i} \oint_T F(z) \frac{(z - b)^n}{(z - \overline{b})^{n+1}} dz = \frac{\sqrt{1 - |b|^2}}{2\pi i} \oint_T F(z) (1 - \overline{b}z)^n \frac{dz}{(z - b)^{n+1}}.
\]

According to Cauchy’s integral formula the Laguerre coefficients can be expressed in the form
\[
l_n = \frac{\sqrt{1 - |b|^2}}{n!} \left. \frac{d^n(1 - \overline{b}z)^n F(z)}{dz^n} \right|_{z=b} \quad (n = 0, 1, 2 \ldots)
\]

As an example let \( F \) be considered as
\[
F(z) = \frac{1}{1 - a z} \quad (a \in \mathbb{D}).
\]

It is obvious that \( F \in H^2(\mathbb{D}) \) and a single pole of the function resides outside the unit circle. The parameter \( a \) can be referred as an inverted pole belonging to \( F \) with the exact definition \( p = 1/\pi \).

The coefficients belonging to the Laguerre-representation of \( F \) according to (3) are given as
\[
l_n = \frac{1 - |a|^2}{n!} \left. \frac{d^n (1 - \overline{b}z)^n}{dz^n} \right|_{z=b}.
\]

In the special case when the parameter \( b \) belonging to the Laguerre-representation is selected to be equal to \( a \), i.e. \( b = a \),
\[
l_n = \frac{1 - |a|^2}{n!} \left. \frac{d^n (1 - |a|^2)^n}{dz^n} \right|_{z=b}.
\]

The function to be differentiated \( n \)-times is a \((n - 1)\)-degree polynomial of \( z \), hence the derivatives equal to zero except if \( n = 0 \). This means that only one coefficient differs from zero, namely \( l_0 \), it is actually
\[
l_0 = \frac{1 - |a|^2}{1 - \overline{a}a} = \frac{1}{1 - |a|^2},
\]

The expansion (2) contains only one term,
\[
F(z) = l_0 \Phi_0(z) = l_0 \frac{1}{1 - \overline{a}a} = \frac{1}{1 - |a|^2},
\]
i.e. the function (4) is represented exactly.

In the case if \( b \neq a \) the Laguerre-coefficients can be computed by the formula (5). The coefficient \( l_0 \) is given immediately as
\[
l_0 = \frac{1 - |a|^2}{1 - \overline{a}a}.
\]
The coefficient \( l_1 \) can be computed from formula
\[
l_1 = \sqrt{1 - |b|^2} \left. \frac{d}{dz} \frac{1 - \overline{b}z}{1 - \overline{a}a} \right|_{z=b}.
\]

By realizing the derivation
\[
\frac{d}{dz} [(1 - \overline{b}z)(1 - \overline{a}a)^{-1}] = \frac{-\overline{b}(1 - \overline{a}a)^{-1} + (1 - \overline{b}z)(1 - \overline{a}a)^{-2}}{(1 - \overline{a}a)^2} = \frac{-\overline{a}(1 - \overline{b}z) - \overline{b}(1 - \overline{a}a)}{(1 - \overline{a}a)^2} = \frac{\overline{a} - \overline{b}}{(1 - \overline{a}a)^2},
\]
the Laguerre-coefficient is given as
\[
l_1 = \sqrt{1 - |b|^2} \frac{\overline{a} - \overline{b}}{(1 - \overline{a}a)^2}.
\]
To obtain a general formula for every \( l_n \) \((n = 0, 1, 2, \ldots)\) Laguerre-coefficient, consider the case when the function \( F(z) \) possesses a single inverted pole \( a \in \mathbb{D} \) of multiplicity \( m \) \((m = 1, 2, \ldots)\), i.e.

\[
F(z) = \frac{1}{(1 - \pi z)^m} \quad (a \in \mathbb{D} \quad m \in \mathbb{N}).
\]

The Laguerre-coefficients \( l_n \) \((n = 0, 1, 2, \ldots)\) can be computed by the form

\[
l_n = \frac{\sqrt{1 - |b|^2}}{n!} \left[ \frac{d^n}{dz^n} \left( \frac{1 - \bar{b}z}{1 - \bar{a}z} \right)^m \right]_{z=b}.
\]  

(6)

By denoting the function to be differentiated as \( f(z) \), i.e.

\[
f(z) = (1 - \bar{b}z)^n(1 - \bar{a}z)^{-m},
\]

it is clear that the \( n \)-th derivative can be computed by applying the generalized Leibnitz rule

\[
(uw)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}.
\]

The \((n-k)\)-th derivative \((k < n)\) of the term \( u = (1 - \bar{b}z)^n \) and the \( k \)-th derivative of \( v = (1 - \bar{a}z)^{-m} \) are given as

\[
u^{(n-k)} = (-1)^{n-k} n(n-1) \ldots (k+1) \bar{b}^{n-k}(1 - \bar{b}z)^k
\]

\[
v^{(k)} = m(m+1) \ldots (m+k-1) \bar{a}^k (1 - \bar{a}z)^{-(m+k)}
\]

respectively. Since

\[
n(n-1) \ldots (k+1) = \frac{n!}{k!},
\]

and

\[
m(m+1) \ldots (m+k-1) = \frac{(m+k-1)!}{(m-1)!},
\]

the \( n \)-th derivative of \( f \) can be expressed as

\[
f^{(n)}(z) = n! \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{(m+k-1)!}{k!(m-1)!} \bar{b}^{n-k}(1 - \bar{b}z)^k \bar{a}^k (1 - \bar{a}z)^{-(m+k)}.
\]

By applying the identity

\[
\frac{(m+k-1)!}{k!(m-1)!} = \binom{m+k-1}{k},
\]

setting out the common factor \((1 - \bar{a}z)^{-(m-1)}\) from the sum, and bringing it to the common denominator \((1 - \bar{a}z)^{n+1}\),

\[
f^{(n)}(z) = \frac{n!}{(1 - \bar{a}z)^{n+1}} \cdot \frac{1}{(1 - \bar{a}z)^{n+1}} \cdot \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{(m+k-1)!}{k!(m-1)!} \bar{b}^{n-k}(1 - \bar{b}z)^k \bar{a}^k (1 - \bar{a}z)^{-(m+k)}
\]

(7)

In the case if \( m = 1 \), i.e. a pole with no multiplicity is considered,

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \bar{b}^{n-k}(1 - \bar{b}z)^k \bar{a}^k (1 - \bar{a}z)^{n-k}
\]

– according to the binomial theorem – forms the \( n \)-th power of the expression

\[
\bar{a}(1 - \bar{b}z) - \bar{b}(1 - \bar{a}z) = \bar{a} - \bar{b}.
\]

Hence, according to (6) the \( n \)-th Laguerre–coefficient \( l_n \) in the case when the function \( F \) to be represented contains a single pole with no multiplicity – can be expressed as

\[
l_n = \frac{\sqrt{1 - |b|^2}}{n!} \left( \frac{\bar{a} - \bar{b}}{1 - \bar{a}b} \right)^n.
\]

(8)

In the cases of poles with multiplicity \( m > 1 \) the binomial theorem cannot immediately be applied be-cause of the presence of the factor

\[
\binom{m+k-1}{k}
\]

in the sum. However it can be applied in an iterative way such that – as it can be shown – a sum containing \( m \) terms with decreasing powers of \( \frac{\bar{a} - \bar{b}}{1 - \bar{a}b} \) is given for the \( n \)-th Laguerre–coefficient:

\[
l_n = \sqrt{1 - |b|^2} \left\{ \frac{\bar{a} - \bar{b}}{1 - \bar{a}b} \right\}^n + \sum_{k=1}^{m-1} C_k n(n-1) \ldots (n-k+1) \left\{ \frac{\bar{a} - \bar{b}}{1 - \bar{a}b} \right\}^{n-k}
\]

(9)

where \( C_k \) \((k = 1, 2, \ldots, (m-1))\) are constant factors. The exact form of these factors is indifferent in respect to the subject of this paper.

### 3 Convergence of the Laguerre–coefficients

In respect of convergence the several terms of the Laguerre–coefficients belonging to multiplicities greater than 1 can be analyzed separately. Two cases should be discriminated, for \( m = 1 \) and \( m > 1 \), respectively

\[
\lambda_1(n) = \left[ \frac{\bar{a} - \bar{b}}{1 - \bar{a}b} \right]^n
\]

\[
\lambda_m(n) = n(n-1) \ldots (n+m-1) \left[ \frac{\bar{a} - \bar{b}}{1 - \bar{a}b} \right]^n
\]

where the constant factors has been disregarded.

The conditions of convergence of \( \lambda_1 \) is obvious,
since $\lambda_1$ forms a geometrical sequence with quotient
\[ q = \frac{\pi - \bar{b}}{1 - \pi b}, \]
and as it is well known, the condition of the absolute convergence is $|q| < 1$. This condition can be verified by using the following proposition:

**Lemma 1** If $a, b \in \mathbb{D}$,
\[ \left| \frac{\pi - \bar{b}}{1 - \pi b} \right| < 1. \tag{10} \]

**Proof:** By considering the square of (10)
\[ \frac{\pi - \bar{b}}{1 - \pi b} a - b = \frac{|a|^2 - (\bar{a} b - \pi b) - |b|^2}{1 - (\bar{a} b - \pi b) + |a|^2|b|^2} < 1. \]
The denominator is positive, hence both sides is multiplied by it, than $(\bar{a} b - \pi b)$-t is added:
\[ |a|^2 + |b|^2 < 1 + |a|^2|b|^2 \]
With some manipulations
\[ 0 < (1 - |a|^2)(1 - |b|^2) \]
\[ 0 < (1 + |a|/(1 - |a|))(1 + |b|)(1 - |b|). \]
Since $|a| < 1$ and $|b| < 1$, the expression in the right-hand side is positive, that proves the proposition (10). $\square$

Convergence of $\lambda_m$ is also based upon the condition $|q| < 1$.

**Lemma 2** If $q \in \mathbb{D}$, for any number $m \in \mathbb{N}$ the sequence
\[ n(n-1)(n-2)\ldots(n-m)q^n \quad (n \in \mathbb{N} > m) \]
absolutely converges.

**Proof:** By applying the quotient criterion, the
\[ \frac{(n+1)n(n-1)\ldots(n-m+1)}{n(n-1)(n-2)\ldots(n-m)} |q| < 1 \]
condition should be satisfied to achieve absolute convergence. After simplification
\[ \frac{n+1}{n-m} |q| < 1, \]
i.e. for indices $n > m$
\[ |q| < \frac{n-m}{n+1} = 1 - \frac{m+1}{n+1} < 1. \tag{11} \]
As $n$ increases toward infinity, $\frac{n-m}{n+1}$ forms a monotonously increasing sequence toward 1, hence there exist an index $n_q$ that makes (11) true for any $n > n_q$, which proves the proposition.

In both cases the quotient $q = \frac{\pi - \bar{b}}{1 - \pi b}$ plays crucial role. By comparing its form with the Blaschke function introduced in definition (1) some correspondence can be observed. To elaborate this correspondence, some group-theoretic considerations are required in association with the Blaschke function.

The Blaschke function upon parameters $(b, \epsilon)$ is defined as
\[ B_b(z) := \frac{z - b}{1 - \overline{b}z} \quad (z \in \mathbb{C}, b = (b, \epsilon) \in B := \mathbb{D} \times \mathbb{T}). \]
If $b \in B$, then $B_b$ is an $1 - 1$ map on $\mathbb{T}$ and $\mathbb{D}$, respectively, which means that the Blaschke function is an inner function in the space $\mathbb{H}^2(\mathbb{D})$. The restrictions of the Blaschke functions on the set $\mathbb{D}$ or $\mathbb{T}$ with the operation of function–composition
\[ (B_{b_1} \circ B_{b_2})(z) := B_{b_1}(B_{b_2}(z)) \]
form a group. A group $(\mathbb{B}, \circ)$ defined in the set of parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ can be obtained if the operation
\[ B_{b_1} \circ B_{b_2} = B_{b_1 \circ b_2} \]
is considered, which is isomorphic to the group $((B_b, b \in B), \circ)$. Either of these isomorphic groups can be referred by the name Blaschke–group.

The neutral and the inverse element of the group $(\mathbb{B}, \circ)$ can be obtained as
\[ \epsilon := (0, 1) \in \mathbb{B} \quad \text{and} \quad b^{-1} = (-\epsilon b, \tau) \]
respectively, if $b = (b, \epsilon) \in \mathbb{B}$.

It can be proved that the map
\[ \rho(z_1, z_2) := \frac{|z_1 - z_2|}{1 - \overline{z_1}z_2} = |B_{z_1}(z_2)| \tag{13} \]
\[ (B_{z_1} := B_{(z_1, 1)}, z_1, z_2 \in \mathbb{D}) \]
is a metric on $\mathbb{D}$. Moreover the Blaschke functions $B_b$ $(b \in \mathbb{D})$ are isometries with respect to this metric, i.e.
\[ \rho(B_b(z_1), B_b(z_2)) = \rho(z_1, z_2) \quad (b \in \mathbb{D}, z_1, z_2 \in \mathbb{D}). \]
The lines in this model are the sets
\[ L_b := \{ B_b(r) : -1 < r < 1 \} \quad (b \in \mathbb{B}), \]
i.e. circles crossing perpendicularly the unit circle. This model is known in the hyperbolic geometry as the Poincaré model of the unit–disc.

Any element of the Blaschke–group $\mathbb{B}$ can be characterized as a transform that maps $\mathbb{T}$ and $\mathbb{D}$ onto themselves, respectively, which can be considered as a hyperbolic transform upon the unit disc. The "hyperbolic" nature of this transforms arises from its distance–preserving property with respect to the hyperbolic metric according to (13).

By returning to the expression of the quotient $q$, it
Recent Researches in System Science

ISBN: 978-1-61804-023-7

Recent Researches in System Science

Recent Researches in System Science

Figure 1: Hyperbolic circles belonging to Laguerre-representations.

Proposition 1 Let \( q \) be the quotient belonging to the Laguerre representation based on parameter \( b \in \mathbb{D} \) of a function \( F \in \mathbb{H}^2(\mathbb{D}) \) containing a single inverse pole \( a \in \mathbb{D} \). A hyperbolic transform belonging to the inverse group element \( b^{-1} = (-b, 1) \) reconstructs \( a \), i.e.

\[
Q = B_a(q).
\]

This relation has great significance in respect to identifying poles: in the case if the quotient \( q \) – by assuming a Laguerre representation parameter \( b \in \mathbb{D} \) – is in any way estimated (the opportunities to do this will be discussed later in this paper), on the basis of this estimate, the location of the pole \( a \) within the unit disc can be derived. The following proposition – that has trivially been proved by the above discussion – has crucial significance:

Proposition 1 Let \( q \) be the quotient belonging to the Laguerre representation based on parameter \( b \in \mathbb{D} \) of a function \( F \in \mathbb{H}^2(\mathbb{D}) \) containing a single inverse pole \( a \in \mathbb{D} \). A hyperbolic transform belonging to the inverse group element \( b^{-1} = (-b, 1) \) reconstructs \( a \), i.e.

\[
a = B_{b^{-1}}(q).
\]

This proposition directly gives an opportunity to identify a single pole on the basis of the Laguerre–representation, since \( q \) can immediately be derived from the Laguerre–coefficients. An extension to multiple poles will be constructed in the subsequent sections.

Fig. 1 illustrates the significance of the quotient \( q \) associated with Laguerre–representations. One pole denoted by \( a \) has been represented by applying two different Laguerre–parameters, \( b_1 \) and \( b_2 \). Two hyperbolic circles has been drawn (the image of a circle can be observed that its conjugate \( \overline{q} \) can be expressed as a hyperbolic transform related to the Blaschke–group element \( b = (b, 1) \), i.e.

\[
\overline{q} = B_b(a).
\]

4 Convergence in the case of multiple poles

Any function \( F \in \mathbb{H}^2(\mathbb{D}) \) can be expressed in partial fraction form, i.e. for a set of inverse poles \( a_k \in \mathbb{D} \) and their multiplicities \( m_k \geq 1 (k = 1, 2, \ldots, N, N \in \mathbb{N}) \)

\[
F(z) = \sum_{k=1}^{N} \frac{A_k}{(1 - \overline{a_k} z)^{m_k}}
\]

with constant coefficients \( A_k \in \mathbb{R} \). Hence the Laguerre–coefficients belonging to a parameter \( b \in \mathbb{D} \) associated to function \( F \) can be expressed term-by-term, i.e. they can be expressed as a linear combination of partial Laguerre–coefficients belonging to every term of the partial fraction.

Concerning the convergence of the Laguerre–coefficients let the case of two identical poles with no multiplicity be considered, i.e. let \( F \) has the form of

\[
F(z) = \frac{A_1}{1 - a_1 z} + \frac{A_2}{1 - a_2 z} \quad (a_{1,2} \in \mathbb{D}, A_{1,2} \in \mathbb{R})
\]

Figure 2: Hyperbolic circles indicating the pole positions.

Concerning the convergence of the Laguerre–coefficients let the case of two identical poles with no multiplicity be considered, i.e. let \( F \) has the form of

\[
F(z) = \frac{A_1}{1 - a_1 z} + \frac{A_2}{1 - a_2 z} \quad (a_{1,2} \in \mathbb{D}, A_{1,2} \in \mathbb{R})
\]

Concerning the convergence of the Laguerre–coefficients let the case of two identical poles with no multiplicity be considered, i.e. let \( F \) has the form of

\[
F(z) = \frac{A_1}{1 - a_1 z} + \frac{A_2}{1 - a_2 z} \quad (a_{1,2} \in \mathbb{D}, A_{1,2} \in \mathbb{R})
\]
The Laguerre–coefficients belonging to parameter \( b \in \mathbb{D} \) get the form

\[
l_n = A_1 c_1 q_1^n + A_2 c_2 q_2^n \quad (n = 0, 1, 2, \ldots)
\]

where

\[
c_j = \frac{\sqrt{1 - |b|^2}}{1 - \alpha_j b} \quad q_j = \frac{\alpha_j - b}{1 - \alpha_j b} \quad (j = 1, 2)
\]

By introducing the notations \( c_j = d_j e^{i\delta_j}, q_j = r_j e^{i\varphi_j} \), and \( f_j = d_j A_j \), \( j = 1, 2 \),

\[
l_n = f_1 r_1^n e^{i(n\varphi_1 + \delta_1)} + f_2 r_2^n e^{i(n\varphi_2 + \delta_2)}.
\]

In respect to convergence the absolute value of \( l_n \) has great significance, let it be denoted by \( \ell_n \). By expressing \( \ell_n^2 \)

\[
\ell^2 = f_1^2 r_1^2n + f_2^2 r_2^2n + 2 f_1 f_2 r_1^nr_2^ncos[n(\varphi_1 - \varphi_2) + (\delta_1 - \delta_2)]
\]

Let this sequence be substituted by a continuous function of parameter \( t \in \mathbb{R}^+ \) with the property for any integer value \( t = n \) let \( \ell(t) = \ell_n \),

\[
\ell^2(t) = f_1^2 e^{2\alpha_1 t} + f_2^2 e^{2\alpha_2 t} + 2 f_1 f_2 e^{(\alpha_1 + \alpha_2) t} cos[t(\varphi_1 - \varphi_2) + (\delta_1 - \delta_2)],
\]

where \( \alpha_j = \log r_j, \alpha_j < 1 \) \( j = 1, 2 \). Furthermore, to consider convergence let the majoring function – that is given by substituting the cos function by constant \( 1 - \)

\[
\hat{\ell}^2(t) = f_1^2 e^{2\alpha_1 t} + 2 f_1 f_2 e^{(\alpha_1 + \alpha_2) t} + f_2^2 e^{2\alpha_2 t}
\]

be used instead, which can be expressed in the simple form

\[
\hat{\ell}(t) = f_1 e^{\alpha_1 t} + f_2 e^{\alpha_2 t}.
\]

It is clear that \( \ell(t) \leq \hat{\ell}(t) \) at least at \( t > t_0 \) for an adequately selected \( t_0 \geq 0 \), hence convergence of \( \hat{\ell}(t) \) involves also convergence of \( \ell(t) \) in \( t \rightarrow \infty \). The function \( \hat{\ell}(t) \) is obviously converges to \( 0 \) if \( t \rightarrow \infty \), since \( \alpha_j < 0 \) \( j = 1, 2 \).

The convergence rate – in comparison with a single exponential corresponding to a geometrical sequence – can be characterized by an equivalent decay rate defined by the function

\[
\alpha(t) = \hat{\ell}(t)/\ell(t),
\]

where dot denotes the derivation by parameter \( t \). The convergence of the Laguerre–coefficients can be characterized by \( \alpha(t) \) values in large \( t \) values. An equivalent convergence quotient \( Q \) – associated with an equivalent geometrical sequence – can be obtained as the exponential function of the limit of \( \alpha(t) \) at \( t \rightarrow \infty \), i.e.

\[
Q = e^{\alpha \infty}, \quad \alpha \infty = \lim_{t \rightarrow \infty} e^{\alpha t}.
\]

Applying function (15) to \( \hat{\ell}(t) \),

\[
\alpha(t) = f_1 \alpha_1 e^{\alpha_1 t} + f_2 \alpha_2 e^{\alpha_2 t}
\]

is given. The limit of this function at \( t \rightarrow \infty \) – as it can easily be verified by factoring out \( e^{\alpha_1 t} \) or \( e^{\alpha_2 t} \) from the sum – is given as

\[
\lim_{t \rightarrow \infty} \alpha(t) = \begin{cases} \alpha_1 \quad \text{if} \quad |\alpha_2| > |\alpha_1|, \\ \alpha_2 \quad \text{if} \quad |\alpha_1| > |\alpha_2| \end{cases}
\]

This result means that in the respect of convergence in the infinity, the exponential with the smallest decay rate will be dominant. Translating it to the sequence of the Laguerre–coefficients: the term consisting of the geometrical sequence with the greatest quotient will dominate in the convergence of the complete sequence.

This assertion can be extended to the case when the function \( F \) contains any number of poles with no multiplicity, since the considerations taken above can easily be expressed for multiple terms in the expression of the Laguerre–coefficients. The exact statement for this case can be expressed in the following form:

**Proposition 2** Let \( F \) a function belonging to the space \( H^2(\mathbb{D}) \) with \( N \in \mathbb{N} \) number of inverse poles \( a_k \in \mathbb{D} \) \( (k = 1, 2, \ldots, N) \) of multiplicity 1. The sequence of the coefficients \( l_k \) belonging to the Laguerre–representation of \( F \) based upon parameter \( b \in \mathbb{D} \)

(i) absolutely converge to zero, and

(ii) its convergence rate is equal to

\[
\max \left\{ \frac{a_1 - b}{1 - ba_1}, \frac{a_2 - b}{1 - ba_2}, \ldots, \frac{a_N - b}{1 - ba_N} \right\}
\]

In the case when poles with multiplicity greater than 1 occur, the situation is more complicated. On the basis of the expression of the Laguerre–coefficients (9) it can be shown that an analogous \( \alpha(t) \) function converges to one of the \( a_k \) values belonging to \( a_k \) that can be that possessing \( \alpha_k \) of minimal absolute value, or has the maximal multiplicity \( m_k \) – depending upon whether \( e^{\alpha_k} \) or \( t^{-(m_k-1)} \) converges faster to zero.

### 5 Identifying poles

The equivalent convergence ratio found on the basis of the limit of \( \alpha(t) \) depends on the selection of the \( b \) parameter of the Laguerre–representation. Different \( b \) values can result in different \( Q \) values of the variety

\[
\left\{ \frac{\pi_1 - b}{1 - \pi_1 b}, \frac{\pi_2 - b}{1 - \pi_2 b}, \ldots, \frac{\pi_N - b}{1 - \pi_N b} \right\}.
\]
According to Proposition 2 from a specific parameterized by the Blaschke–functions that belong to parameters all the poles are identifiable. In Fig. 4 the identifiability domains belonging to a slightly modified pole-set \( \mathbf{a} = [0.75 \, 0.8e^{\frac{i\pi}{4}} \, 0.8e^{-\frac{i\pi}{4}}] \), where the real pole has been displaced to 0.75 can be seen. However the structure of the domains has dramatically been changed: there is no domain belonging to pole No. 1, i.e. this pole is unidentifiable.

The poles that are unidentifiable in the context of the current discussion can be handled by the additional use of rational orthogonal bases, however this topic spreads beyond the subject of the current paper.

The method is illustrated by an example: a function \( F \in \mathbb{H}^2(\mathbb{D}) \) possessing 3 poles, one real pole and a conjugated complex pair, \( \mathbf{a} = [0.8 \, 0.8e^{\frac{i\pi}{4}} \, 0.8e^{-\frac{i\pi}{4}}] \). The parameter \( b \) belonging to the Laguerre–representation selected to be \( b = 0.7e^{\frac{i\pi}{2}} \) that falls in the identifiability region of the third pole, \( a_3 = 0.8e^{-\frac{i\pi}{4}} \), as it can be verified in Fig. 3. Fig. 5 presents the absolute value of the estimated Laguerre–coefficients \( \hat{t}_n \), as well as the absolute value and the phase of the estimated convergence rate \( \hat{q}_n \). The convergence of \( \hat{q}_n \) can be verified both on the latter two diagrams, and in Fig. 6, where the locations of \( \hat{q}_n \)
on the complex plane have been presented by white points. The sequence \( \hat{q}_n \) converges to the pole \( a_3 \) by forming a spiral in the complex plane. The error between the real and the estimated pole locations falls in the magnitude of \( 10^{-4} \) in root mean square sense.

6 Conclusion

This paper discussed the problem of identifying poles of systems with unknown structure on the basis of discrete-time spectral signal representations generated on the basis orthogonal Laguerre systems. After analyzing the convergence of the Laguerre representation coefficients by using the concept of a hyperbolic metrics generated by the Blaschke functions associated with the Laguerre system, the conceptual framework has been presented to solve the task of finding the poles of the system. An algorithm has been proposed to find poles with multiplicity one, and criteria are given to find all of them.

Acknowledgements: This work was supported by the European framework project ADDSAFE (Grant agreement no.: FP7-233815), and the Control Engineering Research Group of HAS at Budapest University of Technology and Economics, which are gratefully acknowledged by the authors. The European Union and the European Social Fund also provided financial support to the project under the grant agreement no. TÁMOP-4.2.1./B-09/1/KMR-2010-0003.

References: