Optimization normal availability of a cold standby system with priority

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Abstract: In this paper we consider an optimization problem for a deteriorating cold standby repairable system that consists of two dissimilar components and one repairman. Suppose that the life of each component satisfies the exponentially distribution and repair time of the component satisfies the general distribution, the component 1 has priority in use and repair. Firstly, a mathematical model is built via the differential and partial differential equations, and some basic result such as the existence and uniqueness of the solution, the non-negative steady-state solution and the exponential stability of the system are derived. Based on the stability result, we discuss an optimization problem about the normal availability of the system.

Key–Words: Cold standby system, $C_0$ Semigroup, steady-state solution, stability, normal availability.

1 Introduction

With the development of the modern technology and extensive use of electronic products, the reliability problem of the reparable systems has become a hot topic. It is well known that the redundancy can enhance the reliability of the system. In the earlier study, the cold standby component doesn't fail in its standby period since they are machine system. However, the electronic product doesn’t satisfy this property. In fact, for an electronic product, it has less failure rate if it is in use, otherwise, it has greater failure rate after certain time, which means that it deteriorates. Therefore, the reliability study of the system consisting of the electronic product with repair is very important. For the reliability problem of system, it has been a hot topic in engineering and mathematics. Using mathematical model and Markov renewal theory, one studied the indices of reliability of the reparable system, for example, see [1], [2], [3]. Notice that, in the earlier research, the components after repair is "as good as new", only recent a deteriorating case is considered in ([8]).

Although some nice results have been obtained, there exists certain difficulty to obtain some properties of the system, for instance, optimal normal availability of a system. The mainly difficulty comes from the mathematical model; this is because one cannot give an exact solution to the model. To overcome this difficulty, many authors have worked on reparable systems using functional analysis method (see [4]-[6], etc), more precisely, semigroup theory of bounded linear operators ([7]) to prove the well-posedness and the stability of the system. In this paper, we will give a new model about a cold standby repairable system, and then discuss the system by functional analysis method.

The rest is organized as follows. In section 2, we describe a mathematical model for the system under consideration and give some results. And then in section 3, we consider an optimal problem about the normal availability of the system. The optimization solution is explicitly given.

2 Description of system and its basic results

Firstly, we describe the system under consideration. Suppose that a system consists of two dissimilar components and one repairman, the component 1 is the main working unit and the component 2 is cold standby unit. The component 1 has priority in use and repair. The system satisfies the following assumptions:

Assumption 1. Initially, the two components are both new, and component 1 is in a working state while component 2 is in a standby state.

Assumption 2. Assume that both components after repair are "as good as new".

Assumption 3. The component 1 has priority in working and repair. When both components are
good, the component 1 has the higher use priority than the
compartment 2, even if component 2 is working, it
must be switched immediately into the standby state
as soon as component 1 after repaired so that the com-
ponent 1 becomes the working state; When both com-
ponents fail (i.e. the system is down), component 1
has the higher repair priority than component 2, even
if the repairman is repairing component 2, he must
switch to repair the component 1. He will work on
the repair of component 2 after completing the repair
on component 1.

Assumption 4. The standby component will per-
haps fail in the standby time for some reason. The
failure and repair times of both components follow ex-
ponential distribution and general distribution respecti-
vely. \( \lambda_j \) and \( \varepsilon \) denote the failure rate of component
\( j (j = 1, 2) \) and the standby component; \( \mu_j(x) \) denote
the repair rate of component \( j (j = 1, 2) \).

Assumption 5. All failures are independent of
each other.

Under these assumptions, we can divide the sys-
tem into the following states:

0: The components 1 and 2 are in good condition;
1: The component 1 is working and component 2 is working;
2: The component 1 is working and component 2
is failure under repair;
3: The components 1 and 2 are failure under re-
pair.

\( P_0(t) \) denotes the probability that component 1 is
in working state and component 2 is in cold standby
state at time \( t \);

\( p_1(t, x)dx \) represents the probability that the re-
pairman is dealing with component 1 with the elapsed
time lying in \( [x, x + dx] \) and component 2 is in work
at time \( t \);

\( p_2(t, x)dx \) represents the probability that the re-
pairman is dealing with component 2 with the elapsed
time lying in \( [x, x + dx] \) and component 1 is in work
at time \( t \);

\( p_3(t, x)dx \) represents the probability that the re-
pairman is dealing with component 1 with the elapsed
time lying in \( [x, x + dx] \) and component 2 is waiting
for repair at time \( t \).

By the supplementary variables technique, the model of the system can be formulated as

\[
\begin{align*}
\frac{dP_0(t)}{dt} &= -\alpha_1 P_0(t) + \int_0^\infty \mu_1(x)p_1(x, t)dx \\
\frac{dp_1(x, t)}{dt} &= -\mu_1(x)p_1(x, t) - \lambda_2 p_2(x, t) \\
\frac{dp_2(x, t)}{dt} &= -\mu_2(x)p_2(x, t) - \lambda_1 p_3(x, t) \\
\frac{dp_3(x, t)}{dt} &= -\mu_3(x)p_3(x, t) - \lambda_1 p_1(x, t).
\end{align*}
\]

The boundary conditions are

\[
\begin{align*}
p_1(0, t) &= \lambda_1 P_0(t), \\
p_2(0, t) &= \varepsilon P_0(t) + \int_0^\infty \mu_1(x)p_3(x, t)dx, \\
p_3(0, t) &= \lambda_1 \int_0^\infty p_2(x, t)dx + \lambda_2 \int_0^\infty p_1(x, t)dx,
\end{align*}
\]

where \( (x, t) \in \mathbb{R}_+ \times \mathbb{R}_+ \).

To discuss the well-posedness of the system, we
formulate the system (1) into a suitable Banach space.
Based on the practical background of our model, we
have assumed that \( \mu_j(x) (j = 1, 2) \) satisfy

\[
M = \sup_{x \in \mathbb{R}_+} \mu_j(x) < \infty; \int_0^\infty \mu_j(s)ds = \infty.
\]

In the sequel, we denote by \( \mathbb{R}_+ = [0, \infty) \). From
the physical meaning of the problem, we take a sta-
tate space \( \mathbb{X} = \mathbb{R} \times [L^1(\mathbb{R}_+)]^3 \), being equipped with
the norm \( ||P|| = ||P_0|| + \sum_{j=1}^3 ||p_j||_{L^1} \), for each
\( P = (P_0, p_1, p_2, p_3) \in \mathbb{X} \). Obviously, \( \mathbb{X} \) is a Banach space.

Define an operator \( A : \mathbb{X} \to \mathbb{X} \) by

\[
A \begin{pmatrix} P_0 \\ p_1(x, t) \\ p_2(x, t) \\ p_3(x, t) \end{pmatrix} = \begin{pmatrix} -(\lambda_1 + \varepsilon)P_0 + \int_0^\infty \mu_1(x)p_1(x, t)dx \\ -\mu_1(x) - \lambda_2 p_2(x, t) \\ -\mu_2(x) - \lambda_1 p_3(x, t) \\ -\mu_3(x)p_3(x, t) \end{pmatrix}
\]

with domain

\[
D(A) = \left\{ P \in \mathbb{X} : p_j(x, t) \text{ is absolutely continuous}, \right. \\
\left. p_1(x, t), p_2(x, t) \in L^1(\mathbb{R}_+), \right. \\
\left. p_1(0) = \lambda_1 P_0, \right. \\
\left. p_2(0) = \varepsilon P_0 + \int_0^\infty \mu_1(x)p_3(x, t)dx, \right. \\
\left. p_3(0) = \lambda_1 \int_0^\infty p_2(x, t)dx + \lambda_2 \int_0^\infty p_1(x, t)dx, \right. \\
\left. j = 1, 2, 3 \right. \}
\]

Obviously, \( A \) is a linear operator in \( \mathbb{X} \).

By the definition of \( A \), the system (1) can be
rewritten as an abstract Cauchy problem in Banach s-
pace \( \mathbb{X} \):

\[
\begin{align*}
\frac{dP(t)}{dt} &= AP(t), \quad t \geq 0, \\
P(0) &= (1, 0, 0, 0).
\end{align*}
\]

where \( P(t) = (P_0(t), p_1(x, t), p_2(x, t), p_3(x, t)) \).

By Theorem II 6.7 and Definition II 6.8 of [9],
we know that the well-posedness of the system (1)
is equivalent to the operator \( A \) being the generator of
some \( C_0 \) semigroup \( T(t) \). Therefore, the main task
of this section is to verify that the operator \( A \) gen-
erates a \( C_0 \) semigroup. In particular, the semigroup
\( T(t) \) generated by the operator \( A \) is also positive and
contractive.

Theorem 1. Let \( A \) be defined by (4) and (5). Then the
following statements are true

1) \( A \) is a closed and densely linear operator in \( \mathbb{X} \).
2) \( A \) is a dissipative operator in \( \mathbb{X} \).
3) \( \gamma \in \rho(A) \) for \( \Re \gamma > 0 \).
As a direct result of Lumer-Phillips Theorem and positive operator theory (see, [7] [11]), we have the following result.

**Theorem 2.** Let $A$ and $X$ be defined as before. Then $A$ generates a positive $C_0$-semigroup of contractions on $X$. Hence the equation (6) has a unique positive solution.

**Proof:** According to the positive semigroup theory (see [11]), $A$ generates a positive $C_0$-semigroup of contractions if and only if $A$ is a dispersive and $\mathcal{R}(I - A) = X$. Since Theorem 1 has asserted that $\mathcal{R}(I - A) = X$, we only need to prove $A$ is a dispersive operator. Take $\Phi = (\text{sign}_+(P_0), \text{sign}_+(P_1(x)), \text{sign}_+(P_2(x)), \text{sign}_+(P_3(x)))$ instead of $Q_P$, where $\text{sign}_+(P) = \max\{0, \text{sign}(P)\}$. The others is similar to proving the dissipativity of the operator $A$, so we omit. \hfill \Box

**Theorem 3.** The semigroup $T(t)$ generated by $A$ is random one.

**Proof:** It is sufficient to prove that $T(t)$ is positive conservative according to definition of random semigroup. For any positive vector $P \in D(A)$, $P = (P_0, p_1(x), p_2(x), p_3(x))$, we have $T(t)P > 0$ and $T(t)P \in D(A)$, since $T(t)$ is positive. Set $P(t, \cdot) = (P_0(t), p_1(x, t), p_2(x, t), p_3(x, t))$, obviously, $T(t)P$ satisfies the equation $\frac{dP(t)}{dt} = AP$, $\forall t > 0$.

Integrating the differential equations in (1) from 0 to $\infty$ with respect to $x$, we have

$$\frac{dP_0(t)}{dt} + \sum_{j=1}^3 \frac{d}{dt} \int_0^\infty p_j(x, t)dx = 0,$$

which shows that $P_0(t) + \sum_{j=1}^3 \int_0^\infty p_j(x, t)dx$ is a constant. Since $T(t)P$ is continuous in $t$, we obtain $||T(t)P|| = P_0(t) + \sum_{j=1}^3 \int_0^\infty p_j(x, t)dx = ||P||$. So $T(t)$ is a random semigroup, which coincides with the physical meaning. \hfill \Box

Next we will introduce the stability result of the system.

**Theorem 4.** $\gamma_0 = 0$ is a simple eigenvalue of $A$ and corresponding an eigenvector is $P = (P_0, p_1, p_2, p_3)$ where

$$\begin{cases}
  p_1(x) = \lambda_1 P_0 e^{-\int_0^t (\mu_1(s) + \lambda_2)ds}, \\
  p_2(x) = P_0 \frac{e^{\lambda_2 \gamma_2(0)}}{1 - \lambda_2 \gamma_2(0)} e^{-\int_0^t (\mu_2(s) + \lambda_1)ds}, \\
  p_3(x) = P_0 \frac{e^{\lambda_1 \gamma_2(0) + \lambda_2 \gamma_3(0)}}{1 - \lambda_1 \gamma_2(0) - \lambda_2 \gamma_3(0)} e^{-\int_0^t \mu_1(s)ds}.
\end{cases}$$

with $P_0 = \frac{1}{2}$, where

$$Z = 1 + \lambda_1 \int_0^\infty e^{-\int_0^t (\mu_1(s) + \lambda_2)ds} + \frac{e^\lambda \lambda_2 \gamma_2(0)}{1 - \lambda_2 \gamma_2(0)} \int_0^\infty e^{-\int_0^t (\mu_2(s) + \lambda_1)ds} + \frac{e^\lambda \lambda_1 \gamma_2(0) + \lambda_2 \gamma_3(0)}{1 - \lambda_1 \gamma_2(0) - \lambda_2 \gamma_3(0)} \int_0^\infty e^{-\int_0^t \mu_1(s)ds}$$

(8)

Moreover if for any $\xi \in \mathbb{R}_+$,

$$\sup_{\xi \geq 0} \int_{-\infty}^\infty e^{-\int_0^t \mu_1(s)ds} dx < \infty.$$  

(9)

There is no other spectrum of $A$ besides zero in the imaginary axis. Hence it holds that $\lim_{t \to \infty} P(t) = \lim_{t \to \infty} T(t)P(0) = (P(0), Q) \hat{P}.

**Remark 5.** The condition (9) is necessary for $i\mathbb{R}\backslash \{0\} \subset \rho(A)$, otherwise the estimate it will be $i\mathbb{R} \subset \sigma(A)$.

Now we define non-negative real number $\hat{\mu}_1$ by

$$\hat{\mu}_1 = \sup\{\eta \geq 0 \mid \sup_{r \geq 0} \int_{-\infty}^\infty e^{nx - \int_0^t \mu_1(s+r)ds} dx < \infty\}.$$  

(10)

and assume that $\hat{\mu}_1 > 0$, which implies that condition (9) holds. Obviously, when $\eta < \hat{\mu}_1$, the integral for $\forall r \geq 0$,

$$\sup_{r \geq 0} \int_{-\infty}^\infty e^{-\int_0^t (\mu_1(s+r) - \eta)ds} dx < \infty,$$

(11)

while for $\eta > \hat{\mu}_1$, it must be

$$\int_{-\infty}^\infty e^{-\int_0^t (\mu_1(s) - \eta)ds} dx = \infty.$$

Set

$$\hat{\mu} = \min\{\hat{\mu}_1, \lambda_1\}.$$  

(12)

**Theorem 6.** Let $X$, $A$ and $\hat{\mu}$ be defined as before. Then we have

(I) The half-plane $\{\gamma \in \mathbb{C} \mid \Re \gamma < \hat{\mu} \}$ is in the spectrum of $A$;

(II) The set $\{\gamma \in \mathbb{C} \mid \Re \gamma + \hat{\mu} > 0, D(\gamma) \neq 0\}$ is in the resolvent set of $A$ and the set $\{\gamma \in \mathbb{C} \mid \Re \gamma + \hat{\mu} > 0, D(\gamma) = 0\}$ consists of all eigenvalues of $A$;

(III) $\forall \delta > 0$, there are at most finitely many eigenvalues of $A$ in the region $\{\gamma \in \mathbb{C} \mid \Re \gamma + \hat{\mu} \geq \delta\}$;

(IV) There exists a constant $\omega_1 > 0$ such that in the region $\{\gamma \in \mathbb{C} \mid \Re \gamma > -\omega_1\}$ has only one eigenvalue $\gamma_0 = 0$, thus it is strictly dominant.

(V) For any initial $P(0)$, we have

$$\|P(t) - (P(0), Q) \hat{P}\| \leq 2e^{-\omega_1 t}\|P(0)\|$$

(13)

where $P(t) = T(t)P(0)$.
Theorem 7. The normal steady-state availability of the system is

\[ A_v^N = \frac{1}{Z} \]  

where \( Z \) is given by (8).

**Proof:** By the expression of steady state (8) and the definition of normal availability \( A_v^N = P_0 \), we know the result holds. \( \square \)

We now consider an optimization problem for the repair rate \( \mu(x) = (\mu_1(x), \mu_2(x)) \). Suppose \( P_0 \) is the expectable probability of zero state in steady-state and \( P_0(\mu) \) is the first component of solution of system corresponding to \( \mu(x) \). Take the index functional

\[ S(\mu) = |P_0(\mu) - \hat{P}_0|^2. \]

Our aim is to find \( \mu(x) = (\mu_1(x), \mu_2(x)) \) such that it minimizes \( S(\mu) \).

According to the assumptions (3), we let the admissible set be

\[ U = \left\{ \left( \mu_1(x), \mu_2(x) \right) \in [L^\infty(\mathbb{R}_+)]^2 \mid \frac{\ln(1+x)}{1+x} \leq \mu_1(x), \mu_2(x) \leq M \right\} \]

Clearly \( U \) is a closed and convex set in \([L^\infty(\mathbb{R}_+)]^2\). Our object is to find \( \mu^* \in U \) such that

\[ S(\mu^*) = \inf_{\mu \in U} S(\mu). \]

\( \mu^* \) is said to be optimal repair rate of the system.

To solve the problem (15), let

\[ W = \left\{ P_0 \in \mathbb{R} \mid \exists \mu \in U \text{ such that } (P_0, p_1(x), p_2(x), p_3(x)) \in \mathbb{R} \times (L^1(\mathbb{R}_+))^3 \right\} \]

is a nonnegative solution corresponding to \( \mu(x) \)

\[ W = \begin{cases} P_0 + \int_0^\infty p_1(x)dx & \text{is a dissipative semigroup in } \mathbb{R}_+ \\ + \int_0^\infty p_2(x)dx + \int_0^\infty p_3(x)dx & = 1 \end{cases} \]

We firstly prove that there exists a \( P_0^* \in W \) satisfying 

\[ S(P_0^*) = \inf_{P_0 \in W} S(P_0) = \inf_{P_0 \in W} |P_0 - \hat{P}_0|^2. \]

**Theorem 8.** \( W \) is a bounded and \( w^* \)-closed set in \( \mathbb{R} \).

**Proof:** From the previous section we know \( W \) is a bounded set. Now we prove that \( W \) is a closed set.

Let \( P_0^{n(1)} \in W \) and \( P_0^{n(2)} \to \tilde{P}_0 \). Then exist a sequence \( \mu^{n(1)}(x) \in U \) such that \( P^{n(1)}(x) \) are the nonnegative solution to (1) with boundary conditions (2). Since \( U \subset [L^\infty(\mathbb{R}_+)]^3 \) is bounded and then \( w^* \)-sequence compact, there exist a subsequence of \( \mu^{n(1)}(x) \) without loss generality such that \( \mu^{n(1)}(x) \to^w \mu(x) = (\mu_1(x), \mu_2(x)) \). It is easy to see that \( \mu_1(x) \) and \( \mu_2(x) \) are nonnegative functions.

For any \( x \in \mathbb{R}_+ \) fixed, the function \( \chi_{[a,x]}(s) \in L^1(\mathbb{R}_+) \), the \( w^* \) convergence of \( \mu^{n(1)}(x) \) deduces that

\[ \lim_{n \to \infty} \int_0^\infty \chi_{[0,x]}(s) \mu_1^{n(1)}(s)ds = \int_0^\infty \mu_1(s)ds, \]

\[ \lim_{n \to \infty} \int_0^\infty \chi_{[0,x]}(s) \mu_2^{n(1)}(s)ds = \int_0^\infty \mu_2(s)ds. \]

Since \( (\mu_1^{n(1)}, \mu_2^{n(1)}) \in U \) implies that

\[ \frac{1}{2} [\ln(1 + x)]^2 \leq \int_0^x \mu_1^{n(1)}(s)ds, \int_0^x \mu_2^{n(1)}(s)ds \leq Mx, \]

so it holds that

\[ \frac{1}{2} [\ln(1 + x)]^2 \leq \int_0^x \mu_1(s)ds, \int_0^x \mu_2(s)ds \leq Mx. \]

Therefore, \( \mu(x) \in U \). On the other hand, from (7) we get that

\[ \begin{align*}
P_1^{n(1)}(x) &= \lambda_1 P_0^{n(1)} e^{-\int_0^x (\mu_1^{n(1)}(s)+\lambda_2)ds} \\
P_2^{n(1)}(x) &= P_0^{n(1)} e^{-\int_0^x (\mu_1^{n(1)}(s)+\lambda_2)ds} \\
P_3^{n(1)}(x) &= P_0^{n(1)} e^{-\int_0^x (\mu_1^{n(1)}(s)+\lambda_2)ds} \\
\end{align*} \]

The convergence of \( P_0^{n(1)}(x) \) implies that for each \( x \in \mathbb{R}_+ \), \( P_1^{n(1)}(x) \), \( P_2^{n(1)}(x) \) and \( P_3^{n(1)}(x) \) are convergent and
the limit are respectively
\[\tilde{p}_1(x) = \lambda_1 e^{-\int_0^b (\mu_1(s) + \lambda_2) ds} \tilde{P}_0,\]
\[\tilde{p}_2(x) = e^{\xi + \lambda_2 G_4(0)} e^{-\int_0^b (\mu_2(s) + \lambda_1) ds} \tilde{P}_0,\]
\[\tilde{p}_3(x) = \frac{e^{\xi \lambda_2 G_4(0) + \lambda_1 G_2(0)}}{1 - \lambda_1 G_2(0)} e^{-\int_0^b \mu_1(s) ds} \tilde{P}_0.\]

Clearly, it holds that
\[\tilde{p}_1(0) = \lambda_1 \tilde{P}_0,\]
\[\tilde{p}_2(0) = e^{\xi} \tilde{P}_0 + \int_0^\infty \tilde{p}_1(x) \tilde{p}_3(x) dx,\]
\[\tilde{p}_3(0) = \lambda_1 \int_0^\infty \tilde{p}_2(x,t) dx + \lambda_2 \int_0^\infty \tilde{p}_1(x) dx.\]

From above expression we see that \(\tilde{p}_1(x), \tilde{p}_2(x), \tilde{p}_3(x)\) are absolute continuous and \(\tilde{P}_0, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3(x)\), satisfy the differential equations in (1).

Finally, we prove that the integral \(\int_0^\infty e^{-\int_0^b \tilde{p}_1(s) ds} dx\) is finite. Note that the relation \(\frac{1}{2} [\ln(1 + x)]^2 \leq \int_0^\infty \mu_1(s) ds \leq M \pi\), we have
\[\int_0^\infty e^{-\int_0^b \mu_1(s) ds} dx \leq \int_0^\infty e^{-\frac{1}{2} [\ln(1 + x)]^2} dx < \infty.\]

The Fatou lemma asserts that
\[\int_0^\infty e^{-\int_0^b \mu_1(s) ds} dx \leq \int_0^\infty e^{-\frac{1}{2} [\ln(1 + x)]^2} dx.\]

Thus,
\[Z = 1 + \lambda_1 \int_0^\infty e^{-\int_0^b (\mu_1(s) + \lambda_2) ds} + \frac{e^{\xi + \lambda_2 G_4(0)}}{1 - \lambda_1 G_2(0)} \int_0^\infty e^{-\int_0^b (\mu_2(s) + \lambda_1) ds} + \frac{e^{\xi \lambda_2 G_4(0) + \lambda_1 G_2(0)}}{1 - \lambda_1 G_2(0)} \int_0^\infty e^{-\int_0^b \mu_1(s) ds} \tag{19}\]

has meaning. Therefore, \((\tilde{P}_0, \tilde{p}_1(x), \tilde{p}_2(x), \tilde{p}_3(x))\) is a nonnegative solution of (1) corresponding to \(\mu(x) = (\mu_1(x), \mu_2(x))\), and satisfies condition
\[\tilde{p}_0(\mu) + \sum_{j=1}^3 \int_0^\infty \tilde{p}_j(x) dx = 1.\]

So \(\tilde{P}_0 \in W\), and \(W\) is a closed set.

**Theorem 9.** \(W\) is a convex set, and \(S(P_0)\) is a strictly convex functional on \(W\).

**Proof:** Let \(P_0^{(1)}, P_0^{(2)} \in W\) and \(P_0^{(1)} \neq P_0^{(2)}\). By the definition of \(W\) there exist \(\mu_i^{(i)} = (\mu_1^{(i)}, \mu_2^{(i)}), i = 1, 2\) corresponding to \(P_0^{(i)}\). For any \(0 < \tau < 1\), we set \(P^{(i)} = (P_0^{(i)}, p_1^{(i)}(x), p_2^{(i)}(x), p_3^{(i)}(x))\) and
\[P_\tau(x) = \tau P^{(1)}(x) \tau P^{(2)}(x).\]

A direct verification shows that \(P_\tau\) satisfy the differential equation (1) corresponding to \(\mu_\tau = \tau \mu^{(1)} + (1 - \tau) \mu^{(2)}\). So \(W\) is a convex set. Further, we have
\[S(\tau P_0^{(1)} + (1 - \tau) P_0^{(2)}) = |\tau P_0^{(1)} + (1 - \tau) P_0^{(2)} - \tilde{P}_0| \]
\[< |\tau P_0^{(1)} - \tilde{P}_0|^2 + (1 - \tau)|P_0^{(2)} - \tilde{P}_0|^2 = \tau S(P_0^{(1)}) + (1 - \tau) S(P_0^{(2)}).\]

Therefore, \(S(P_0)\) is a strictly convex functional in \(W\).

**Theorem 10.** There exists a unique \(P_0^* \in W\) such that
\[S(P_0^*) = \inf_{P_0 \in W} S(P_0).\]

**Proof:** Since \(W \subset \mathbb{R}_+^n\) is a bounded and closed set, the existence and uniqueness of \(P_0^* \in W\) follows from the theory of the convex function on convex set.

**Theorem 11.** If \(\tilde{P}_0 \notin W\), then there exists a unique \(\mu^* \in U\) such that
\[S(\mu^*) = \inf_{\mu \in U} S(\mu).\]

**Proof:** Note that \(S(P_0) = |P_0 - \tilde{P}_0|^2\), \(S(\mu) = |P_0(\mu) - \tilde{P}_0|^2\) and \(P_0(\mu) = \frac{1}{Z(\mu)}\) where
\[Z = 1 + \lambda_1 \int_0^\infty e^{-\int_0^b (\mu_1(s) + \lambda_2) ds}
+ \frac{e^{\xi + \lambda_2 G_4(0)}}{1 - \lambda_1 G_2(0)} \int_0^\infty e^{-\int_0^b (\mu_2(s) + \lambda_1) ds}
+ \frac{e^{\xi \lambda_2 G_4(0) + \lambda_1 G_2(0)}}{1 - \lambda_1 G_2(0)} \int_0^\infty e^{-\int_0^b \mu_1(s) ds} \tag{20}\]

has meaning. Obviously, \(Z(\mu)\) depends continuously on \((\mu_1, \mu_2) \in U\). In particular, it holds that
\[Z(\mu) \geq 1 + \lambda_1 \frac{M + \lambda_2}{M + \lambda_1} + \lambda_2 \frac{M + \lambda_1}{M + \lambda_2} + \frac{e^{\lambda_1(M + \lambda_2) + \lambda_2 (M + \lambda_1)}}{M^2(M + \lambda_2)} \tag{21}\]

where \(M\) is defined as in \(U\). Therefore, \(\forall \mu \in U\),
\[P_0(\mu) \leq \frac{1}{Z(\mu)} \leq \frac{M^2(M + \lambda_2)(M + \lambda_1)(M + \lambda_2)(M + \lambda_1)}{M^2(M + \lambda_2)(M + \lambda_1)(M + \lambda_2)(M + \lambda_1)}.\]

Therefore, when \(\tilde{P}_0 \notin W\), it must have
\[\tilde{P}_0 > \frac{M^2(M + \lambda_2)^2}{M^2(M + \lambda_2)} = M(1, 1)^2\]

so \(S(\mu)\) arrives minimum at \(\mu^* = M(1, 1)^2\).

**Acknowledgements:** The research is supported by the National Science Natural Foundation in China (NSFC-60874034).
References:


