Robust stabilization of fractional systems

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Abstract—In this paper, we study the asymptotic stabilization of multivariable fractional system using a robust linear controller with fractional order. The multivariable system are either linear or non linear. The Gronwall-Bellman is employed to investigate the robust stability conditions which are based on the upper norm bounds of the incertitude.

Keywords—Fractional system, Gronwall-bellman, robust linear controller.

I. INTRODUCTION

Fractional order system have been studied by many authors in engineering sciences [10]. Many systems can be described with the help of fractional derivation. The stability analysis and stability proof of fractional order control system [12], [16] are still consider yet as open problem this is due to the fact that existing theory developed so far for stability proof manly exists for integer order and generally is not applicable to fractional order control system [17].

In this paper, we generalize the results presented in [2] about robust linear integer controller design, for fractional order controller.

The robust fractional controller is introduced for the multivariable dynamic fractional system with parametrical as well as structural linear or non linear time varying model uncertainties. The Gronwall lemma is used to investigate the robust stability conditions which are based on the upper norm bounds of the uncertainties. The parameters of a dynamic controller are selected to satisfy the requirement of robust stability under plant uncertainties.

The paper is organized as follow: in section II, we introduce the definition of fractional derivative in brief; we present also some mathematical results. In section III, we propose robust fractional controller for the stabilization of multivariable system.

II. PRELIMINARY DEFINITION

A. Definition of fractional derivation

The fractional calculus is a generalization of integration and derivation to non-integer order operators. At first, we generalize the differential and integral operators into one fundamental operator $D^\alpha_t$ which is known as fractional calculus:

$$\textstyle D^\alpha_t f(t) = \begin{cases} \frac{d^\alpha}{dt^\alpha} f(t) & \text{Re}(\alpha) > 0, \\ 1 & \text{Re}(\alpha) = 0, \\ \int_t^\infty (\tau - t)^{-\alpha} f(\tau) d\tau & \text{Re}(\alpha) < 0. \end{cases}$$

The two definitions used for the general fractional differintegral are the Grünwald definition and the Riemann-Liouville(RL) definition. The Grünwald definition is given here:

$$\textstyle D^\alpha_t f(t) = \lim_{k \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[t-k]} (-1)^j \binom{\alpha}{j} f(t-jh)$$

Where $[x]$ means the integer part of $x$. the (RL) definition is given as:

$$\textstyle D^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty \frac{f(\tau)}{(\tau-t)^{n+1-\alpha}} d\tau$$

For $(n-1<\alpha<n)$ and $\Gamma(x)$ is the well known Euler’s gamma function.

The Laplace transform method is used for solving engineering problems. the formula for the Laplace transform of the (RL) fractional derivative (2) has the form:

$$\textstyle \int_0^\infty e^{-pt} D^\alpha_t f(t) dt = p^\alpha F(p) - \sum_{k=0}^{n-2} p^k D^\alpha_0 f(t) \bigg|_{t=0}$$

For $(n-1<\alpha \leq n)$.

For simplicity, we will note $D^\alpha$ for $D^\alpha_t$. 
B. Definition of the two parameter Mittag-Leffler function

The two parameter Mittag-Leffler function is defined as follows:

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + k\beta)}, \quad \alpha > 0, \beta > 0 \] (5)

Where

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + k\beta + 1)} = E_{\alpha}(z) \]

is the one-parameter Mittag-Leffler function.

The Laplace transform of the two parameter Mittag-Leffler function is:

\[ \int_0^\infty e^{-st}z\alpha+k\beta-1E_{\alpha,\beta}(z\alpha)dt = \frac{\Gamma_{\alpha+k\beta}}{(s^{\alpha}-a)^{k\beta+1}} \] (6)

Where \( E_{\alpha,\beta} \) is a nondecreasing function on \( \mathbb{R}^+ \).

C. Some mathematical inequalities

Lemma 1 [3]: for \( \alpha < 1, z \in \mathbb{C} \)

\[ L_{\alpha}(z) = \text{Re}(z^{\alpha}) \quad \text{if} \quad \left| \text{Arg}(z) \right| \leq \frac{\pi}{2} \alpha \]

(7)

\[ L_{\alpha}(z) = 0 \quad \text{if} \quad \frac{\pi}{2} \alpha < \left| \text{Arg}(z) \right| \leq \pi \]

In particular, \( L_{\alpha} \geq 0 \), moreover, for every \( \varepsilon > 0 \), there is a constant \( C_{\varepsilon} > 0 \) such that

\[ \left\| E_{\alpha,\beta}(z) \right\| \leq C_{\varepsilon} \exp(L_{\alpha}(z) + \varepsilon|z|^{\alpha}) \] (8)

Lemma 2 [13]: there exist finite real constant \( K_{\alpha,\alpha} > 1 \) such that for any \( 0 < \alpha < 1 \) and it is constant matrix

\[ E_{\alpha,\alpha}(A_{\alpha}^{\alpha}) \leq K_{\alpha,\alpha} \left\| \exp(A_{\alpha}^{\alpha}) \right\| \] (9)

D. Theorem of Gronwall Bellman [8]

\[ \beta > 0, a(t) \] is nonnegative function locally integrable on \( 0 \leq t < T \) (some \( T \), \( t \) is nonnegative, nondecreasing continuous function with \( \left\| g(t) \right\| < N \) and \( u(t) \) is nonnegative and locally integrable on \( 0 \leq t < T \) with:

\[ u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s)ds \] (10)

Then on \( 0 \leq t < T \):

\[ u(t) \leq a(t) + \left[ \sum_{s=1}^{\infty} \frac{g(s)}{\Gamma(s\beta)} (t-s)^{\beta-1} a(s) \right]ds \] (11)

If \( a(t) \) is a nondecreasing function on \( [0, T] \), then

\[ u(t) \leq a(t)E_{\beta}(g(t)\Gamma(\beta))^\beta \] (12)

Where \( E_{\beta} \) is the Mittag Leffler function:

\[ E_{\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + 1)} \] (13)

III. THE MAIN RESULTS

Consider the following multivariable dynamic system with parametical uncertainties:

\[ \begin{align*}
\mathbf{D}^\alpha x &= A x + B u + A\mathbf{A}(x) + A\mathbf{B}(u) \\
y &= C x + D u + A\mathbf{C}(x) + A D(u), \quad 0 < \alpha < 1
\end{align*} \] (14)

where \( x(t) \) is the state vector, \( u(t) \) is the input vector, \( y(t) \) is the output vector, and \( A, B, C \) and \( D \) are constant matrices. \( A\mathbf{A}(x), A\mathbf{B}(u), A\mathbf{C}(x) \) and \( A D(u) \) are nonlinear time-varying parametrical uncertainties with the following known upper norm-bounds.

\[ \left\| A\mathbf{A}(x) \right\| \leq \delta_1 \] \( \left\| A\mathbf{B}(u) \right\| \leq \delta_2 \] \( \left\| A\mathbf{C}(x) \right\| \leq \delta_3 \]

(15)

where \( \delta_1, \delta_2, \delta_3 \) and \( \delta_4 \) are constant positives.

This system (14) yields an approximate dynamic system as follows:

\[ \begin{align*}
\mathbf{D}^\alpha x &= A x + B u \\
y &= C x + D u \quad 0 < \alpha < 1
\end{align*} \] (16)

Thus, system (16) represents the “approximate model” of the plant. Without loss of generality, we will assume that \( (A, B) \) is controllable and \( (C, A) \) is observable.

In this note the dynamic controller has the following structure

\[ \begin{align*}
\mathbf{D}^\alpha x &= A_{\mathbf{L}} x_{\ell} + B_{\ell} y \\
u(t) &= L_{\ell} x_{\ell}
\end{align*} \] (17)

where \( x_{\ell} \in \mathbb{R}^n \) and \( A_{\mathbf{L}}, B_{\ell} \) and \( L_{\ell} \) are constant matrices with appropriate dimensions.

The first design problem is to choose the parameters \( A_{\mathbf{L}}, B_{\ell} \) and \( L_{\ell} \) in the dynamic controller (17) such that the closed-loop system with uncertainties of (14) and (17) is asymptotically stable, the uncertainties can be tolerated in our design and the controller is a robust controller.

The closed-loop system with nonlinear parametrical uncertainties is described by (14) and (17). Combining (14) and (17) we get:

\[ \begin{align*}
\mathbf{D}^\alpha x_{\ell} &= A_{\mathbf{L}} x_{\ell} + B_{\ell} y \\
y &= C_{\ell} x_{\ell} + A_{\ell} C(x) + A_{\ell} D(u)
\end{align*} \] (18)

where \( 0 < \alpha < 1 \).

We define

\[ \begin{align*}
\mathbf{T} &= \begin{bmatrix} x_{\ell} \\ x_{\ell} \end{bmatrix} \\
\mathbf{D}^\alpha \mathbf{T} &= \begin{bmatrix} \mathbf{D}^\alpha x_{\ell} \\ \mathbf{D}^\alpha x_{\ell} \end{bmatrix}, \quad \mathbf{T} &= \begin{bmatrix} A_{\mathbf{L}} & B_{\ell} \\ B_{\ell} C & A + B_{\ell} D_{\ell} \end{bmatrix}
\end{align*} \] (19)
Theorem 1: suppose the nonlinear parametrical uncertainties are bounded by (15) and if we choose the control parameters of (17) such the following inequality is satisfied:
\[
(H \Gamma (\alpha))^a (1 + \epsilon') + \sigma < 0
\]
Then, the nonlinear parametrical perturbed closed-loop system (21) is also asymptotically stable.

Proof:

Using the Laplace transform to the system (21), we obtain:
\[
\overline{x}(s) = \left( I_n s^{-\alpha} - \overline{A} \right)^{-1} \left( s^{-\alpha} \overline{x}_0 + L(\Delta \overline{x}(\overline{x})) \right)
\]
Using (2) we obtain:
\[
\left\| \overline{x}(t) \right\| \leq \left\| \overline{x}_0(t) \right\| + \left\| \int_0^t \left( t - \tau \right)^{\alpha - 1} \right\| E_{\alpha,\alpha} \left( \overline{A}(t - \tau)^{\alpha} \right) d\tau \overline{A}(t - \tau) d\tau
\]
and
\[
\overline{x}_2(t) = \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha,\alpha} \left( \overline{A}(t - \tau) \right) \Delta \overline{A}(\overline{x}(\tau)) d\tau
\]
So
\[
\left\| \overline{x}(t) \right\| \leq \left\| \overline{x}_0(t) \right\| + \left\| \overline{x}_2(t) \right\|
\]
Using (2) we obtain:
\[
\left\| \Delta \overline{x}(t) \right\| \leq \delta \left\| \left( t + \delta_2 \right) \right\| \left( \overline{y} \right) + \delta_3 \left\| \left( \overline{y} \right) \right\| + \delta_4 \left\| \left( \overline{y} \right) \right\|
\]
\[
\leq \delta_1 + \delta_2 + \delta_3 \left\| \overline{y} \right\| + \delta_4 \left\| \overline{y} \right\| L \right\| \overline{x}(t) \right\|
\]
we put
\[
R = \left\{ \delta_1 + \delta_2 + \delta_3 \right\} \left\| \overline{y} \right\| + \delta_4 \left\| \overline{y} \right\| L \right\| \overline{x}(t) \right\|
\]
we put
\[
\left\| \overline{A} \overline{x}(t) \right\| \leq R \left\| \overline{x}(t) \right\|
\]
Using the lemma 2, we obtain:
\[
\exists K_{\alpha,\alpha} > 1, \left\| E_{\alpha,\alpha} \left( \overline{A}(t - \tau)^{\alpha} \right) \right\| \leq K_{\alpha,\alpha} \left\| \overline{x}(t) \right\|
\]
Then
\[
\left\| \overline{x}_2(t) \right\| \leq K_{\alpha,\alpha} \left\| \overline{x}_0(t) \right\| + \left\| \int_0^t \left( t - \tau \right)^{\alpha - 1} \right\| \left\| \overline{A}(t - \tau) \right\| d\tau
\]
And we have
\[
\exp(\overline{A}(t - \tau)) \leq M \exp(\omega(t - \tau)) , M > 1, \omega \in IR
\]
and
\[
\left\| \overline{x}_2(t) \right\| \leq H \int_0^t (t - \tau)^{\alpha - 1} \exp(\omega(t - \tau)) d\tau, \quad H = M R K_{\alpha,\alpha}
\]
Using the lemma 1 we obtain
\[
\left\| E_{\alpha,\alpha} \left( \overline{A} t^{\alpha} \right) \right\| < C_{\alpha} \exp \left( \frac{L}{\alpha} \left( \left\| \overline{x} \right\|^{\alpha} \right) \right)
\]
So
\[
\left\| \overline{x}_1(t) \right\| \leq C_{\alpha} \left\| \overline{x} \right\| \exp(\omega t)
\]
From (30) and (32) we obtain:
\[
\exp(-\omega t) \leq F_\alpha \exp(\omega t) + H \int_0^t (t - \tau)^{\alpha - 1} \exp(\omega(t - \tau)) d\tau,
\]
\[
F = C_{\alpha} \left\| \overline{x} \right\| \sup(\omega, t)
\]
By multiplying the both side of inequality, we obtain
\[
\exp(-\omega t) \leq F_\alpha \exp(\omega t)\exp(H(\alpha))t^\alpha
\]
We apply the lemma of Gromwell Bellman , we have:
\[
\left\| \overline{x}(t) \right\| \exp(-\omega t) \leq \left\| \overline{x}(t) \right\| \exp(-\omega t)
\]
Using the lemma 1 , we find
\[
\left\| \overline{x}(t) \right\| \exp(-\omega t) \leq FC_{\alpha} \exp \left( \frac{L}{\alpha} \left( \left\| \overline{x} \right\|^{\alpha} \right) \right)
\]
So the system is asymptotically stable if:
\[
\frac{1}{\left( H \Gamma (\alpha))^a (1 + \epsilon') + \sigma < 0
\]
IV. CONCLUSION

We presented a robust fractional controller for the multivariable fractional system with linear, nonlinear, or time varying model uncertainties. The Gronwall inequality was employed to investigate the robust stability condition. Our work is simplified to choose the dynamical control parameters to satisfy the requirement of robust stability. More general multivariable fractional system should be considered in the future work with another form of uncertainties.
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