On state space energy controlled systems with quantum chaotic-like behavior

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Abstract: - The presented contribution is motivated by some fundamental issues arising in nonlinear nanostructures such as Chua’s memristance effect, the Hall-effect or the famous Josephson effect. Some general features of standard approaches, such as classical mechanics, deterministic chaos theory and that of quantum dynamics have been investigated and compared from a unifying state space energy based point of view. A new possibly nonlinear strictly causal model structure has been derived and analyzed. Some theoretical consequences concerning stability analysis and synthesis of state space energy controlled quantum-chaotic behavior are discussed, too. Theoretical results are illustrated and verified by computer simulations.

Key-Words: State space energy, unpredictability, quantum-chaotic behavior, nonlinear, state energy feedback.

1 Introduction

The period characterized as the era of semiconductor based microelectronics obviously reaches its limits and undoubtedly will be replaced by some fundamentally new functional principles [1] in a near future. Recently a typical class of nonlinear structures being able to explain such issues as e.g. the Chua’s memristance effect, the Hall-effect and the Josephson effect has been presented in [1, 2]. For instance nanoelectronic devices such as the Josephson junctions are capable of generating voltage oscillations of extraordinary high frequencies, typically $10^{10}$-$10^{11}$ Hz. They have also been successfully used to detect far infrared radiation from distant galaxies.

There is no doubt that the chosen class of nonlinear nanostructures is not only extraordinary challenging from a system-theoretical point of view, but it is also of increasing importance for effective applications of new devices of nano-electronics - the electronics of the future.

In this paper a new state space energy based nonlinear control algorithm intended as a tool for future nano-electronic applications is proposed.

2 State space energy conservation principle for dissipative systems

We start with presentation of some basic ideas of the state space energy based approach [3]. Let $P_0(t)$ denotes the output dissipation power of a zero input causal system with an informational output $y(t)$ defined by:

$$y(t) = \pm \sqrt{P_0(t)} , P_0(t) \geq 0 \quad (1)$$

Let $E(t)$ denotes the instantaneous value of the state space energy (stored in a state vector $x(t)$):

$$E(t) = \int_0^t P_0(\tau) d\tau , \forall t : t = t_0 \quad (2)$$

Let a concept of the state space hyper-energy be defined by integration:

$$J = \int_{t_0}^{t_1} E(\tau) d\tau , \quad t_1 \rightarrow \infty \quad (3)$$

The state space energy conservation principle holds

$$\frac{dE}{dt} = \langle \psi(x), f(x) \rangle = -P_0(t) \quad (4)$$

where $\psi$ is the gradient vector of the state space potential field $E$, $f$ is the state space velocity vector, and $\langle , \rangle$ denotes the operation of dual product.

Because the choice of origin and of the state space coordinate system is free we can define the gradient $\psi(x)$ of the energy $E$ in its most simple form:

$$\psi(x) = x^T \rightarrow E = \frac{1}{2} \sum_{i=1}^{n} x_i^2 \quad (5)$$

where $n$ is the order of the system representation. It is important to note that the new concept of the state space hyper-energy $J$, divided by the length of the interval $T=[t_0,t_1]$, defines a mean value of $E(x)$. 

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3 General structure of a class of the dissipative system representations

Expressing the algebraic structure of the state space representation in a vector-matrix form as follows:

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

\[ y(t) = Cx(t) \]

it becomes obvious that it can be generalized for a general class of strictly causal systems of any finite order \( n \) as follows:

\[ \Delta_1 | 0 0 0 ... 0 0 \]

\[ -\sigma_3 0 \omega_2 0 ... 0 0 \]

\[ 0 0 \omega_4 0 ... 0 0 \]

\[ 0 0 0 -\sigma_3 0 ... 0 0 \]

\[ ... ... ... ... ... ... \]

\[ 0 0 0 0 -\sigma_3 | 0 0 \Omega \]

\[ A = \begin{bmatrix} -\Delta_1 & \omega_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\omega_2 & \sigma_3 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\sigma_3 & \omega_3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\omega_4 & \sigma_4 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\sigma_3 & 0 & \cdots & 0 & 0 \\ ... & ... & ... & ... & ... & \cdots & ... & ... \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\omega_3 | 0 \omega_2 \]

\[ C = \begin{bmatrix} \pm \sqrt{\Delta_1} & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \]

Example 1: In order to illustrate the idea of the state space energy conservation a 4th-order system is introduced:

\[ \dot{x}_1 = -\Delta_1 x_1 + \omega_2 x_2 \\
\dot{x}_2 = -\omega_2 x_1 + \sigma_3 x_3 \\
\dot{x}_3 = -\sigma_3 x_2 + \omega_4 x_4 \\
\dot{x}_4 = -\omega_4 x_3 + \sigma_4 x_5 \\
y = \pm \sqrt{\Delta_1} x_1 \]

The state space energy \( E(t) \) is given by:

\[ \psi(x) = [x_1, x_2, x_3, x_4]^{\top} \rightarrow E = \frac{1}{2} [x_1^2 + x_2^2 + x_3^2 + x_4^2] \]

the dissipation power \( P_0(t) \) is defined by:

\[ P_0(t) = y^2(t) = \Delta_1 x_1^2 \]

and the state space energy conservation law reads:

\[ \frac{dE}{dt} = -\Delta_1 x_1^2 + (\omega_2 - \omega_2)x_1 x_2 + (\sigma_3 - \sigma_3)x_2 x_3 + (\omega_4 - \omega_4)x_3 x_4 = -P_0(t) \]

where \( \Delta_1 \) is the dissipation parameter, \( \omega_2, \omega_4 \) are eigen-frequencies of two oscillating subsystems and \( \sigma_3 \) is the interaction parameter, representing the interaction between both the 2nd-order subsystems.

Observation 1: It is worthwhile to notice that for all values of the frequency parameters \( \omega_2, \omega_4 \) and for arbitrary value of the interaction parameter \( \sigma_3 \) the positivity condition of the dissipation parameter \( \Delta_1 \) is necessary and sufficient for strict dissipativity of the complete system representation (6).

Observation 2: Notice that for \( \sigma_3=0 \) the system degenerates to two isolated 2nd-order subsystems. The first subsystem remains dissipative, but the other one will become conservative.

Observation 3: Notice further that for a special case \( \Delta_1=0 \) the system representation becomes conservative for any value of the interaction parameter \( \sigma_3 \) without any regard to values of \( \omega_2, \omega_4 \). The possible courses of the state space energy for \( \Delta_1 = 0.1, |\sigma_3| = 1, |\omega_2| = 1, |\omega_4| = 1 \) are displayed in the Fig. 1.

The topological structure of the 4th-order system representation (6) is displayed in the Fig. 2.
The reason is that for any non-vanishing constant energy \( E(x) = \text{const.} > 0 \), the definition integral would diverge. Hence due to mathematical reasons, the dissipativity axiom is necessary.

**Observation 5**: It is worthwhile to notice that the last equation in (6) says that no information about the internal system behavior can be gained by measurement the output, if dissipation parameter \( \Delta_1 \) vanishes. It means that strictly Hamiltonian systems have to be excluded because the class of conservative systems, as a limiting case of the structure for \( \Delta_1 \rightarrow 0 \), is from the system-theoretical point of view an impermissible idealization.

**Observation 6**: That the last equation in eqn. (6) should be seen as a signature of necessity of the Heisenberg uncertainty principle of quantum theory.

If the state representation (6) is given, it is easy to determine the corresponding 4th-order ordinary homogenous differential equation:

\[
 x^{(4)}(t) + a_4 x^{(3)}(t) + a_3 x^{(2)}(t) + a_2 x(t) + a_1 x(t) = 0
\]  

(13)

where the parameters \( a_k \) for \( k=1,2,3,4 \) are given by nonlinear parameter space transformation

\[
 a_1 = \Delta_1 \\
 a_2 = \sigma_1^2 + \omega_3^2 \\
 a_3 = \Delta_2 (\sigma_1^2 + \omega_4^2) \\
 a_4 = \omega_2^2 \omega_3^2 
\]

(14)

On the other hand, the energy motivated state space representation (6) exists if and only if the parameter space transformation is real and invertible.

**Example 1** (Continued): Let’s illustrate some consequences of the non-unicity, manifested by spin structure changing. The dissipation rate of the eight possible state space energy courses, shown in the Fig. 1., is evaluated by means of the hyper-energy \( J \) (see Table 1.), in dependence on possible spin structures.

<table>
<thead>
<tr>
<th>Spin structure ( {s_1, s_2, s_3} )</th>
<th>( \omega_2 )</th>
<th>( \sigma_3 )</th>
<th>( \omega_4 )</th>
<th>Hyper-energy ( J )</th>
<th>Normalized inverse of ( J ) ( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>27.6</td>
<td>1</td>
</tr>
<tr>
<td>S2</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>49.5</td>
<td>0.56</td>
</tr>
<tr>
<td>S3</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>68.5</td>
<td>0.4</td>
</tr>
<tr>
<td>S4</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>52.6</td>
<td>0.53</td>
</tr>
<tr>
<td>S5</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>47.6</td>
<td>0.58</td>
</tr>
<tr>
<td>S6</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>71.5</td>
<td>0.39</td>
</tr>
<tr>
<td>S7</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>50.5</td>
<td>0.55</td>
</tr>
<tr>
<td>S8</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>32.6</td>
<td>0.85</td>
</tr>
</tbody>
</table>

Table 1. Hyperenergy based system evolution and its pseudo probabilistic interpretation

Notice that the set of binary parameters \( s_i \), \( s_i \in \{-1, +1\} \) appeared as a consequence of the fact that the solution of the nonlinear set of equations (14) resulting from the state equivalence relation was not unique. In the given context the logical variables \( s_i \) with values in a binary set play a similar role like spins in the quantum theory. Notice further that another set of parameters \( \Delta_k \), \( k = 1, 2, \ldots, n \) represents the well known set of diagonal minors of the corresponding Hurwitz determinant important in the asymptotic stability theory [3, 4, 5, 6].

It is easy to prove that it is the case if and only if all the diagonal minors \( \Delta_1, \Delta_2, \Delta_3, \ldots \) of the corresponding Hurwitz determinant \( H_n \)

\[
 H_n \triangleq \left| \begin{array}{cccccc}
 a_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\
 a_3 & a_2 & a_1 & 1 & 0 & \cdots & 0 \\
 a_5 & a_4 & a_3 & a_2 & a_1 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & 0 & 0 & & a_n \\
 \end{array} \right| 
\]

(15)

are real and positive.

### 4 Non-unicity of the dissipative state space system representations

Suppose a 6-th order system representation described by

\[
 \Re(S): \quad \dot{x}_1(t) = -\Delta_1 x_1(t) + \omega_3 x_2(t) \\
 \dot{x}_2(t) = -\omega_2 x_1(t) + \sigma_3 x_3(t) \\
 \dot{x}_3(t) = -\sigma_3 x_3(t) + \omega_4 x_4(t) \\
 \dot{x}_4(t) = -\omega_4 x_4(t) + \sigma_4 x_5(t) \\
 \dot{x}_5(t) = -\sigma_4 x_5(t) + \omega_5 x_6(t) \\
 \dot{x}_6(t) = -\omega_5 x_6(t) + \sigma_5 x_1(t) + \ldots 
\]

\[
y(t) = \pm \sqrt{-\Delta_1} x_1(t) 
\]

is given. The corresponding transformation of the parameter space reads:

\[
 a_1 = \Delta_1 \\
 a_2 = \omega_2^2 + \sigma_3^2 + \omega_4^2 + \sigma_5^2 + \omega_6^2 \\
 a_3 = \Delta_2 (\sigma_1^2 + \omega_3^2 + \omega_4^2) \\
 a_4 = \omega_2^2 (\sigma_1^2 + \omega_3^2 + \omega_4^2) \\
 a_5 = \Delta_3 (\sigma_2^2 + \omega_4^2 + \omega_5^2) \\
 a_6 = \omega_2^2 \omega_3^2 \omega_4^2 
\]

(17)
It is easy to find the unique solution of the set of the algebraic equations (17) with respect to the dissipation parameter $\Delta_i$, as well as with respect to squares of the unknown frequency parameters $\omega_i$, $\omega_4$, $\omega_6$, . . . , and of the unknown interaction parameters $\sigma_3$, $\sigma_5$, $\sigma_7$, . . . in the following form:

\[
\Delta_1 = a_1, \quad \omega_2 = \frac{a_1a_3 - a_3}{a_1} = \frac{\Delta_2}{\Delta_1}, \quad \sigma_3 = \frac{a_1a_2 - a_3 - a_1^2}{(a_1a_2 - a_3)a_1} = \frac{\Delta_3}{\Delta_1},
\]

\[
\omega_4 = \frac{\Delta_1\Delta_3}{\Delta_2}, \quad \sigma_5 = \sigma_6 = \frac{\Delta_4}{\Delta_3}, \quad \omega_6 = \frac{\Delta_1\Delta_5}{\Delta_4}, \quad \sigma_4 = \frac{\Delta_6}{\Delta_5}, \quad \sigma_2 = \frac{\Delta_8}{\Delta_6}, \quad \sigma_1 = \frac{\Delta_9}{\Delta_8}, \quad \Delta_4 > 0, \quad \Delta_6 > 0, \quad \Delta_8 > 0, \quad \Delta_9 > 0,
\]

Consequently the matrices $A$ and $C$ take the form

\[
A = \begin{bmatrix}
-\alpha_1 & 0 & 0 & 0 & 0 \\
-s_1\alpha_2 & 0 & 0 & 0 & 0 \\
0 & -s_2\alpha_3 & 0 & 0 & 0 \\
0 & 0 & -s_{n-2}\alpha_{n-1} & 0 & 0 \\
0 & 0 & 0 & -s_{n-4}\alpha_{n-2} & 0 \\
0 & 0 & 0 & 0 & -s_{n-4}\alpha_{n-1}
\end{bmatrix}
\]

(21)

\[
C = \begin{bmatrix}
\pm\sqrt{\Delta_1}, 0, 0, \ldots, 0
\end{bmatrix}
\]

(22)

and the general form of the representation (16) can be explicitly expressed as follows

\[\mathcal{R}(S): \quad \dot{x}_i(t) = -s_i\alpha_i x_i(t) + s_i\alpha_i x_j(t)\]

\[\dot{x}_j(t) = -s_j\alpha_j x_j(t) + s_j\alpha_j x_k(t)\]

\[\dot{x}_k(t) = -s_k\alpha_k x_k(t) + s_k\alpha_k x_l(t)\]

\[\dot{x}_l(t) = -s_l\alpha_l x_l(t) + \ldots \]

(23)

\[y(t) = \pm\sqrt{\Delta_1} x_1(t), \quad s_1 \in \{-1, +1\}, \quad \Delta_i = \alpha_i, \quad \alpha_k > 0, \quad k \in \{1, 2, \ldots, n\}\]

\section{5 State space transformations}

In order to be able to interpret the obtained results for any given $n$-th order ordinary differential equation like (13), we need an effective tool to transform any given state space representation, such as (16), but of arbitrary finite order, into the form like (13), and back.

One straightforward approach suggests to compute successive derivatives of the output equation up to the order $n$, combined with successive elimination all of the state variables as follows (for $n=6$):

\[
y(t) = \pm\sqrt{\Delta_1} x_1(t)\]

\[
\dot{y} + \alpha_1 y = \pm\sqrt{\Delta_1} s_1\alpha_2 x_2
\]

\[
\ddot{y} + \alpha_2 \dot{y} + \alpha_2^2 y = \pm\sqrt{\Delta_1} s_2s_3\alpha_2 x_3, \ldots
\]

\[
y^{(5)} + \alpha_1 y^{(4)} + (\alpha_2^2 + \alpha_3^2 + \alpha_3^2) y^{(3)} + \alpha_1(\alpha_2^2 + \alpha_3^2 + \alpha_3^2) y^{(2)} + \alpha_1(\alpha_2^2 + \alpha_3^2 + \alpha_3^2) y + \alpha_1(\alpha_2^2 + \alpha_3^2 + \alpha_3^2) y
\]

As a by-product the following state space transformation relations have been obtained:
\[
x_1 = \frac{1}{\gamma_1} y, \quad \gamma_1 = \pm \sqrt{\Delta_1}, \Delta_1 = \alpha_1 > 0 \\
x_2 = \frac{1}{\gamma_1 s_2 \alpha_2} (\alpha_1 y + \dot{y}), \quad \gamma_1 = \pm \sqrt{\Delta_1}, s_1 = \pm 1, \alpha_2 > 0 \\
x_3 = \frac{1}{\gamma_1 s_3 s_4 s_5 s_6 \alpha_3 \alpha_4 \alpha_5 \alpha_6} \left[ \alpha_1 \alpha_2^2 \alpha_3^2 \alpha_4^2 y + \right. \\
+ \left. (\alpha_2^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2) y + + \alpha_1 (\alpha_2^2 + \alpha_4^2 + \alpha_5^2) y + \\
+ \left. (\alpha_2^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2) y + + \alpha_1 y^{(4)} + y^{(5)} \right], \\
\gamma_1 = \pm \sqrt{\Delta_1 s_1 \in \{-1, 1\}}, \alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0, \ldots, \alpha_6 > 0. 
\]

which can be expressed in a vector-matrix form

\[
x = T \bar{x}, \quad \bar{x} = T^{-1} x
\]

where the signature matrix \( T_s \) is defined by

\[
T_s = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & s_1 & 0 & 0 & \cdots \\
0 & 0 & s_1 s_2 & 0 & \cdots \\
0 & 0 & 0 & s_1 s_2 s_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

with \( s_1, s_2, \ldots, s_n = \pm 1 \), and the parameter matrix \( T_\alpha \) is defined by

\[
T_\alpha = \pm \frac{1}{\sqrt{\Delta}} \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \\
\frac{\alpha_1}{\alpha_2} & 1 & 0 & 0 & \cdots \\
\frac{\alpha_2}{\alpha_3} & \frac{\alpha_3}{\alpha_4} & 1 & 0 & \cdots \\
\frac{\alpha_3}{\alpha_4} & \frac{\alpha_4}{\alpha_5} & \frac{\alpha_5}{\alpha_6} & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

For the inverse matrix \( T^{-1} \) we get immediately

\[
T^{-1} = (T_s T_\alpha)^{-1} = (T_\alpha)^{-1} (T_s)^{-1} = T_s^{-1} T_\alpha^{-1}
\]

where the parameter matrix \( T_\alpha \) equals to the inverse transpose of observability matrix of the system representation given by

\[
H_o = \begin{bmatrix}
C^T, \quad A^T C^T, (A^T)^2 C^T, \ldots, (A^T)^{n-1} C^T
\end{bmatrix}
\]

6 Nonlinear state energy feedback

In this part, the example of switching system based on energy feedback is presented. In this example the 4th order system described by eq. like (23) is used and only sign of parameter \( \alpha_1 \) is switched so that state space energy values are changed between \( E_1 \) and \( E_2 \) and mean \( E_M \) (desired value) is

\[
E_M = 0.5 (E_1 + E_2)
\]

In this example \( E_1 = 2 \), \( E_2 = 3 \) and \( E_M = 2.5 \). The block diagram of energy feedback system is shown in Fig. 3 and simulation results are in Fig. 4 - 9.

![Fig. 3. The block diagram of energy feedback system.](image)

![Fig. 4. The state space energy evolution of the state energy feedback controlled system.](image)

![Fig. 5. 3-D projection of a chaotic-like state space trajectory.](image)
7 Conclusion

In the contribution a new approach to nonlinear system synthesis and state space energy feedback control algorithms is presented. It seems natural and may be useful from practical applications point of view to transform some standard nonlinear control problems into the proposed paradigm.

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