Channel Capacity Restoration of Noisy Optical Quantum Channels

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Abstract: - This paper defines a fundamentally new approach of capacity recovery of very noisy, practically completely useless optical quantum channels. The transmission of information through classical channels and quantum channels differs in many ways. The capacity recovery of very noisy communication channels cannot be imagined for classical systems, and this effect has no analogue in classical systems. The capacity recovery of very noisy quantum channels makes it possible to use two very noisy optical-fiber based quantum channels with a positive joint capacity at the output. Our method gives an algorithmic solution to the capacity recovery problem, and provides an efficient algorithmic solution for finding recoverable very noisy optical quantum channels.

Key-Words: - Noisy Optical Communications, Quantum Channels, Quantum Communications

1 Introduction
The capacity recovery of very noisy, practically completely useless quantum channels makes it possible to use two very noisy quantum channels with a positive joint capacity at the output. As derived in [1], two very noisy communication links can be used to transmit information through the quantum communication channel and it is possible to “activate” one channel with the other channel, so the capacity of very noisy quantum channels can be increased [1]. The process of information transmission through an optical quantum communication channel can be described in three phases. 

In the first phase, the sender has to encode his information, according to properties of the physical channel - this step is called source encoding. After the sender has encoded his information into the appropriate form, it has to put on the optical quantum channel, which transforms it according to its channel map - this second phase is called the channel evolution. The optical quantum channel conveys the quantum state to the receiver, however this state is still a superposed quantum state. To extract the information which is encoded in the state, the receiver has to make a measurement - this measurement process is the third phase of the communication over a quantum channel [2], [8], [17].

In Fig. 1, we illustrate the source coding phase. The sender encodes his information into a physical attribute of a physical particle, such as the spin of the particles. For example, in the case of an electron or a half-spin particle, this can be an axis spin. The half-spin particles could take two possible states, hence these particles are two-level quantum systems - hence qubits.

The channel transformation represents the noise of the quantum channel. Physically, the optical quantum channel is the medium, which moves the particle from the sender to the receiver. The noise disturbs the state of the particle, in the case of a half-spin particle, it causes spin precession. For a noisy optical quantum channel, the channel transforms the original state into a mixed state [2]. The channel evolution phase is illustrated in Fig. 2.

Fig. 1. The source coding phase.

Fig. 2. The channel evolution phase.
Finally, the measurement process responsible for the decoding, or the extraction of the encoded information. The previous phase determines the success probability of the recovery of the original data. If the channel is a completely noisy channel, then the receiver will get a maximally mixed quantum state - which state is geometrically positioned at the origin of the Bloch sphere. The output of the measurement of a maximally mixed state is completely undeterministic: it tells us nothing about the original information encoded by the sender [2], [8].

A general quantum channel transforms the original pure quantum state into a mixed quantum state, but not into a maximally mixed state - which makes it possible to recover the original message with a high - or low - probability, according to the level of the noise of the quantum channel. The measurement phase is illustrated in Fig. 3.

We present an algorithmic solution to the capacity recovery problem of very noisy optical quantum channels. Currently, we have no theoretical results for describing all possible combinations of recoverable very noisy channels, hence there should be many other possible combinations. We analyze the capacity recovery of the amplitude damping channel, which is an important channel in optical physical implementations. This channel describes the effect of energy dissipation of the quantum states. In practical optical or quantum communications, where quantum states or quantum bits are used, the loss of energy from the quantum system causes amplitude damping. In many practical applications, energy dissipation is an unavoidable phenomenon, hence analysis of the amplitude damping quantum channel is a relevant issue [2], [8], [17].

1.1 Quantum Informational Distance

In our work, we apply computational geometry in quantum space, between pure and mixed quantum states. In Fig. 4, we illustrate the logical structure of the analysis and the cooperation of classical and quantum systems. Since, currently we have no quantum computers, we would like to find recoverable noisy optical quantum channels using current classical computer architectures and the most efficient currently available algorithms. To this day, the most efficient classical algorithms for this purpose are computational geometric methods.

Unlike ordinary geometric distances, the quantum informational distance is not a metric and is not symmetric, hence this pseudo-distance features as a measure of informational distance.

2 Capacity of a Noisy Quantum Channel

The general classical informational theoretic model for a noisy quantum channel is illustrated in Fig. 5. Our geometrical analysis is focused on the mixed quantum state, received by Bob. Alice’s pure state is denoted by \( \rho_A \), the noise is modeled by an affine map \( \mathcal{N} \) and Bob’s mixed input state is denoted by \( \mathcal{N}(\rho_A) = \sigma_B \). For random variables \( X \) and \( Y \), \( H(X,Y) = H(X) + H(Y|X) \), where \( H(X), H(X,Y) \) and \( H(Y|X) \) are defined by probability distributions.

We measure in a geometrical representation the information which can be transmitted in the presence of noise on the quantum channel.

In the classical communication model, we seek to maximize \( H(X) \) and minimize \( H(X|Y) \) in order to maximize the radius of the smallest enclosing ball of Bob, since the radius can be computed as

\[
r^* = \max_{\text{all possible } x} H(X) - H(X|Y).
\]

To compute the radius \( r^* \) of the smallest informational ball of quantum states and the entropies between mixed...
quantum states, instead of the classical Shannon entropy, we use the Holevo-Schumacher-Westmoreland (HSW) channel capacity [15], [16]. According to the HSW theorem, the single use capacity \( C^{(1)}(\mathcal{N}) \) of a quantum channel \( \mathcal{N} \), can be defined as follows [15], [16]:

\[
C^{(1)}(\mathcal{N}) = \max_{\{p_i \}} S(\rho) = \max_{\{p_i \}} \sum_{i} p_i S(\rho_i),
\]

where \( S(\rho) = -\text{Tr} (\rho \log \rho) \) is the von Neumann entropy, and \( \rho \) represents the output density matrix obtained from the quantum channel input density matrix \( \rho_i \).

Using the result of the HSW theorem [15], we will refer to the single use channel capacity as the radius of the smallest enclosing ball as follows:

\[
r^* = C^{(1)}(\mathcal{N}) = \max_{\{p_i \}} \rho \| \sigma \).
\]

In this paper, we use the geometrical interpretation of HSW channel capacity, using quantum relative entropy as a distance measure function.

### 2.1 Geometrical Interpretation of Quantum Channel Capacity

The authors of [15] have shown that the capacity of a quantum channel can be measured geometrically, using quantum relative entropy function as a distance measure. Schumacher and Westmoreland have shown that the channel capacity of every optimal output state \( \rho_k \) can be expressed as [15]

\[
C^{(1)}(\mathcal{N}) = D(\rho_k \| \sigma),
\]

where \( \sigma = \sum p_i \rho_i \) is the optimal average output state and the relative entropy function of two density matrices can be defined as

\[
D(\rho_k \| \sigma) = \text{Tr} [\rho_k \log(\rho_k) - \rho_k \log(\sigma)].
\]

In this definition, \( \text{Tr} \) is the trace operator. In conclusion, for non-optimal output states \( \delta \) and optimal average output state \( \sigma = \sum p_i \rho_i \), we have

\[
C^{(1)}(\mathcal{N}) = D(\delta \| \sigma) \leq D(\rho_k \| \sigma).
\]

Moreover, in [15], Schumacher and Westmoreland have also shown that there exists at least one optimal output state \( \{p_i, \rho_i\} \) which achieves the optimal capacity \( C^{(1)}(\mathcal{N}) = D(\rho_k \| \sigma) \). The geometrical interpretation of quantum channel capacity was introduced in [15], using the quantum relative entropy function as a distance measure as follows:

\[
C^{(1)}(\mathcal{N}) = r^* = \min_{\{\rho\}} \max_{\{\sigma\}} D(\mathcal{N}(\rho) \| \mathcal{N}(\sigma)) = \lim_{n \to \infty} \frac{1}{n} \left( \sum_{i=1}^{n} r_i^* \right) = \lim_{n \to \infty} \frac{1}{n} \max_{\{\rho_i \}} (X_{\rho_i}^{\infty}),
\]

where \( r_i^* \) is the single use capacity of the \( i \)th use of quantum channel \( \mathcal{N} \), \( \rho_i^{\infty} \) is the optimal output channel state, and \( \sigma^{\infty} \) is the average state. We analyze the superactivation property of the quantum channel, using the mini-max criterion for states \( \rho_i^{\infty} \) and \( \sigma^{\infty} \). The radius \( r_{sup}^* \) of the superball is equal to the asymptotic classical capacity [6], [11].

In Fig. 6, we illustrate the superball representation for the analysis of two quantum channels, however it naturally can be extended to \( n \) different quantum channel models. The geometrical structure of quantum informational balls differs from the geometrical structure of ordinary Euclidean balls.
In the superactivation problem, we have to use different quantum channel models [1]. For two different quantum channels \( \mathcal{N}_1^{\otimes n} \) and \( \mathcal{N}_2^{\otimes n} \), the asymptotic HSW channel capacity \( C(\mathcal{N}_1 \otimes \mathcal{N}_2) \) is equal to the sum of the radii \( r_{\text{super}}^*(\mathcal{N}_1) \) and \( r_{\text{super}}^*(\mathcal{N}_2) \) of the quantum informational superballs, whose radii form a new quantum superball with radius
\[
\lim_{n \to \infty} \frac{1}{n} C^{(n)}(\mathcal{N}_1 \otimes \mathcal{N}_2) = r_{\text{super}}^*(\mathcal{N}_1) + r_{\text{super}}^*(\mathcal{N}_2). \tag{9}
\]
In Fig. 7 we show the measurement setting to analyze the capacity recovery of very noisy optical quantum channel-pair \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \).

\[
\begin{array}{c|c|c}
\mathcal{N}_1 & \mathcal{N}_2 & \text{Our geometrical approach} \\
\hline
& & \text{Superball radius} \\
\end{array}
\]

Fig. 7. The capacity recovery of optical quantum channels.

The geometrical method computes the joint capacity, based on the clustering of channel output quantum states and a convex hull calculation [12], [13], [14].

### 3 The Quantum Informational Ball

We use the Delaunay tessellation, since it is the fastest known tool to seek the center of the smallest enclosing ball of points. The circumcircle of the given quantum states is the circle that passes through the quantum states \( \rho_1 \) and \( \rho_2 \) of the edge \( \rho_1 \rho_2 \) and endpoints \( \rho_1 \), \( \rho_2 \) and \( \rho_i \) of the triangle \( \rho_1 \rho_2 \rho_i \). The triangle \( t \) is said to be Delaunay, when its circumcircle is empty [3], [7]. For an empty circumcircle, the circle passing through the quantum states of a triangle \( t \in T \) and encloses no other vertex of the set \( S \). The quantum Delaunay diagrams between mixed quantum states are different from Euclidean diagrams, as we have illustrated Fig. 8.

![Quantum Delaunay tessellation](image)

Fig. 8. Classical Euclidean (a), and quantum Delaunay tessellation (b).

The problem of clustering in quantum space, using the quantum informational distance as a distance function, is a completely new area in quantum information theory. The properties of Voronoi diagrams [3] in quantum space have been studied by Kato et al. [10], however the problem of clustering was not analyzed in their work. The coreset method for different distances has been studied in the literature [4], [7], [9].

### 4 The Optical Quantum Channel

The optical quantum channel \( \mathcal{N} \) can be described in the Kraus representation [2], [5], using a set of Kraus matrices \( \mathcal{A} = \{ A_i \} \), in the following form
\[
\mathcal{N}(\rho) = \sum_i A_i \rho A_i^\dagger, \tag{10}
\]
where \( \sum_i A_i^\dagger \rho A_i = I \), and
\[
A_i = \begin{bmatrix} \sqrt{p} & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \sqrt{1-p} \\ \sqrt{1-p} & 0 \end{bmatrix}, \tag{11}
\]
where \( p \) represents the probability that the channel leaves the \( |0\rangle \) input state unchanged. In practical applications, this parameter represents the probability of energy loss from losing a particle. The channel flips the input state from \( |0\rangle \) to \( |1\rangle \) with probability \( 1-p \). In the Bloch sphere representation, the effect of the amplitude damping channel on the initial input state
\[
\rho = \frac{1}{2}(1+|r_i\rangle \langle r_i|), \quad |r_i| = \text{length of the initial Bloch vector},
\]
can be analyzed. The output state is denoted by \( \mathcal{N}(\rho) = \frac{1}{2}(1+|r_{\text{out}}\rangle \langle r_{\text{out}}|) \), hence the amplitude damping channel can be expressed using Bloch vectors \( r_{\text{in}} \) and \( r_{\text{out}} \) in the following way:
\[
\begin{bmatrix}
|r_{\text{out}}|^{(x)} \\
|r_{\text{out}}|^{(y)} \\
|r_{\text{out}}|^{(z)}
\end{bmatrix} = \begin{bmatrix}
\sqrt{1-p} & 0 & 0 \\
0 & \sqrt{1-p} & 0 \\
0 & 0 & 1-p/2
\end{bmatrix} \begin{bmatrix}
|r_{\text{in}}|^{(x)} \\
|r_{\text{in}}|^{(y)} \\
|r_{\text{in}}|^{(z)}
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
p/2
\end{bmatrix} \tag{12}
\]

The smallest value of \( D(\rho\|\sigma) \) corresponds to the contour closest to the location of the density matrix. In Fig. 9(a), the Euclidean distances from the origin of the Bloch sphere to center \( c \) and to point \( \rho \) are denoted by \( m_{\rho} \) and \( m_\rho \), respectively. To determine the optimal length of vector \( r_{\rho} \), the algorithm moves point \( \sigma \). As we move vector \( r_{\rho} \) from the optimum position, the larger contour corresponding to a larger value of quantum relative entropy \( D \) will intersect the channel ellipsoid surface, thereby increasing \( \max_{r_{\rho}} D(\rho\|r_{\rho}) \).

The optimal quantum informational ball is illustrated in light-grey in Fig. 9(b).
From our geometrical analysis, it can be concluded that the optimum input states for an optical quantum channel are unentangled, non-orthogonal quantum states [17]. In Fig. 10, we show the results for the capacity recovery analysis of amplitude damping channel. The capacity recovery of amplitude damping channel \( \mathcal{N} \) cannot be described by the relation derived for unital quantum channels. The radii of the smallest quantum informational balls of channels \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) are denoted by \( r_1^* \) and \( r_2^* \).

Our analysis has shown that the optimum input states are non-orthogonal input quantum states [16].

### 4.1 Analysis of Channel Output States

A coreset of a set of output quantum states has the same behavior as the larger input set, so clustering and other approximations can be made with smaller coresets. The coreset can be viewed as a smaller input set of channel output states, hence it can be used as the input to an approximation algorithm. The weighted sum of errors of the smaller coreset is a \((1 \pm \varepsilon)\) approximation of the larger input set. These coresets are called weak coresets [9] and this method can be applied in quantum space between quantum states. The weak coresets include all the relevant information required to analyze the original extremely large input set. The coreset approach has significantly lower computational complexity, hence it can be applied very efficiently in the quantum space [9].

### 4.2 Clustering Channel Output States

Using \( \mu \)-similar quantum informational distances and the \( \mathcal{W} \)-weak coreset of quantum states, the capacity recovery of very noisy optical quantum channels can be analyzed by an \((1 + \varepsilon)\)-approximation algorithm in a run time

\[
\mathcal{O}
\left( d^2 2^r \log^{k+2} n + dkn \right),
\]

where \( k \) is the number of quantum states in set \( S_{\text{OUT}} \), \( n \) is the number of input states and \( d \) is the dimension of the points.

To summarize, our algorithmic capacity recovery of very noisy optical quantum channels combines the weak coreset methods and the clustering algorithms.

In Fig. 12, we illustrate the clustering method of channel output states. In the clustering process, our algorithm computes the median-quantum states denoted by \( \sigma_i \), using a fast weak coreset and clustering algorithm. In the next step, we compute the convex hull of the median quantum states and, from the convex hull, the radius of the smallest quantum informational ball can be obtained. The convex-hull calculation is based on the quantum Delaunay diagrams. The smallest superball measures the channel capacity, hence the radius of the superball is equal to the sum of radii of quantum balls of independent channel outputs. The output states are measured by a joint measurement setting.
Using the modified weak coreset algorithm and the \((1 + \varepsilon)\)-approximation algorithm, the capacity recovery of quantum channels can be analyzed relative to \(\mu\)-similar quantum informational distances and \(k\) median-quantum states with error 
\[
\text{error} (S_{\text{IN}}, S_{\text{OUT}}) \leq (1 + 7\varepsilon) \text{opt}_k (S_{\text{IN}}),
\]
where \(\text{opt}_k (S_{\text{IN}})\) is the error of the optimal solution for set of input quantum states \(S_{\text{IN}}\).

5 Conclusion

This paper shows a fundamentally new algorithmic solution for capacity recovery of very noisy optical quantum channels. To analyze channel capacity recovery, we introduced the quantum informational “superball” representation. The iterations are based on the computed radius of the superball. The algorithm presented has lower complexity in comparison with other existing coreset and approximation algorithms, which can also be applied in quantum space. This paper is intended to be an introduction to the basic properties of the proposed framework for finding recoverable very noisy optical quantum channels. The proposed method can be a very valuable tool for improving the results of fault-tolerant quantum computation and possible communication techniques over noisy optical quantum channels.

In future work, we would like to extend our results to other possible very noisy quantum channel models and we would like to show some typical results on recovered very noisy quantum channels.

References


