Optimization of elastic and inelastic conical shells of piece wise constant thickness

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Abstract: - Elastic and inelastic conical shells with stepped thickness subjected to the distributed transverse pressure and loaded by a rigid central boss are studied. The optimization problem is posed in a general form involving as particular cases several different problems.

Resorting to the variational methods necessary optimality conditions are derived. The problems regarding to the maximization of the plastic limit load and to the minimum weight design are studied in a greater detail.

Key-Words: - conical shell, optimization, anisotropy, yield condition, limit load, elastic material

1 Introduction
Optimal designs of conical shells made of inelastic materials are established by Lellep and Puman [10, 11] whereas spherical caps made of von Mises material are studied by Lellep and Tungel [12]. It is well known that the especially high ratio of the strength-to-weight ratio is achieved by the use of composite materials which exhibit the orthotropic and anisotropic behaviour. Exploiting the upper bound theorem of limit analysis the anisotropic behaviour of structures was studied by Capsoni, Corradi, Vena [2]; Corradi, Luzzi, Vena [4]; Pan, Seshadri [13].

In the present paper conical shells with piece wise constant thickness are considered. An optimal design method is developed for conical shells made of elastic or inelastic materials.

2 Problem formulation
Let us study the behaviour of an axisymmetric conical shell (Fig. 1) subjected to an axisymmetric loading. We confine our attention to sandwich shells with the total thickness \( H \) and the thickness of rims \( h \). Let the inner radius of the mid surface be \( a \) and the outer radius \( R \), respectively. Different combinations of end conditions will be investigated in this paper.

First of all, the case when the outer edge is clamped or simply supported and the inner edge is free is studied in a greater detail. In this case the shell is loaded by a uniformly distributed transverse pressure of intensity \( P \). In the alternative case the shell is loaded by a central rigid boss.

It is assumed that the face sheet thickness is piece wise constant, e.g. \( h = h_j \) for \( r \in (a_j, a_{j+1}) \) where \( j = 0, \ldots, n \). It is reasonable to denote \( a_0 = a \), \( a_{n+1} = R \). The parameters \( a_j \) and \( h_j (j = 1, \ldots, n) \) will be treated as preliminarily unknown design parameters.

The aim of the paper is to determine the design parameters so that the given cost function attains its minimum value. In the particular case of minimization of the weight of the shell the cost function can be presented as

\[
V = \frac{\pi}{\cos \varphi} \sum_{j=0}^{n} h_j (a_{j+1} - a_j) \tag{1}
\]

where \( \varphi \) stands for the angle of inclination of the middle surface. However, we shall consider herein a more general case of the optimization problem which involves a series of particular problems which can be solved from a unique point of view.
The cost function to be minimized will be presented as
\[ J = G(a_1, \ldots, a_n, h_0, \ldots, h_n, W(r), U(r), M_i(r), N_i(r)) + \int_a^r F(W, U, N_i, M_i) \, dr \] (2)
were \( F \) and \( G \) are given differentiable functions whereas \( W \) and \( U \) stand for displacements in the two orthogonal directions. Here and henceforth \( M_i, M_2 \) stand for bending moments and \( N_i, N_2 \) for membrane forces in the radial and tangential direction, respectively. The quantity \( r \) is assumed to be a given value of the current radius \( r \in [a, R] \).

If, for instance, \( G = W(r), F = 0 \), then the optimization problem consists in the minimization of the radial deflection at \( r = r_0 \). If, however, \( F = 0 \) and \( G = V \), then one has the minimum weight problem for a stepped conical shell.

At the outer edge of the shell following boundary conditions must be satisfied
\[ U(R) = 0, W(R) = 0, M_j(R) = 0, N_j(R) = 0 \] (3)
When minimizing the cost function (3) one has to take into account additional constraints (4) as well as the governing equations which consist of the equilibrium equations and of constitutive relations. The constitutive equations can be presented via strain components. In the case of a conical element the strain components have the form [5, 7]
\[
\begin{align*}
\varepsilon_i &= \frac{d}{dr} U \cos \phi, \\
\varepsilon_2 &= \frac{1}{r} \left( U \cos \phi + W \sin \phi \right), \\
\kappa_1 &= \frac{d}{dr} \frac{d}{dr} \cos^2 \phi, \\
\kappa_2 &= -\frac{1}{r} \frac{d}{dr} \cos^2 \phi,
\end{align*}
\] (4)
where \( \varepsilon_1, \varepsilon_2 \) stand for linear extension ratios and \( \kappa_1, \kappa_2 \) are curvatures of the middle surface of the shell.

In what follows we will treat the shells made of different materials including elastic and inelastic materials. It is well known that in the case of an elastic material the Hooke’s law holds good. The latter can be presented as (see Hodge [6], Ventsel and Krauthammer [15])
\[
\begin{align*}
N_i &= D_{ij} (\varepsilon_i + \nu \varepsilon_2), \\
N_2 &= D_{ij} (\varepsilon_2 + \nu \varepsilon_1), \\
M_i &= D_{ij} (\kappa_1 + \nu \kappa_2), \\
M_2 &= D_{ij} (\kappa_2 + \nu \kappa_1)
\end{align*}
\] (5)
for \( r \in (a_j, a_{j+1}), j = 0, \ldots, n \), where \( \nu \) stands for the Poisson’s modulus and
\[
\begin{align*}
D_j &= \frac{EH_j h_j}{2(1-\nu^2)}, \\
D_{0j} &= \frac{2Eh_j}{1-\nu^2}.
\end{align*}
\] (6)
Here \( E \) denotes the Young modulus.

Conical shells loaded beyond the elastic limit are investigated, as well. In this case it is assumed that the shell is fully plastic and the material obeys the Hill’s yield criterion which can be presented as [2 - 4]
\[
\begin{align*}
\frac{1}{M_{0j}} (M_j^2 - \alpha M_j M_2 + \beta M_2^2) + \frac{1}{N_{0j}^2} \cdot (N_j^2 - \alpha N_j N_2 + \beta N_2^2) - Y_j^2 &\leq 0
\end{align*}
\] (7)
for the section \( r \in (a_j, a_{j+1}) \). In (7) the quantities \( \alpha, \beta \) and \( Y_j \) are certain material parameters whereas \( N_{0j} \) and \( M_{0j} \) stand for the yield force and the yield moment for \( r \in (a_j, a_{j+1}) \). Evidently, in the case of a sandwich shell
\[ M_{0j} = \sigma_o H h_j, \quad N_{0j} = 2\sigma_o h_j \]
where \( \sigma_o \) is the yield stress of the material. It is known from the theory of plasticity that in a plastic region the associated flow law holds good. According to the associated gradientality law and the Hill’s yield criterion (7) one has
\[
\begin{align*}
\varepsilon_1 &= \frac{\lambda_j}{N_{0j}} (2N_j - \alpha N_j), \\
\varepsilon_2 &= \frac{\lambda_j}{N_{0j}} (2\beta N_j - \alpha N_j), \\
\kappa_1 &= \frac{\lambda_j}{M_{0j}} (2M_2 - \alpha M_2), \\
\kappa_2 &= \frac{\lambda_j}{M_{0j}} (2\beta M_2 - \alpha M_2),
\end{align*}
\] (8)
for \( r \in (a_j, a_{j+1}), j = 0, \ldots, n \). Here \( \lambda_j \) stands for a non-negative scalar multiplier. Combining the obtained equations with (4) results in the system of equations
\[
\begin{align*}
\frac{dW}{dr} &= -\frac{\lambda_j r}{\cos^2 \phi M_{0j}} (2\beta M_2 - M_1), \\
\frac{d^2 W}{dr^2} &= -\frac{\lambda_j}{\cos^2 \phi M_{0j}^2} (2M_2 - \alpha M_2), \\
\frac{dU}{dr} &= \frac{\lambda_j}{\cos \phi N_{0j}^2} (2N_j - \alpha N_j),
\end{align*}
\] (9)
and

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The equilibrium of a shell element furnishes equations (see Hodge [5])

\[ N_j' = \frac{1}{r} \left( N_j - N_j \right) \]

\[ M_j' = -\frac{1}{r} (M_j - M_j) - N_j \frac{\sin \varphi}{\cos^2 \varphi} - \frac{P(r^2 - \alpha^2)}{2 \cos^2 \varphi \cdot r} \]  

(11)

where prims denote the differentiation with respect to \( r \).

In the case of an elastic material the governing equations can be expressed as differential equations depending on displacements \( U \) and \( W \). Indeed, it follows from (4), (5) that

\[ N_i = D_i \left( U' \cos \varphi + \frac{U}{r} \left( U \cos \varphi + W \sin \varphi \right) \right) \]

\[ N_j = D_j \left( \frac{1}{r} \left( U \cos \varphi + W \sin \varphi \right) + vU' \cos \varphi \right) \]

\[ M_j = -D_j \left( W' \cos^2 \varphi + \frac{v}{r} W' \cos^2 \varphi \right) \]

\[ M_j = -D_j \left( \frac{W'}{r} \cos^2 \varphi + vW' \cos^2 \varphi \right) \]

Substituting these quantities in (11) results in

\[ (rU')' + (vU + vU') \cos \varphi + W \sin \varphi = 0 \]

\[-\frac{1}{r} \left( U \cos \varphi + W \sin \varphi \right) = 0, \]

\[ D_j \left( -rW' - vW' + \frac{W'}{r} + vW' \right) \cos^2 \varphi - \]

\[ -\frac{1}{r} \frac{\sin \varphi}{\cos^2 \varphi} \left( U' \cos \varphi + \frac{U}{r} \left( U \cos \varphi + W \sin \varphi \right) \right) - \]

\[ \frac{P}{2 \pi} = 0. \]

The last system of equations holds good for each \( r \in (a_j, a_{j+1}) \) for \( j = 0, \ldots, n \) in the case of shells made of an elastic material.

\[ \frac{dW}{dr} = Z, \]

\[ \frac{dZ}{dr} = \frac{Z(2M_j - \alpha M_j)}{r(2\beta M_j - \alpha M_j)} \]

\[ \frac{dU}{dr} = -\frac{M_j' \cdot Z(2N_j - \alpha N_j)}{N_j' \cdot r(2\beta M_j - \alpha M_j)} \]  

(12)

where \( Z \) is an auxiliary variable and

\[ \lambda_j = -\frac{\cos^2 \varphi \cdot M_j' \cdot Z}{r(2\beta M_j - \alpha M_j)} \]

Substituting (13) in (10) yields the equation

\[ U \cos \varphi + W \sin \varphi + \frac{M_j'^2 \cos^2 \varphi}{N_j'} \left( \frac{2\beta N_j - \alpha N_j}{2\beta M_j - \alpha M_j} \right) = 0 \]

(14)

In order to minimize the cost function (2) under constraints (11) - (14) let us introduce the extended (augmented) functional \( [1, 8 - 12] \)

\[ J_* = G + \int_a^b \sum_{i=1}^m \left[ \psi_i \left( \frac{dW}{dr} - Z \right) + \right. \]

\[ + \psi_j \left( \frac{dZ}{dr} - \frac{Z(2M_j - \alpha M_j)}{r(2\beta M_j - \alpha M_j)} \right) + \]

\[ + \psi_j \left( \frac{dU}{dr} + \frac{M_j'^2(2N_j - \alpha N_j)Z}{N_j' r(2\beta M_j - \alpha M_j)} \right) + \]

\[ + \left. \psi_j \left( \frac{d}{dr} N_j - \frac{N_j - N_j}{r} \right) \right] \]

\[ + \psi_j \left( \frac{d}{dr} M_j - \frac{M_j - M_j}{r} \right) - \frac{1}{r \cos^2 \varphi} \left( rN_j \sin \varphi - \frac{P}{2\pi} \right) \]

\[ + v_i \left( U \cos \varphi + W \sin \varphi + \frac{M_j'^2 \cos^2 \varphi}{N_j'} Z(2\beta N_j - \alpha N_j) \right) \]

\[ + \psi_j \left( \frac{1}{M_j' \alpha M_j} + \beta M_j^2 + \lambda_j \right) \]

\[ + \frac{1}{N_j'} (N_j - \alpha N_j) + \left. \left( N_j' - \alpha N_j N_j' + \beta N_j'^2 - Y_i \right) \right] \]  

(15)

Here \( \psi_j, \ldots, \psi_j \) stand for conjugate (adjoint) variables whereas \( v_i, \lambda_j, (j = 0, \ldots, n) \) and \( \psi_{ai}, (i = 1, \ldots, m) \) are Lagrangean multipliers. It is well known that \( \psi_{ai} = \text{const} \) (see Bryson [1], Hull [6]) as multipliers corresponding to isoperimetric constraints must be constant.

Calculating the total variation of (15) and applying the optimality condition \( \Delta J_* = 0 \) leads to a set of differential and algebraic equations with integral terms. First of all, one obtains the system of adjoint equations
\[
\frac{d\psi_j}{dr} = \frac{\partial F}{\partial N} + v_j \sin \varphi,
\]
\[
\frac{d\psi_j}{dr} = -\psi_j \left( \frac{2M_j - \alpha M_j}{r} \right) + \psi_j \left( \frac{M_j}{r} \right) - \frac{2N_j - \alpha N_j}{2} + \frac{v j M_j^2 \cos \varphi}{N_{0j}} + \frac{2\beta N_j - \alpha N_j}{2} \frac{\cos \varphi}{M_{0j}},
\]
\[
\frac{d\psi_j}{dr} = \frac{\partial F}{\partial U} + v_j \cos \varphi,
\]
\[
\frac{d\psi_j}{dr} = \frac{\partial F}{\partial M_j} \left( \psi_j \frac{M_j}{r} \right) - \frac{2Z}{r} \frac{\beta - \alpha}{r} \left( \frac{M_j}{r} \right) + \frac{\psi_j \sin \varphi}{\cos \varphi} + \frac{v_j M_j^2 \cos \varphi}{N_{0j}} \left( \frac{\varphi}{2} \right) \left( \frac{2N_j - \alpha N_j}{2} \right),
\]
\[
\frac{d\psi_j}{dr} = \frac{\partial F}{\partial M_j} \left( \psi_j \frac{M_j}{r} \right) - \frac{2Z}{r} \frac{\beta - \alpha}{r} \left( \frac{M_j}{r} \right) + \frac{\psi_j \sin \varphi}{\cos \varphi} + \frac{v_j M_j^2 \cos \varphi}{N_{0j}} \left( \frac{\varphi}{2} \right) \left( \frac{2N_j - \alpha N_j}{2} \right),
\]
\[
\frac{d\psi_j}{dr} = \frac{\partial F}{\partial M_j} \left( \psi_j \frac{M_j}{r} \right) - \frac{2Z}{r} \frac{\beta - \alpha}{r} \left( \frac{M_j}{r} \right) + \frac{\psi_j \sin \varphi}{\cos \varphi} + \frac{v_j M_j^2 \cos \varphi}{N_{0j}} \left( \frac{\varphi}{2} \right) \left( \frac{2N_j - \alpha N_j}{2} \right),
\]
\[
\frac{d\psi_j}{dr} = \frac{\partial F}{\partial M_j} \left( \psi_j \frac{M_j}{r} \right) - \frac{2Z}{r} \frac{\beta - \alpha}{r} \left( \frac{M_j}{r} \right) + \frac{\psi_j \sin \varphi}{\cos \varphi} + \frac{v_j M_j^2 \cos \varphi}{N_{0j}} \left( \frac{\varphi}{2} \right) \left( \frac{2N_j - \alpha N_j}{2} \right),
\]
\[
\frac{d\psi_j}{dr} = \frac{\partial F}{\partial M_j} \left( \psi_j \frac{M_j}{r} \right) - \frac{2Z}{r} \frac{\beta - \alpha}{r} \left( \frac{M_j}{r} \right) + \frac{\psi_j \sin \varphi}{\cos \varphi} + \frac{v_j M_j^2 \cos \varphi}{N_{0j}} \left( \frac{\varphi}{2} \right) \left( \frac{2N_j - \alpha N_j}{2} \right),
\]
\[
\frac{d\psi_j}{dr} = \frac{\partial F}{\partial M_j} \left( \psi_j \frac{M_j}{r} \right) - \frac{2Z}{r} \frac{\beta - \alpha}{r} \left( \frac{M_j}{r} \right) + \frac{\psi_j \sin \varphi}{\cos \varphi} + \frac{v_j M_j^2 \cos \varphi}{N_{0j}} \left( \frac{\varphi}{2} \right) \left( \frac{2N_j - \alpha N_j}{2} \right),
\]
Note that the system (16) holds good for 
\[
r \in \{a_j, a_{j+1}\}, \text{ where } j = 0, \ldots, n.
\]

Making use arbitrariness of increments \(\Delta h_j\) \((j = 0, \ldots, n)\), \(\Delta a_j\) \((j = 1, \ldots, n)\), also the arbitrariness of variations \(\delta M_j, \delta N_j, \delta W(r_j), \delta U(r_j), \delta Z(r_j), \delta N_j(r_j), \delta M_j(r_j)\) one obtains the system of necessary optimality conditions. It follows from this system that adjoint variables are continuous at \(r = a_j\), but can be discontinuous at \(r = r\).

Note that in the case of an elastic material optimality conditions can be derived in the similar manner with a modified extended functional.

4 Numerical results

Numerical results are presented for conical shells of constant thickness and stepped shells with the unique step. Here following notations are used:

\[
\gamma_{0j} = \frac{h_j}{h}, \quad \alpha_j = \frac{a_j}{R}, \quad k = \frac{M_j \cos \varphi}{RN \sin \varphi}, \quad p = \frac{PR}{N \sin \varphi}
\]

In these formulae \(h\) stands for the thickness of carrying layers of the reference shell of constant thickness whereas \(N_j = 2\delta h_j; M_j = \delta h_j H\).

The curves depicted in Fig. 2 – Fig. 5 regard to an elastic shell clamped at the outer edge and absolutely free at the inner edge. The dimensions of the shell are: \(h = 0.02; R = 1; \varphi = 16^\circ\). The elastic moduli of the material are \(v = 0.3\) and \(E = 210\, \text{GPa}\). The results are obtained with a finite element technique using beam elements.

In Fig. 2 the distributions of transverse deflections are presented in the cases of various values of the inner radius.

Fig. 2. Transverse deflections

Figures 3, 4, 5 portray the bending moments \(m_j = \frac{M_j}{M_*}\) and the membrane force \(n_j = \frac{N_j}{N_*}\). It can be seen from Fig. 3 - 5 that the generalized stresses achieve their maximal values at an intermediate point of the interval \((a, R)\) as might be expected. Minimal values of bending moments are achieved at the clamped end at \(r = R\).

Fig. 3. Radial bending moments
In Fig. 6, 7 similar results are depicted for stepped shells with $h_i = 0.02$. Solid lines in Fig. 6, 7 correspond to shells of piece wise constant thickness whereas the dashed lines are associated with stepped shells. Here $a = 0.1R$ and the step is located at the center of the interval $(a,R)$.

It reveals from Fig. 6 that removing a little of the material at the outer region diminishes essentially transverse deflections of the whole shell.

In Fig. 8 the optimal radius of the step is presented versus $a/R$ for different values of the internal thickness $h_i = \gamma_i h$. in the case of the shell loaded by the rigid central boss.

It can be seen from Fig. 8 that when the thickness $h_i$ tends to $h_i$ then the step location tends to unity.

5 Conclusion
Methods of analysis and optimization of circular conical shells are established. The cases of elastic and inelastic (ideal plastic) materials are investigated. Resorting to the variational methods of the theory of optimal control necessary optimality conditions are obtained.

Calculations carried out showed that in the case of the minimum weight problem a considerable amount of the material can be saved even when using the design of a unique step. When increasing
the number of steps one can achieve the more efficient design of the shell.

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