Constrained $H_\infty$ Control of Discrete-Time Systems

ANNA FILASOVÁ, DUŠAN KROKAVEC
Technical University of Košice
Department of Cybernetics and Artificial Intelligence
Letná 9, 421 00 Košice
SLOVAKIA
anna.filasova@tuke.sk, dusan.krokavec@tuke.sk

Abstract: In this paper the design problem is addressed for linear discrete-time systems with state variables equality constraints. Based on an equivalent linear matrix inequality representation of bounded real lemma for discrete-time systems a new formulation is provided for the control law gain computation to circumvent a generally ill conditioned singular design task. The approach is successfully illustrated on a simulation example, where validity of the proposed method is demonstrated.

Key–Words: Discrete-time systems, equality constraints, state feedback, linear matrix inequalities, control algorithms, quadratic stability.

1 Introduction

In the last years many significant results have spurred interest in the problem of determining the control laws for the systems with constraints. For the typical case where a system state reflects a certain physical entities this class of constraints often rises because of physical limits, and these constraints usually keep the system state in the region of the technological conditions. Subsequently this problem can be formulated using technique dealing with system state constraints directly, where it can be coped with efficiently using linear system techniques [6]. Therefore, a special form of the constrained problems can be so formulated with the goal to optimize the state feedback controller while the system state variables satisfy the equality constraints [2], [8]. To optimize the state feedback controller parameters while the system state variables satisfy the equality constraints the design task is specified as a singular one, and associated methods have to be used to design the controller parameters in the proposed control law structure.

A number of problems that arise in the state feedback control can be reduced to a handful of standard convex and quasi-convex problems that involve matrix inequalities. It is known that the optimal solution can be computed by using interior point methods [9] which converge in polynomial time with respect to the problem size, and efficient interior point algorithms have recently been developed for, and further development of algorithms for these standard problems is an area of active research. For this approach, the stability conditions may be expressed in terms of linear matrix inequalities (LMI), which have a notable practical interest due to the existence of the numerical solvers [4], [10]. Some progress review in this field can be found in [5], [11], and the references therein.

This paper aims at providing controller design conditions with closed-loop state equality constraints for discrete-time systems and with given quadratic performance introduced in the bounded real lemma (BRL) form [1]. Motivated by the underlying idea in [2], [3] in this note it is presented new criteria to circumvent ill conditioned singular task concerning the discrete-time systems control design with state equality constraints. Due to the introduction of an enhanced LMI representation of BRL which exhibits a kind of decoupling between the Lyapunov matrix and the system dynamic matrix the task is well conditioned. This condition still impose some common matrices to obtain control that assures quadratic stability.

The paper is organized as follows. Starting with problem formulation presented in Section 2, then in Section 3 basic preliminaries are introduced together with an adapted version of discrete BRL form, referred to as equivalent form. These results are used in Section 4 to derive a convex formulation of design conditions where closed-loop state equality constraints are considered. The proposed approach leads to a well conditioned LMI to obtain a feasible solution in such singular task. Subsequently, in Section 5 a numerical example illustrates basic properties of this approach. Section 6 is finally devoted to a brief overview of the method properties demonstrating accepted conservatism of the proposed approach.
2 Problem Formulation

Through the paper the task is concerned with design of the state feedback (3) which controls a discrete-time linear dynamic system given by the set of state equations

\[ q(i + 1) = Fq(i) + Gu(i) \]  
\[ z(i) = Cq(i) + Du(i) \]

where \( q(i) \in \mathbb{R}^n \), \( u(i) \in \mathbb{R}^r \), and \( y(i) \in \mathbb{R}^m \) are vectors of the state, input and objective variables, respectively, and nominal system matrices \( F \in \mathbb{R}^{n \times n} \), \( G \in \mathbb{R}^{n \times r} \), \( C \in \mathbb{R}^{m \times n} \), and \( D \in \mathbb{R}^{m \times r} \) are real matrices. Problem of the interest is to design a stable closed-loop system using a linear memoryless state feedback controller of the form

\[ u(i) = -K q(i) \]  

where all state variables are measurable, \( K \in \mathbb{R}^{r \times n} \) is the feedback controller gain matrix, and design constraint in the next equality form

\[ q(i) \in \mathcal{N}_E = \{ q : E q = 0 \} \]

is considered, with \( E \in \mathbb{R}^{k \times n} \), rank\( E = k \leq r \).

3 Basic Preliminaries

Proposition 1 (Matrix pseudoinverse) Let \( \Theta \) is a matrix variable and \( A, B, \Lambda \) are known non-square matrices of appropriate dimensions such the equality

\[ A \Theta B = \Lambda \]

is set. Then all solution to \( \Theta \) means

\[ \Theta = A^{-1} \Lambda B^{-1} + \Theta^o - A^{-1} \Lambda \Theta^o B^{-1} \]

where

\[ A^{-1} = A^T (A A^T)^{-1}, B^{-1} = (B^T B)^{-1} B^T \]

is Moore-Penrose pseudoinverse of \( A, B \), respectively, and \( \Theta^o \) is an arbitrary matrix of appropriate dimension.

Proof: (see e.g. [3], [6], [11])

\[ q_c^T(i) = \begin{bmatrix} q^T(i) & u^T(i) \end{bmatrix} \]

it can be obtained

\[ \Delta v(q_c(i)) = q_c^T(i) P_c q_c(i) < 0 \]

where

\[ P_c = \begin{bmatrix} P_{c11} & P_{c12} \\ * & A_c \end{bmatrix} \]  
\[ P_{c11} = F^T P F + C^T C - P \]
\[ P_{c12} = F^T P G + C^T D \]
\[ P_{c22} = G^T P G + D^T D - \gamma I_r \]

It is evident that (15)-(18) can be rewritten into the composite form

\[ \begin{bmatrix} F^T P F - P & F^T P G \\ * & G^T P G - \gamma I_r \end{bmatrix} + \begin{bmatrix} C^T C & C^T D \\ * & D^T D \end{bmatrix} < 0 \]

Since

\[ \begin{bmatrix} C^T C & C^T D \\ * & D^T D \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \geq 0 \]
Therefore, adding (25), as well as the transposition of where $I$ compactly as (8). This concludes the proof.

Direct application of BRL given by (9), (8) in constrained control design is generally ill conditioned owing to singular design conditions [7], [8]. To circumvent this problem an equivalent representation of BRL is proposed in the next, where new design condition proof in the paper is based on the next lemma.

**Lemma 1 (Equivalent LMI representation of BRL)** There exists a matrix $P > 0$ and a scalar $\gamma > 0$ satisfying (8) if and only if there exists a positive definite symmetric matrix $P > 0$, $P \in \mathbb{R}^{n \times n}$, a general matrix $W \in \mathbb{R}^{n \times n}$, and a positive scalar $\gamma > 0$, $\gamma \in \mathbb{R}$ such that

$$P = P^T > 0, \quad \gamma > 0$$

$$\begin{bmatrix}
-P & 0 & F^T W^T & C^T \\
* & -\gamma I_r & G^T W^T & D^T \\
* & * & -W - W^T & 0 \\
* & * & * & -I_m
\end{bmatrix} < 0$$

(23)

where $I_r \in \mathbb{R}^{r \times r}$. $I_m \in \mathbb{R}^{m \times m}$ are identity matrices, respectively.

**Proof:** Since (1) can be rewritten as

$$q(i+1) - Fq(i) - Gu(i) = 0$$

with an arbitrary square matrix $X \in \mathbb{R}^{n \times n}$ it yields

$$q^T(i+1)X(q(i+1) - Fq(i) - Gu(i)) = 0$$

(25)

Therefore, adding (25), as well as the transposition of (25) to (11) gives

$$\Delta v(q(i)) = q^T(i+1)Pq(i+1) - q^T(i)Pq(i) + z^T(i)z(i) - \gamma u^T(i)u(i) + (q(i+1) - Fq(i) - Gu(i))^T X^T q(i+1) + q^T(i+1)X(q(i+1) - Fq(i) - Gu(i)) < 0$$

(26)

Now (26) can be compactly written as

$$q^T(i)J^Tq(i) < 0$$

(27)

where

$$J^T = \begin{bmatrix}
C^T C - P & C^T D - F^T X^T \\
* & D^T D - \gamma I_r - G^T X^T \\
* & * & P + X + X^T
\end{bmatrix}$$

(28)

Exploiting the composite form then (28) implies

$$\begin{bmatrix}
-S & 0 & VF^T \\
* & -\gamma I_r & G^T \\
* & * & S - V - V^T
\end{bmatrix} \begin{bmatrix}
C^T C & C^T D \\
* & D^T D
\end{bmatrix} < 0$$

(30)

Thus, equivalently using (20), (21), and setting $X = -W$ then (28) implies (23). This concludes the proof.

The state-feedback control problem is to find for an optimized (or prescribed) scalar $\gamma > 0$ the state-feedback gain $K$ such that the control law guarantees an upper bound of $\gamma$ to $H_{\infty}$ norm of the closed-loop transfer function.

**Lemma 2** System (1), (2) under control (3) is stable with quadratic performance $\|C_c(zI - F_c)^{-1}G\|_{\infty} \leq \gamma$, $F_r = F - GK$, $C_c = C - DK$ if there exist a positive definite symmetric matrix $S \in \mathbb{R}^{n \times n}$, a regular square matrix $V \in \mathbb{R}^{n \times n}$, a matrix $U \in \mathbb{R}^{r \times n}$, and a scalar $\gamma > 0$, $\gamma \in \mathbb{R}$ such that

$$S = S^T > 0, \quad \gamma > 0$$

(31)

$$\begin{bmatrix}
-S & 0 & VF^T & VC^T \\
* & -\gamma I_r & G^T & D^T \\
* & * & S - V - V^T & 0 \\
* & * & * & -I_m
\end{bmatrix} < 0$$

(32)

The control law gain matrix is now given as

$$K = U^T V^{-T}$$

(33)

**Proof:** Considering that det $W \neq 0$ then the congruence transform matrix $T$ can be defined as follows

$$T = \text{diag} \begin{bmatrix} W^{-1} & I_r & W^{-1} & I_m \end{bmatrix}$$

(34)

and multiplying left-hand side of (23) by $T$, and right-hand side of (23) by $T^T$ gives

$$\begin{bmatrix}
-S & 0 & VF^T & VC^T \\
* & -\gamma I_r & G^T & D^T \\
* & * & S - V - V^T & 0 \\
* & * & * & -I_m
\end{bmatrix} \begin{bmatrix}
C^T C - P & C^T D - F^T X^T \\
* & D^T D - \gamma I_r - G^T X^T \\
* & * & P + X + X^T
\end{bmatrix} < 0$$

(35)

$$S = W^{-1}PW^{-T}, \quad W^{-1} = V$$

(36)

Inserting $F \leftarrow F_c = F - GK$, $C \leftarrow C_c = C - DK$ into (35) gives

$$\begin{bmatrix}
-S & 0 & V(F - GK)^T & V(C - DK)^T \\
* & -\gamma I_r & G^T & D^T \\
* & * & S - V - V^T & 0 \\
* & * & * & -I_m
\end{bmatrix} < 0$$

(37)
and with
\[ U = VK^T \]  \hspace{1cm} (38)

(37) implies (32).

Note, \( S \) is a symmetric matrix owing to that \( W \) is a regular square matrix. This concludes the proof. \[ \Box \]

4 Constrained Control Design

4.1 Constrained Control

Using control law (3) the equilibrium control equations take the form
\[ q(i + 1) = (F - GK)q(i) \]  \hspace{1cm} (39)
\[ z(i) = (C - DK)q(i) \]  \hspace{1cm} (40)

and prescribed by a matrix \( E \in \mathbb{R}^{k \times n} \),\( \text{rank}E = k \leq r \) it is now considered the design constraint
\[ q(i) \in \mathcal{N}_E \{ q : Eq = 0 \} \]  \hspace{1cm} (41)

where the state-variable vectors have to satisfy equalities
\[ Eq(i + 1) = E(F - GK)q(i) = 0 \]  \hspace{1cm} (42)
for \( i = 1, 2, \ldots \). It is supposed that \( E \) is chosen by such way that
\[ E(F - GK) = 0 \]  \hspace{1cm} (43)
\[ EF = EGK \]  \hspace{1cm} (44)

respectively, as well as that the closed-loop system matrix \( (F - GK) \) be stable (all its eigenvalues lie in the unit circle in the complex plane \( \mathbb{C} \)). Therefore, \( \mathcal{N}_E \) is the constrain subspace, and the states be constrained in this subspace (the null space of \( E \)). Under these conditions the system state stays within the constrain subspace, i.e. \( q(i), Fq(i) \in \mathcal{N}_E \) \cite{6}.

Solving (44) with respect to \( K \) then (6) implies all solutions of \( K \) as follows
\[ K = (EG)^{\otimes 1}EF + (I - (EG)^{\otimes 1}EG)K^o \]  \hspace{1cm} (45)
where \( K^o \) is an arbitrary matrix with appropriated dimension and
\[ (EG)^{\otimes 1} = (EG)^T(EG(EG)^T)^T \]  \hspace{1cm} (46)

Thus, it is possible to express (45) as
\[ K = J + LK^o \]  \hspace{1cm} (47)
where
\[ J = (EG)^{\otimes 1}EF \]  \hspace{1cm} (48)
and
\[ L = I - (EG)^T(EG(EG)^T)^T \]  \hspace{1cm} (49)
is the projection matrix (the orthogonal projector onto the null space \( \mathcal{N}_{EG} \) of \( EG \) \cite{7}).

4.2 Control Parameter Design

Theorem 3 System (1), (2) under control (3) satisfying the constrain (4) is stable with quadratic performance \( \|C_c(zI - F)^{-1}G\|_{\infty} \leq \gamma \), \( F_c = F - GK \), \( C_c = C - DK \) if there exist a positive definite symmetric matrix \( S \in \mathbb{R}^{n \times n} \), a regular square matrix \( V \in \mathbb{R}^{n \times n} \), a matrix \( U \in \mathbb{R}^{l \times n} \), and a scalar \( \gamma > 0 \), \( \gamma \in \mathbb{R} \) such that
\[ S = S^T > 0, \quad \gamma > 0 \]  \hspace{1cm} (50)
\[ \begin{bmatrix} -S & 0 & VF^oT - UG^oT & VC^oT - UD^oT \\ -\gamma I_r & G^oT & D^oT \\ \ast & \ast & S - V - V^T & \ast \\ \ast & \ast & \ast & -I_m \end{bmatrix} < 0 \]  \hspace{1cm} (51)

where
\[ F^o = F - GJ, \quad G^o = GL \]  \hspace{1cm} (52)
\[ C^o = C - DJ \]  \hspace{1cm} (53)

Then
\[ K^o = UTV^{-T}, \quad F_c = F^o - G^oK^o = F - GK \]  \hspace{1cm} (54)

and the control law gain matrix \( K \) is given as in (47), \( \Box \)

Proof: Inserting (3), (47) into (1), (2) gives
\[ q(i + 1) = F^oq(i) + G^o\mathbf{u}^o(i) \]  \hspace{1cm} (55)
\[ z(i) = C^oq(i) + D^o\mathbf{u}^o(i) \]  \hspace{1cm} (56)

Since now (55) can be rewritten as
\[ q(i + 1) - F^oq(i) - G^o\mathbf{u}^o(i) = 0 \]  \hspace{1cm} (57)

with an arbitrary square matrix \( X \in \mathbb{R}^{n \times n} \) it yields
\[ q(i + 1) = F^oq(i) - G^o\mathbf{u}^o(i) = 0 \]  \hspace{1cm} (58)

Defining Lyapunov function as follows
\[ v(q(i)) = q^T(i)Pq(i) + \sum_{i=0}^{i-1}(z^T(l)z(l) - \gamma \mathbf{u}^oT(l)\mathbf{u}^o(l)) > 0 \]  \hspace{1cm} (59)

then the forward difference along a solution of the system (57), (58) is
\[ \Delta v(q(i)) = q^T(i + 1)Pq(i + 1) - q^T(i)Pq(i) + z^T(i)z(i) - \gamma \mathbf{u}^oT(i)\mathbf{u}^o(i) < 0 \]  \hspace{1cm} (60)

and adding (58), as well as the transposition of (58) to (60) results in
\[ \Delta v(q(i)) = q^T(i + 1)Pq(i + 1) - q^T(i)Pq(i) + z^T(i)z(i) - \gamma \mathbf{u}^oT(i)\mathbf{u}^o(i) + + (q(i + 1) - F^oq(i) - G^o\mathbf{u}^o(i))^T X^T q(i + 1) + + q^T(i + 1)X(q(i + 1) - F^oq(i) - G^o\mathbf{u}^o(i)) < 0 \]  \hspace{1cm} (61)
Since (61) can now be compactly written as
\[ q^T(i)J^*q^*(i) > 0 \] (62)
where
\[ q^*(i) = [q^T(i) \ u^T(i) \ q^T(i+1)] \]
\[ J^* = \begin{bmatrix} C^T - P & C^TD & -F^TX^T \\ D^T - \gamma I_r & -G^TX^T & \ast \\ \ast & \ast & P + X + X^T \end{bmatrix} < 0 \]
(64)
it is evident that (64) takes the same structure as (28), and so due to (28) replacing the matrices \((F, G, C, D)\) in (32) by \((F^o, G^o, C^o, D^o)\) the inequality (51) is obtained. This concludes the proof.

4.3 Constrained tracking problem

**Theorem 4** Considering the state control in a forced mode then the state constrain (41) given on the system state variables attains the steady-state value of the common state variable \(q_d(i) = Eq(i)\), i.e.
\[ q_d = EW_ww_s \] (65)
where the forced mode is defined by the control policy
\[ u(i) = -Kq(i) + W_ww(i) \] (66)
m = r, \( w(i) \in \mathbb{R}^r \) is desired output vector, and \( W_w \in \mathbb{R}^{r \times r} \) is the gain matrix.

**Proof:** See [3].

5 Illustrative example

To demonstrate properties of proposed approach, the system with two-inputs and two-outputs is used in the example. The parameters of the system are
\[ F = \begin{bmatrix} 0.9993 & 0.0987 & 0.0042 \\ -0.0212 & 0.9612 & 0.0775 \\ -0.3875 & -0.7187 & 0.5737 \end{bmatrix}, \ D = 0 \]
\[ G = \begin{bmatrix} 0.0010 & 0.0010 \\ 0.0206 & 0.0197 \\ 0.0077 & -0.0078 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix} \]
respectively, for sampling period \(\Delta t = 0.1\) s. The state constraint was specified as
\[ \frac{q_1(t) - 0.4 \ q_3(t)}{q_2(t)} = 0.1 \]
which implies
\[ E = \begin{bmatrix} 1 & -0.1 & -0.4 \end{bmatrix} \]
and subsequently it yields
\[ (EG)^{\oplus 1} = \begin{bmatrix} -38.2302 \\ 19.6977 \end{bmatrix}, \ L = \begin{bmatrix} 0.2098 & 0.4072 \\ 0.4072 & 0.7902 \end{bmatrix} \]
\[ J = \begin{bmatrix} -44.2102 & -11.0891 & 8.9088 \\ 22.7789 & 5.7135 & -4.5902 \end{bmatrix} \]
Solving (50), (51) with respect to the LMI matrix variables \(S, U, V\) and \(\gamma\) using Self–Dual–Minimization (SeDuMi) package for Matlab [10], the feedback gain matrix design problem in the constrained control was solved as feasible with the results
\[ U = \begin{bmatrix} 0.0147 & 0.0285 \\ 0.3985 & 0.7733 \\ -0.0445 & -0.0864 \end{bmatrix}, \ \gamma = 0.7324 \]
\[ V = \begin{bmatrix} -0.0043 & -0.0100 & -0.0059 \\ -0.0124 & 0.1389 & 0.1049 \\ -0.0072 & 0.0868 & 0.2694 \end{bmatrix} \]
\[ S = \begin{bmatrix} -0.0049 & -0.0121 & -0.0087 \\ -0.0121 & 0.1522 & 0.1035 \\ -0.0087 & 0.1035 & 0.3180 \end{bmatrix} \]
Inserting \(U\) and \(V\) into (54) there were computed the feedback gain matrices as follows
\[ K^o = \begin{bmatrix} 13.4769 & 5.1827 & -1.4741 \\ 26.1567 & 10.0588 & -2.8609 \end{bmatrix} \]
\[ K = \begin{bmatrix} -30.7333 & -5.9064 & 7.4347 \\ 48.9355 & 15.7723 & -7.4511 \end{bmatrix} \]
Control law such defined rises up a stable control with the closed-loop system matrix eigenvalue spectrum
\[ \rho(F - GK) = \{ 0.0000, 0.0934, 0.8271 \} \]
Note that one eigenvalue of \(F_c = F - GK\) is zero (rank(\(E\)) = 1) since the constrained control design task is a singular problem [8].

In the presented two figures the simulation results are shown of the closed-loop system response in the forced mode [12], where Figure 1 is concerned with the state variables response, and Figure 2 with the output variable response. The initial state vector, the desired steady-state values of the objective variables, and the signal gain matrix were set as: \(q(0) = 0\),
\[ w(i) = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}, \ W = \begin{bmatrix} -6.2124 & -19.5270 \\ 6.6807 & 37.3138 \end{bmatrix} \]
It is evident, that the constrain (42) is satisfied at all time instant in such way that

\[ q_d = EW_ww_s = 0.0601 \]

( see common variable \( q_d(i) \) in the Figure 1).

6 Concluding Remarks

In this paper there was developed a new method based on a classical memoryless feedback \( H_{\infty} \) control of discrete-time systems if equality constraints tying together state variables are prescribed. The quadratic stability of the control scheme is established in the sense of equivalent representation of BRL to circumvent ill conditioned singular design task. Such matrix inequality is linear with respect to the system variables, and does not involve any product of the Lyapunov matrix and the system matrices. This provides one way for determination of parameter-independent Lyapunov function by solving singular LMI problems. The validity of the proposed method is demonstrated by a numerical example with the equality constraint tying together all state variables in a prescribed way.

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