A single-turn coil with alternating current inside a cylindrical region with varying electric conductivity and magnetic permeability

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Abstract: - The formula for the change in impedance of a single-turn coil located inside a cylindrical region with varying electric conductivity and magnetic permeability is derived in the present paper. It is assumed that the electric conductivity and magnetic permeability of the conducting medium depend on the radial coordinate. The solution is expressed in terms of improper integrals containing Bessel’s functions of complex argument. Results of numerical calculations are presented.

Key-Words: - vector potential, change in impedance, Bessel function

1 Introduction

Eddy current testing methods are widely used in order to control properties of conducting materials of a cylindrical shape [1]-[4]. Theoretical investigation of the interaction of a coil located inside a multilayer tube is conducted in [5]-[6]. The analysis in [5] and [6] is based on the assumption that the electric conductivities and magnetic permeabilities of conducting cylindrical layers are constant.

In some cases, however, the properties of a conducting medium may depend on geometrical coordinates. Examples include surface hardening and decarbonization [7], [8]. Mathematical models for such cases can be constructed using the solution for a coil inside a multilayer tube with constant properties. From a practical point of view any variation of the electric conductivity and magnetic permeability in the radial direction can be modeled by piecewise-constant functions. The number of layers in the model should be large enough in order to take into account the changes in the electrical conductivity and magnetic permeability.

Another approach is based on an attempt to construct a closed-form solution of the problem assuming profiles of the electric conductivity and magnetic permeability of a relatively simple form. The profiles are usually dependent on some parameters so that by choosing proper values of these parameters one can model the variation of the electric conductivity and magnetic permeability in the radial direction. Such an approach is used in [9] where analytical solutions are constructed for some eddy current problems with varying properties of conducting media.

A closed-form solution is found in this paper for the case where a single-turn coil is located inside a cylindrical region with varying electric conductivity and magnetic permeability. The change in impedance of a single-turn coil is expressed in terms of improper integral containing Bessel’s functions of complex argument.

2 Mathematical formulation of the problem

Consider a single-turn coil with alternating current located in region \( R_0 \) inside a conducting cylindrical region \( R \) (see Fig. 1). The radius of the coil is, \( r_c \), is chosen as the measure of length. The dimensionless radii of the conducting cylindrical region and coil are \( R \) and 1, respectively.
The current in the coil is
\[ i(t)\hat{e}_\varphi = I \exp(j\omega t)\hat{e}_\varphi, \quad (1) \]
where \( I \) is the amplitude of the current, \( \omega \) is the frequency and \( \hat{e}_\varphi \) is the unit vector in the \( \varphi \)-direction (here \((r, \varphi, z)\) is the system of cylindrical polar coordinates with the origin at the coil’s center).

Due to the axial symmetry the vector potential is independent on \( \varphi \) and has the following structure
\[ \tilde{A}_i(r, \varphi, z, t) = A_i(r, z) \exp(j\omega t)\hat{e}_\varphi, \quad i = 0, 1, \quad (2) \]
where the subscripts 0 and 1 correspond to regions \( R_0 \) and \( R_1 \), respectively.

The amplitudes, \( A_i(r, z) \), \( i = 0, 1 \), of the vector potential satisfy the following system of equations in regions \( R_0 \) and \( R_1 \), respectively [9]:
\[ \frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I r^2 \delta(r-1)\delta(z), \]
\[ \frac{\partial^2 A_1}{\partial r^2} + \left( \frac{1}{r} - \frac{1}{r \mu dr} \right) \frac{\partial A_1}{\partial r} - \left( \frac{1}{r^2} + \frac{d \mu}{r \mu dr} + r^2 j \omega \sigma \mu_0 \mu \right) A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \quad (3) \]
where \( \delta(z) \) is the Dirac delta function, \( \mu_0 \) is the magnetic constant, \( \sigma \) and \( \mu \) are the electric conductivity and magnetic permeability of region \( R_1 \), respectively. Examples of analytical solutions of (4) are presented in [9] for some relatively simple functions of the form \( \mu = \mu(r) \). In particular, the case \( \mu(r) = r^\alpha \), where \( \alpha \) is a given constant, is analyzed in [9]. In the present paper the results of [9] are generalized for the case where both functions, \( \mu(r) \) and \( \sigma(r) \), depend on the radial coordinate \( r \). The functions \( \mu(r) \) and \( \sigma(r) \) are assumed to be of the form
\[ \mu = \mu_r r^\alpha, \quad \sigma = \sigma_0 r^\beta, \quad (5) \]
where \( \alpha \) and \( \beta \) are given constants.

Substituting (5) into (4) we obtain the following equation in region \( R_1 \):
\[ \frac{\partial^2 \tilde{A}_1}{\partial r^2} + \frac{(1 - \alpha)}{r} \frac{\partial \tilde{A}_1}{\partial r} - \left( \frac{1 + \alpha}{r^2} + p^2 r^{\alpha+\beta} \right) \tilde{A}_1 + \frac{\partial^2 \tilde{A}_1}{\partial z^2} = 0, \quad (6) \]
where \( p = \eta \sqrt{\mu_0 \mu}, \quad \eta = r \sqrt{\mu_0 \mu_0 \mu_0} \).

The boundary conditions are
\[ A_0 \big|_{r=R} = \tilde{A}_1 \big|_{r=R}, \quad \frac{\partial A_0}{\partial z} \big|_{r=R} = -\frac{1}{\mu_i} \frac{\partial \tilde{A}_1}{\partial z} \big|_{r=R}, \quad (7) \]
\[ A_1 \to 0, \quad r \to \infty, \quad (8) \]
\[ A_0 \to 0, \quad A_1 \to 0, \quad z \to \pm \infty, \quad (9) \]
where \( \mu_i = \mu_i R^\alpha \).

### 3 Mathematical analysis

Problem (3), (6)-(9) is solved by means of the Fourier cosine transform of the form
\[ \tilde{A}_i(\lambda, r) = \int_0^\infty A_i(r, z) \cos \lambda zdz, \quad i = 0, 1. \quad (10) \]
Applying (10) to (3), (6)-(9) we obtain
\[ \frac{d^2 \tilde{A}_0}{dr^2} + \frac{\lambda^2 - \lambda^2}{r^2} \tilde{A}_0 = -\mu_0 I r^2 \delta(r-1), \quad (11) \]
\[ \frac{d^2 \tilde{A}_1}{dr^2} + \frac{(1 - \alpha)}{r} \frac{d \tilde{A}_1}{dr} - \left( \frac{1 + \alpha}{r^2} + p^2 r^{\alpha+\beta} + \lambda^2 \right) \tilde{A}_1 = 0. \quad (12) \]

The boundary conditions are
\[ \tilde{A}_0 \big|_{r=R} = \tilde{A}_1 \big|_{r=R}, \quad \frac{d \tilde{A}_0}{dr} \big|_{r=R} = -\frac{1}{\mu_i} \frac{d \tilde{A}_1}{dr} \big|_{r=R}, \quad (13) \]
\[ \tilde{A}_1 \to 0, \quad r \to \infty. \quad (14) \]
In order to solve (11) we consider two subregions of region $R_0$, namely, $0 < r < 1$ and $1 < r < R$. The solutions in these two regions are denoted by $\tilde{A}_{00}(\lambda, r)$ and $\tilde{A}_{01}(\lambda, r)$, respectively. The bounded solution to (11) in region $0 < r < 1$ is

$$\tilde{A}_{00}(\lambda, r) = C_1 I_1(\lambda r),$$  \hspace{1cm} (15)$$

where $I_1(\lambda r)$ is the modified Bessel function of the first kind of order 1.

The general solution to (11) in region $1 < r < R$ is given by

$$\tilde{A}_{01}(\lambda, r) = C_2 I_1(\lambda r) + C_3 K_1(\lambda r)$$  \hspace{1cm} (16)$$

where $K_1(\lambda r)$ is the modified Bessel function of the second kind of order 1.

The solution to (12) for different values of $\alpha$ and $\beta$ can be expressed in terms of different special functions. In this paper we consider the case $\alpha = -1$, $\beta = -1$ in detail.

The solution to (12) for the case $\alpha = -1$, $\beta = -1$ is (see [10]):

$$\tilde{A}_1(\lambda, r) = C_4 \frac{K_\nu(\lambda r)}{\sqrt{r}},$$  \hspace{1cm} (17)$$

where $K_\nu(\lambda r)$ is the modified Bessel function of order $\nu$ and $\nu = \sqrt{\sigma^2 + 1/4}$.

Two additional conditions at $r = 1$ are needed to determine the constants of integration. First, we assume that the functions $\tilde{A}_{00}(\lambda, r)$ and $\tilde{A}_{01}(\lambda, r)$ are continuous at $r = 1$. Integrating (11) with respect to $r$ from $1 - \varepsilon$ to $1 + \varepsilon$ and considering the limit in the resulting equation as $\varepsilon \to 0+$ the second condition is obtained. The two additional conditions are

$$\tilde{A}_{00} \big|_{r=1} = \tilde{A}_{01} \big|_{r=1}, \quad \frac{d\tilde{A}_{00}}{dr} \big|_{r=1} - \frac{d\tilde{A}_{01}}{dr} \big|_{r=1} = -\mu_0 \frac{I_{c}^2}{2}.$$  \hspace{1cm} (18)$$

The boundary conditions (7) can be rewritten in the form

$$\tilde{A}_{01} \big|_{r=R} = \tilde{A}_1 \big|_{r=R}, \quad \frac{d\tilde{A}_{01}}{dr} \big|_{r=R} = \frac{1}{\mu_1} \frac{d\tilde{A}_1}{dr} \big|_{r=R}. \hspace{1cm} (19)$$

Using (18) and (19) we determine the constants $C_1, C_2, C_3$ and $C_4$ in (15)-(17). In particular, the constant $C_2$ is

$$C_2 = \frac{\mu_0 I_{c}^2 I_1(\lambda)}{2} \frac{D}{E},$$  \hspace{1cm} (20)$$

where

$$D = 2R \lambda K_1(\lambda R)K_{\nu}'(\lambda R) - K_{\nu}(\lambda R)[K_1(\lambda R) + 2\mu_1 \lambda K_1(\lambda R)]$$

and

$$E = 2R \lambda \mu_1 I_1(\lambda R)K_{\nu}(\lambda R) + \lambda I_1(\lambda R)[K_1(\lambda R) - 2R \lambda K_1(\lambda R)].$$

The Fourier cosine transform of the induced vector potential in free space (region $R_0$) due to the presence of electrically conducting cylindrical region has the form

$$\tilde{A}_{0}^{\text{ind}}(r, \lambda) = C_4 I_1(\lambda r).$$  \hspace{1cm} (21)$$

Using the inverse Fourier cosine transform of the form

$$A_i(r, z) = \frac{2}{\pi} \int_0^\infty \tilde{A}_i(r, \lambda) \cos \lambda z d\lambda, \quad i = 0, 1,$$  \hspace{1cm} (22)$$

we obtain the induced component of the vector potential in region $R_0$:

$$A_{0}^{\text{ind}}(r, z) = \frac{2}{\pi} \int_0^\infty C_4 I_1(\lambda r) \cos \lambda z d\lambda.$$  \hspace{1cm} (23)$$

The induced change in impedance in the coil due to the presence of the conducting cylindrical region $R_1$ is given by the formula

$$Z^{\text{ind}} = \frac{L}{I} \int_{\text{contour}} A_{0}^{\text{ind}}(r, z) dl,$$  \hspace{1cm} (24)$$

where $L$ is the contour of the coil. Substituting (20), (23) into (24) we obtain the change in impedance of the form

$$Z^{\text{ind}} = 2\sigma r^2 \mu_0 Z,$$  \hspace{1cm} (25)$$

where

$$Z = \int_0^\infty \frac{D}{E} I_1^2(\lambda) d\lambda.$$  \hspace{1cm} (26)$$
4 Numerical results

Formula (26) is used to compute the change in impedance of the coil due to the presence of the conducting cylindrical region for different values of the parameters. Calculations are done with “Mathematica” since there are built-in functions in the package that allow one to compute modified Bessel functions of complex order. The results are shown in Fig. 2.

The calculated points (from top to bottom) for each curve correspond to the following values of \( \eta \): 1, 2, ..., 10. The three curves (from right to left) are for the values of \( R = 1.2, 1.4 \) and 1.6, respectively. It is seen from Fig. 2 that the modulus of the change in impedance increases as the parameter \( \eta \) increases (that is, if the frequency of the current in the coil increases). The change in impedance is stronger if the cylindrical region is closer to the coil (for small values of \( R \ ).

5 Conclusion

Analytical solution for the change of impedance in a single-turn coil is found in the present paper for the case where a coil is located inside a cylindrical region. The electric conductivity and magnetic permeability of the cylindrical region are power functions of the radial coordinate. The solution is found in terms of improper integral containing modified Bessel functions of complex order. Analytical solution can be used for the solution of the inverse problem in the case where the properties of a conducting cylindrical region depend on the radial coordinate.

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References:


