Multitime optimal control and equilibrium deformations

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Abstract: This paper uses optimal control theory in order to describe the deforming processes which, at some future moment optimize a certain energy. Section 1 analyses an optimal control problem with interior cost and terminal cost differential forms. The proof of the corresponding multitime maximum principle relies on a certain temporal order and a corresponding type of needle-shaped control variations. The main result in Section 2 refers to a variational problem with mixed Lagrangian. In Section 3, the equilibrium equations of an elastic body are derived from the mixed Euler-Lagrange PDEs of a variational problem with mixed Lagrangian. Section 4 applies again the results of Section 2 in order to obtain the multitime maximum principle for a control problem with terminal running cost differential form. Again, we use needle-shaped control variations and an adapted temporal order. In the last section, we analyze the evolution of a curve which, at some given future moment, minimizes the energy.

Keywords: multitime maximum principle, multitime needle-shaped control variations, terminal cost differential form, terminal running cost differential form, elasticity.

Mathematics Subject Classification (2000): 49J20, 49J40, 70H06, 74G65.

1 Optimal control with terminal cost differential forms

We are already familiarized with the single-time optimal control problem (see [1], [3], [5], [7], [8], [21])

$$\max_{u(\cdot)} J[u(\cdot)] = \int_0^{t_0} X(t, x(t), u(t))dt + X_0(x(t_0))$$

subject to

$$\frac{dx^i}{dt}(t) = X^i(t, x(t), u(t)), \; i = 1, \ldots, n,$$

$$u(t) \in U, \; t \in [0, t_0], \; x(0) = x_0.$$ 

Here, $X$ denotes the interior running cost and $X_0$ is the terminal cost.

In this section, we give a generalization for the previous problem by considering a multitime optimal control problem, for which the terminal cost function is replaced by a terminal cost functional defined as a multiple integral associated to a terminal cost m-form.

Let us consider the global coordinates $(t^1, \ldots, t^m)$ on the Euclidean space $R^m$. Also, let $M = R^n$ be endowed with global coordinates $(x^1, \ldots, x^n)$, $R^k$ be considered having the coordinates $(u^1, \ldots, u^k)$ and let $U \subset R^k$. We also take the hyper-parallelepiped $T = \Omega_{t_0} \subset R^m$ defined by the opposite diagonal points $0 = (0, \ldots, 0)$ and $t_0 = (t^1_0, \ldots, t^m_0)$ and, if $\tau_0 \in R_+$, we introduce the set $N = [0, \tau_0] \times \Omega_{t_0}$. The natural order from the interval $[0, \tau_0]$ induces an order on $[0, \tau_0] \times \Omega_{t_0}$ as follows: for the multi-times $s = (\sigma, s^1, \ldots, s^m)$ and $t = (\tau, t^1, \ldots, t^m)$ we denote $s \leq (\tau, t)$ if and only if $\sigma \leq \tau$ and $s^i \leq t^i$ if and only if $\sigma \leq \tau$. We also consider the set $[\sigma, \tau] \times \Omega_{t_0}$ (a vertical band) and the following (block-type) multi-intervals:

$$([\sigma, \tau]) = (\sigma, \tau) \times \Omega_{t_0}; \; [[[\sigma, \tau]]] = [\sigma, \tau] \times \Omega_{t_0}.$$ 

Let $X_\alpha = (X^i_\alpha), X^i_\alpha : N \times M \times U \rightarrow R$ be $C^1$ vector fields. For a given $C^1$ control function $u : N \rightarrow R^k$, suppose the evolution PDEs system (controlled $(m+1)$-flow)

$$(PDE) \quad \frac{\partial x^i}{\partial \tau^\alpha}(\tau, t) = X^i_\alpha(\tau, t, x(\tau, t), u(\tau, t)),$$

$$i = 1, \ldots, n, \; \alpha = 0, \ldots, m, \; x(0, t) = \Phi(t),$$

$$x(\tau, \cdot)|_{\partial \Omega_{t_0}} = \varphi(\tau, \cdot), \; \tau \in [0, \tau_0], \; t \in \Omega_{t_0}$$

has solution, where \( \frac{\partial}{\partial \tau^\alpha} = \frac{\partial}{\partial \tau} \), i.e. the complete integrability conditions

$$(CIC) \quad \frac{\partial X^i_\alpha}{\partial \tau^\beta} + \frac{\partial X^i_\alpha}{\partial x^j} \chi^j_\beta + \frac{\partial X^i_\alpha}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \tau^\beta}$$
subject to
$$\frac{\partial x^i}{\partial t^\alpha}(\tau, t) = X^i_\alpha(\tau, t, x(\tau, t), u(\tau, t)),$$
$$i = 1, \ldots, n, \quad \alpha = 0, \ldots, m,$$
$$u(\tau, t) \in \mathcal{U}, \quad x(0, t) = \Phi(t), \quad x(\cdot, \cdot)|_{\partial \Omega_0} = \varphi(\cdot, \cdot).$$

The multitime maximum principle (necessary condition) asserts that the existence of an optimal control $u^*(\cdot)$ implies the existence of the costate functions $(p^0_0, p^1_0, \ldots, p^m_0, p^0_1, p^1_1, \ldots, p^m_1)^\ast(\cdot)$, which, together with the optimal $(m + 1)$-sheet $x^*(\cdot)$, satisfy a suitable PDEs system. Similar to single-time theory, this multitime maximum principle involves an appropriate control Hamiltonian
$$H(t, x, p_0, p, u) = p_0 X(t, x, u) + p_\alpha X^1_\alpha(t, x, u).$$

The general proofs of multitime maximum principle rely on a special type of variations, called $(m + 1)$-needle-shaped control variation, which induces also a variation of the reference $(m + 1)$-sheet.

Suppose $u^*(\cdot)$ is a candidate optimal control and that $x^*(\cdot)$ is the corresponding $(m + 1)$-sheet. Fixing a time $\sigma \in (0, \tau_0)$ and an admissible control $u(t)$, we set $\epsilon \in [0, \sigma]$ and define the modified control
$$u_\epsilon(\tau, t) = \begin{cases} 
    u(\tau, t) & \text{if } (\tau, t) \in ([\sigma - \epsilon], [\sigma]); \\
    u^*(\tau, t) & \text{otherwise.}
\end{cases}$$

Let $x_\epsilon(\cdot)$ be the corresponding response of our system, i.e.
$$\frac{\partial x^i}{\partial t^\alpha}(\tau, t) = X^i_\alpha(\tau, t, x_\epsilon(t), u_\epsilon(t), u_\epsilon(t)),$$
$$x_\epsilon(0, t) = \Phi(t), \quad x_\epsilon(\cdot, \cdot)|_{\partial \Omega_0} = \varphi(\cdot, \cdot).$$

We denote by $y^\ast(\tau, t)$ the infinitesimal deformation of the $(m + 1)$-sheet $x^*(\tau, t)$ induced by the previous control variation.

**Lemma 1.2** Let $\varphi : [[0, \sigma]] \times (\delta, \delta) \rightarrow R$, $\varphi = \varphi(\tau, t, \epsilon)$ be a differentiable function with parameter. Then
$$\frac{d}{d\epsilon} \int_{[\sigma - \epsilon], [\sigma]} \varphi(\tau, t, \epsilon) d\epsilon d\tau|_{\epsilon = 0} = \int_{[\sigma]} \varphi(\tau, t, 0) dt$$
$$= \int_{\Omega_0} \varphi(\sigma, t, 0) dt.$$

**Lemma 1.3** The infinitesimal deformation $y$ induced by the needle-shaped control variation satisfies the following relations:

1. $y^i(\tau, t) = 0$, if $(\tau, t) \in [[0, \sigma]],$
2. $\int_{\Omega_0} y^i(\sigma, t) dt = \int_{\Omega_0} \left[ X^i_0(\sigma, t, x^*(\sigma, t), u(\sigma, t)) - X^i_0(\sigma, t, x^*(\sigma, t), u^*(\sigma, t)) \right] d\sigma$, if $(\tau, t) \in ([\sigma], [\sigma])$.

**Proof.** We recall $y^i(\epsilon, \tau, t) = \frac{\partial x^i}{\partial \epsilon}(\tau, t)$. Since $x_\epsilon(\tau, t) = x^*(\tau, t)$, $\forall (\tau, t) \in [[0, \sigma - \epsilon]]$, we have
and, by applying Lemma 1.2, it follows
\[
\int_{\Omega_{t_0}} y^i(\sigma, t) dt = \int_{\Omega_{t_0}} [X^i_0(\sigma, t, x^*(\sigma, t), u(\sigma, t)) - X^i_0(\sigma, t, x^*(\sigma, t), u^*(\sigma, t))] dt.
\]

On \([\{\sigma\}, [\tau_0]]\), we have again
\[
\frac{\partial y^i}{\partial t^\alpha}(\tau, t) = \frac{\partial X^i}{\partial x^j} \left(\tau, t, x^*(\tau, t), u^*(\tau, t)\right) y^j(\tau, t).
\]

\[\tag{1} \frac{\partial(p^i_0 x^i)}{\partial t^\alpha}(\tau, t) = \frac{\partial p^i_0}{\partial t^\alpha}(\tau, t)x^i(\tau, t) + H(\tau, t, x^*(\tau, t), p_0(\tau, t), p(\tau, t), u^*(\tau, t)) - p_0(\tau, t) X(\tau, t, x^*(\tau, t), u^*(\tau, t))
\]

The adjoint system (ADJ) has solutions since it contains \(n\) PDEs with \(nm\) unknown functions \(p^i_0\).

**Theorem 1.5 (multitime maximum principle)** Suppose \(u^*(\cdot)\) is optimal for (PDE), (J) and that \(x^*(\cdot)\) is the corresponding optimal \((m + 1)\)-sheet. Then there exist the dual functions \(p^i_0, p^i_1 : [0, \tau_0] \to R\) such that
\[
(\text{PDE}) \quad \frac{\partial x^i}{\partial t^\alpha}(\tau, t) = \frac{\partial H}{\partial x^i}(\tau, t, x^*, p^*_0, p^*, u^*),
\]

\[
(\text{ADJ}) \quad \frac{\partial p^i_0}{\partial t^\alpha}(\tau, t) = -\frac{\partial H}{\partial x^i}(\tau, t, x^*, p^*_0, p^*, u^*)
\]

and
\[
(\text{M}) \quad \frac{\partial H}{\partial u^i}(\tau, t, x^*(\tau, t), p^*(\tau, t), u^*(\tau, t)) = 0,
\]

\[
\forall(\tau, t) \in [0, [\tau_0]].
\]

Finally, the boundary conditions
\[
(\tau_0) \quad p^i_0(\tau_0, t) = \frac{\partial y}{\partial x^i}(\tau_0, t), \forall t \in \Omega_{t_0}
\]
are satisfied.

We call \(x^*(\cdot)\) the state of the optimally controlled system and \((p^i_0, p^i_1(\cdot))\) the costate functions. Even more, we may consider \(p^i_0 = 1\).

**Proof.** The control Hamiltonian satisfies
\[
\frac{\partial(p^i_0 x^i)}{\partial t^\alpha}(\tau, t) = \frac{\partial p^i_0}{\partial t^\alpha}(\tau, t)x^i(\tau, t) + H(\tau, t, x^*(\tau, t), p_0(\tau, t), p(\tau, t), u^*(\tau, t)) - p_0(\tau, t) X(\tau, t, x^*(\tau, t), u^*(\tau, t))
\]

\[
\frac{\partial p^i_0}{\partial t^\alpha}(\tau, t) = -\frac{\partial H}{\partial x^i}(\tau, t, x^*, p^*_0, p^*, u^*)
\]

\[
\frac{\partial(p^i_0 x^i)}{\partial t^\alpha}(\tau, t) = \frac{\partial p^i_0}{\partial t^\alpha}(\tau, t)x^i(\tau, t)
\]

Remark 1.4 If \(T\) and \(M\) are two differentiable manifolds, of dimension \(m\), respectively \(n\), and \(X : M \to R\) is a differentiable function on \(M\), then the variational flow
\[
\frac{\partial y^i}{\partial t^\alpha}(t) = y^i(t) \frac{\partial X^i}{\partial x^j}(x(t))
\]
on the jet bundle of order one \(J^1(T, M)\), determines a dual \(m\)-flow (adjoint \(m\)-flow)
\[
(\text{ADJ}) \quad \frac{\partial p^i_0}{\partial t^\alpha}(\tau, t) = -p^i_0(\tau, t) \frac{\partial X^i}{\partial x^j}(x(t))
\]
on the dual space \(J^1*(T, M)\). These PDEs systems are called adjoint, meaning that, if taking the vector fields \(Q = p^i_0(t)y^i(t) \frac{\partial}{\partial x^i}\) and \(y = y^i \frac{\partial}{\partial x^i}\), then
\[
(\text{Div} Q)(t) = -y(L)(x(t)).
\]
+H(τ, t, x, (τ, t), p₀(τ, t), p(τ, t), u*(τ, t))
−p₀(τ, t)X(τ, t, x, (τ, t), u*(τ, t)),
∀(τ, t) ∈ ([σ], [τ₀]).

Therefore, by taking the difference (2) − (1) on ([σ − ε], [σ]) and integrating afterwards, we obtain
\[
\int _{[σ]} [(x^i − x^i_i) p^i_0] dt = \int _{[σ−ε],[σ]} \{ H(τ, t, x, p₀, p, u)
−H(τ, t, x*, p₀, p, u*) + \frac{∂p^α}{∂t^α}(x^i_i − x^i_i)
−p₀[X(τ, t, x, u) − X(τ, t, x*, u*)]\} dt.
\]

Computing the derivative with respect to ε (see Lemma 1.2), we find
\[
\int _{[σ]} y_i^i p^i_0 dt = \int _{[σ]} \{ H(τ, t, x, p₀, p, u)
−H(τ, t, x*, p₀, p, u*)
−p₀[X(τ, t, x, u) − X(τ, t, x*, u*)]\} dt.
\]

We chose the costate vector p* as solution for the adjoint PDEs (ADJ) with boundary conditions (τ₀). On the multi-interval ([σ], [τ₀]), we have
\[
\frac{∂(p^α_i y^i)}{∂t^α} = −p₀ \frac{∂X}{∂x^i}(τ, t, x*, u*)y^i\text{ on }([σ], [τ₀]).
\]

Then,
\[
\int _{[σ],[τ₀]} p₀ \frac{∂X}{∂x^i}(τ, t, x*, u*)y^i(τ, t) dτ dt
= −\int _{[σ],[τ₀]} \frac{∂(p^α_i y^i)}{∂t^α} dτ dt
= \int _{[σ]} (y_i^i p^i_0) dt − \int _{[τ₀]} (y_i^i p^i_0) dt
= \int _{[σ]} (y_i^i p^i_0) dt − \int _{Ω_{τ₀}} \frac{∂q}{∂x^i}(τ₀, t)y^i(τ₀, t) dt,
\]

or
\[
\int _{[σ]} [H(τ, t, x, p₀, p, u) − H(τ, t, x*, p₀, p, u*)] dt
= \int _{[σ]} p₀[X(τ, t, x, u) − X(τ, t, x*, u*)] dt
− \int _{[σ],[τ₀]} p₀ \frac{∂X}{∂x^i}(τ, t, x*, u*)y^i dτ dt
− \int _{Ω_{τ₀}} \frac{∂q}{∂x^i}(τ₀, t)y^i(τ₀, t) dt.
\]

Since u* is an optimal control, it follows that ϵ = 0 is a maximum point for the function
\[
eq \int _{[[0],[σ−ε]]} X(τ, t, x*, u*) dτ dt
+ \int _{[[σ−ε],[σ]]} X(τ, t, x, u) dτ dt
+ \int _{[[σ],[τ₀]]} X(τ, t, x, u*) dτ dt + \int _{Ω_{τ₀}} g(t, x(τ₀, t)) dt,
\]

therefore,
\[
\int _{[σ]} X(τ, t, x*, u*) dt + \int _{[σ]} X(τ, t, x, u) dt
+ \int _{[[σ],[τ₀]]} \frac{∂X}{∂x^i}(τ, t, x*, u*)y^i dτ dt
+ \int _{Ω_{τ₀}} \frac{∂q}{∂x^i}(τ₀, t)y^i(τ₀, t) dt \leq 0.
\]

Combining (5) and (6) and taking p₀ = 1, it follows
\[
\int _{[σ]} [H(τ, t, x, p, u) − H(τ, t, x*, p, u*)] dt \leq 0.
\]

We have obtained the maximum principle inequality in functional integral form. By writing the Euler-Lagrange equations, we obtain
\[
\frac{∂H}{∂u}(σ, t, x*, σ, t, p*, σ, t, u*(σ, t)) = 0, \forall t ∈ Ω_{τ₀},
\forall σ ∈ (0, τ₀).
\]

□

2 Variational problems with interior and boundary energies

We start with a closed (m + 1)-dimensional differentiable manifold N with smooth piece-wise boundary ∂N and an n-dimensional differentiable manifold M. On their first order jet bundle J¹(N, M), endowed with the induced local coordinates (tⁿ, xⁿ, xⁿ'), we consider a differentiable Lagrangian L. Also, we take a boundary Lagrangian g : J¹(∂N, M) → R. The pair (L, g) is called mixed Lagrangian. We study next the variational problem consisting in finding
\[
(Mix.J) \min J(x(·)) = \int _{N} L(t, x(t), \nabla x(t)) dt
+ \int _{Ω_{t}} g(t, x(t), \nabla x(t)) dσ
\]
constraint by
\[(Ω_2) \quad x|_{Ω_2} = Φ,\]
where the vector \(\nabla x(t) = (x^i_\alpha(t))\) denotes the partial velocities of \(x(t), Ω_1 ∪ Ω_2 = ∂N\) and \(∂Ω_1 = ∂Ω_2\).

**Theorem 2.1 (Mixed Euler-Lagrange PDEs)** If the \((m+1)\)-sheet \(x^*(\cdot)\) is an optimal solution for the variational problem \((M|x, J, Ω_2)\), then \(x^*(\cdot)\) is solution for the mixed Euler-Lagrange PDEs

\[
(Mix(E - L))
D_α \left( \frac{∂g}{∂x^i_α} \right) (t, x^*, \nabla x^*) - \frac{∂L}{∂x^i_α}(t, x^*, \nabla x^*) = 0 \text{ in } N;
\]

\[
D_α \left( \frac{∂g}{∂x^i_α} \right) (t, x^*, \nabla x^*) - \frac{∂g}{∂x^i_α}(t, x^*, \nabla x^*)
= \frac{∂L}{∂x^i_α}(t, x^*, \nabla x^*) n_α \text{ on } Ω_1.
\]

**Remark 2.2 (1)** If \(Ω_1\) is of measure zero, the variational problem is

\[
(J_L)
\min_{x} J[x(\cdot)] = \int_N L(t, x(t), \nabla x(t))dt, \ x|_{Ω_2} = Φ,
\]
where \(Ω_2\) is the boundary of \(N\) minus a subset of measure zero. By applying the previous Theorem, we regain, as expected, the classic multitime Euler-Lagrange PDEs

\[
\frac{∂L}{∂x^i} - D_α \left( \frac{∂L}{∂x^i_α} \right) = 0.
\]

\[
(2) \text{ If } L = 0, \text{ the variational problem refers to }

(J_g)
\int_{Ω_1} g(t, x(t), \nabla x(t))dσ; \ x|_{Ω_2} = Φ
\]
and we obtain the Euler-Lagrange PDEs

\[
\frac{∂g}{∂x^i} - D_α \left( \frac{∂g}{∂x^i_α} \right) = 0 \text{ on } Ω_1.
\]

We remark that these Euler-Lagrange PDEs describe the behavior of a solution on the boundary only. Again, this result was to be expected, since the variational problem \((J_g)\) is of the same type as \((J_L)\), via the change of domain and an adjustment related to the constraint (in the relation \(x|_{Ω_2} = Φ\) we eliminate the unnecessary information and keep only the boundary restriction \(x|_{Ω_2} = x|_{Ω_1} = Φ|_{Ω_2}\)). We meant to say previously that, for studying such a problem we have two possibilities: to apply the Mixed Euler-Lagrange PDEs Theorem as above, or to resume the problem, from the start, to the boundary and to apply the classic multitime variational theory. Either way, we obtain a description of the solution only on the boundary.

**Proof.** For solving this problem, we suppose that \(x^* : N → M\) is an optimal \((m+1)\)-sheet. For an arbitrary vector field \(y : N → M\) satisfying the boundary constraint \(y|_{Ω_2} = 0\), we consider a variation \(ϵ \rightarrow x^*\) of \(x^*\), defined by \(x^*(t) = x^*(t) + ϵg(t), \ ϵ ∈ (-δ, δ)\). Then \(x^*(\cdot)\) is an optimal solution iff \(ϵ = 0\) is a critical point for the function

\[
J(ϵ) = \int_N L(t, x^*(t), \nabla x^*(t))dt + \int_{Ω_1} g(t, x^*(t), \nabla x^*(t))dσ,
\]
meaning

\[
0 = \frac{d}{dϵ} J(ϵ)|_{ϵ=0} = \int_N \left( \frac{∂L}{∂x^i} y^i + \frac{∂L}{∂x^i_α} y^i_α \right) dt + \int_{Ω_1} \left( \frac{∂g}{∂x^i} y^i + \frac{∂g}{∂x^i_α} y^i_α \right) dσ
\]

\[
= \int_N \left( \frac{∂L}{∂x^i} - D_α \left( \frac{∂L}{∂x^i_α} \right) \right) y^i dt + \int_{Ω_1} \left( \frac{∂L}{∂x^i} y^i + \frac{∂g}{∂x^i_α} y^i_α \right) dσ + \frac{∂L}{∂x^i_α}(t, x^*, \nabla x^*) n_α dσ
\]

\[
= \int_N \left( \frac{∂L}{∂x^i} y^i - D_α \left( \frac{∂L}{∂x^i_α} \right) \right) y^i dt + \int_{Ω_1} \left( \frac{∂L}{∂x^i} y^i + \frac{∂g}{∂x^i_α} y^i_α \right) dσ.
\]

In order to obtain the expected result, we use further optimal control theory with boundary costs (see [27]). For that, let us consider the functions

\[
φ_i : N → R, \ ψ_i : Ω_1 → R
\]

\[
φ_i(t) = \left( \frac{∂L}{∂x^i} - D_α \left( \frac{∂L}{∂x^i_α} \right) \right) (t, x^*(t), \nabla x^*(t));
\]

\[
\psi_i(t) = \frac{∂L}{∂x^i_α} n_α + \frac{∂g}{∂x^i} n_α - D_α \left( \frac{∂g}{∂x^i_α} \right) (t, x^*(t), \nabla x^*(t)).
\]

The previous relation rewrites

\[
0 = \int_N (φ_i y^i) dt + \int_{Ω_1} (ψ_i y^i) dσ
\]
and this equality holds for each \(y(\cdot)\). If taking the optimal control problem with boundary cost

\[
\max_{y(\cdot)} \int_N (φ_i y^i) dt + \int_{Ω_1} (ψ_i y^i) dσ
\]

The optimal control problem with boundary cost

\[
\max_{y(\cdot)} \int_N (φ_i y^i) dt + \int_{Ω_1} (ψ_i y^i) dσ
\]
satisfying the evolution system $\frac{\partial y^i}{\partial t} = u^i_0$, obviously, the Pontryagin maximum principle is identically satisfied. We have

$$H = \varphi_i y^i + p^a_{i\alpha} u^i_\alpha; \quad 0 = \frac{\partial H}{\partial u^i_{\alpha}} = p^a_{i\alpha};$$

$$0 = \frac{\partial p^a_{i\alpha}}{\partial t} - \frac{\partial H}{\partial y^i} = \varphi_i; \quad 0 = p^a_{i\alpha} n_\alpha|_{\Omega_1} = \psi_i.$$

Reconsidering our notations, the previous relations become

$$\frac{\partial L}{\partial x^i} - D_\alpha \left( \frac{\partial L}{\partial x^i_\alpha} \right) = 0;$$

$$\frac{\partial L}{\partial x^i_\alpha}|_{\Omega_1} n_\alpha + \frac{\partial g}{\partial x^i} - D_\alpha \left( \frac{\partial g}{\partial x^i_\alpha} \right) = 0.$$

### 3 The equations of nonlinear elasticity

Let there be given a body occupying the closed domain $N \subset \mathbb{E}^3$ with global coordinates $(t^1, t^2, t^3)$. The set $N$ is the reference configuration of the body. We also consider a partition $\Omega_1 \cup \Omega_2 = \partial N$ of the boundary, $\partial \Omega_1 = \partial \Omega_2$. The body is subject to applied body forces in its interior, of density $f = f^1 e_i : \text{Int}(N) \rightarrow \mathbb{R}^3$ per unit volume and to applied surface forces on the portion $\Omega_2$ of the boundary, of density $h = h^i e_i : \Omega_1 \rightarrow \mathbb{R}^3$ per unit area. The unknown is the displacement field $x = x^i e_i : N \rightarrow \mathbb{E}^3$ that the body undergoes when it is subject to applied forces. It is assumed that the displacement vanishes on $\Omega_2$, i.e. satisfies the boundary condition of place

$$x|_{\Omega_2} = 0.$$

The equations of equilibrium write

$$- \frac{\partial}{\partial t} \left( \sigma^{ij} + \sigma^{kj} x^k_i \right) = f^i \text{ in } \text{Int}(N);$$

$$(\sigma^{ij} + \sigma^{kj} x^k_i) n_j = h^i \text{ on } \Omega_1;$$

$$\sigma^{ij} = \sigma^{ji},$$

where $\Sigma = (\sigma^{ij})$ denotes the second Piola-Kirchhoff stress tensor field.

If the body material is elastic, then there is a response map $\mathcal{R} = (\mathcal{R}^{ij})$ such that the stress tensor $\Sigma$ satisfies the constitutive equations

$$\sigma^{ij}(t) = \mathcal{R}^{ij}(t, E(\nabla x)(t)),$$

where $E = (E_{kl})$, defined by

$$E_{kl}(\nabla x) = \frac{1}{2} \left( \delta_{mn} \delta^m_{kl} x^n_i + \delta_{mn} \delta^m_{ik} x^n_l + \delta_{mn} x^m_i x^n_l \right)$$

denotes the Green-St. Venant strain tensor field (or the change of metric tensor field associated with the displacement field $x(\cdot)$)

Furthermore, an elastic material is said to be hyperelastic if there is a stored energy function $W$ depending on $E_{a\beta}$, such that

$$\mathcal{R}^{kl}(t, E) = \frac{\partial W}{\partial E_{kl}}(t, E).$$

We prove next, by using the theory of the previous section, that the equilibrium equations are precisely the Mixed Euler-Lagrange PDEs corresponding to the mixed Lagrangian $(L, g)$ = $(W(t, E(\nabla x)) - \delta_{ij} f^i x^j, -\delta_{ij} h^i x^j)$ and the boundary constraint $(\Omega_2)$. Indeed, let us consider the functional

$$J[x(\cdot)] = \int_N W(t, E(\nabla x)(t)) dt$$

$$- \int_N \delta_{ij} f^i x^j dt - \int_{\Omega_1} \delta_{ij} h^i x^j d\omega.$$

By computation, we obtain

$$\frac{\partial L}{\partial x_j} = \frac{\partial W}{\partial E_{kl}}(t, E) \frac{\partial E_{kl}}{\partial x_j}$$

$$= \mathcal{R}^{kl}(t, E) \left[ \frac{1}{2} (\delta_{mn} \delta^m_{kl} x^n_i + \delta_{mn} \delta^m_{ik} x^n_l + \delta_{mn} x^m_i x^n_l) \right] \Rightarrow$$

$$\frac{\partial L}{\partial x_j}(t, x(t), \nabla x(t)) = \delta_{kl} \delta_{mi} \sigma^{kl}(t)[\delta^m_{ij}(t) + x^m_i(t)].$$

Therefore, the first set of equations from (Mix (E-L)) give

$$\delta_{mi} \left\{ \frac{\partial}{\partial t} \left[ \sigma^{kl}(t) [\delta^m_{ij}(t) + x^m_i(t)] \right] + f^m(t) \right\} = 0,$$

$$\forall i = 1, 2, 3, \forall t \in N,$$

which represents the covectorial expression of the first set of equilibrium equations.

The second part of the mixed Euler-Lagrange PDEs gives

$$\delta_{mi} \left[ \sigma^{kl}(t) | \delta^m_{ij}(t) + x^m_i(t) | n_k(t) - h^m(t) \right] = 0,$$

$$\forall t \in \Omega_1,$$

representing the second set of equilibrium equations. Finally, relations (3) come out due to the symmetries of the Green-St. Venant strain tensor field $E$. 


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4 Optimal control problem with interior running cost and terminal running cost

In this section, we generalize the result obtained in the first section, by replacing the terminal cost m-form \( g = g(t, x(\tau_0, t))dt \) with a terminal running cost m-form \( g = g(t, x(\tau_0, t), u(\tau_0, t))dt \). Therefore, our new optimal control problem reads

\[
\max_{u(\cdot)} J[u(\cdot)] = \int_{[0, \tau_0]} X(\tau, t, x(\tau, t), u(\tau, t))d\tau dt + \int_{\Omega_{\tau_0}} g(t, x(\tau_0, t), u(\tau_0, t))dt
\]

subject to

\[
(MixJ) \quad \frac{\partial x^i}{\partial t^\alpha}(\tau, t) = X^i_\alpha(\tau, t, x(\tau, t), u(\tau, t)), \\
\quad i = 1, ..., n, \quad \alpha = 0, ..., m,
\]

\( u(\tau, t) \in U, \quad x(0, t) = \Phi(t), \quad x(\tau, \cdot)|_{\partial\Omega_{\tau_0}} = \varphi(\tau, \cdot) \).

For solving this problem, we define a new temporal order and new intervals on which we take needle-shaped control variations: for two multi-times \((\sigma, s)\) and \((\tau, t)\) in \([0, \tau_0] \times \Omega_{\tau_0}\), we denote \((\sigma, s) \leq (<) (\tau, t)\) if and only if \(s^\lambda \leq (<) t^\lambda\), \(\lambda = 1, ..., m\) (that is, \(s \leq (<) t\) in \(R^m\)). We also consider the \(L\)-type sheet

\[
[s] = [0, \tau_0] \times \{t \in R^m_+ | t \leq s \text{ and } \exists \lambda = 1, ..., m \text{ such that } t^\lambda = s^\lambda\}
\]

and the block intervals, respectively \(L\)-type intervals:

\([0, [s]] = [0, \tau_0] \times \Omega_{\tau_0}; \quad ([s], [t]) = ([0, t]) \setminus ([0, [s]])\).

Furthermore, we denote the sections at \(\tau = \tau_0\) of the previous sets (the projections on \(\{\tau_0\} \times \Omega_{\tau_0}\) with \((s), ([0), (s)]\), respectively \((s), ([0), (t)]\).

Suppose \(u^*(\cdot)\) is a candidate optimal control and that \(x^*(\cdot)\) is the corresponding \((m + 1)\)-sheet. Fixing a multi-time \(s \in \Omega_{\tau_0}\) and a control \(u(\cdot) \in U\), and considering \(0 \leq \epsilon < s\), a needle variation is a family of controls \(u_\epsilon\) obtained by replacing \(u^*\) with \(u\) on \([s - \epsilon, [s]]\). In other words,

\[
\epsilon(\tau, t) = \begin{cases} 
  u(\tau, t) & \text{if } (\tau, t) \in ([s - \epsilon, [s]) \\
  u^*(\tau, t) & \text{otherwise.}
\end{cases}
\]

We denote by \(x_\epsilon(\cdot)\) the corresponding response of our system, i.e., the \((m + 1)\)-sheet satisfying

\[
\frac{\partial x^i_\epsilon}{\partial t^\alpha}(\tau, t) = X^i_\alpha(\tau, t, x_\epsilon(\tau, t), u_\epsilon(\tau, t)),
\]

\(x_\epsilon(0, t) = \Phi(t), \quad x_\epsilon(\tau, \cdot)|_{\partial\Omega_{\tau_0}} = \varphi(\tau, \cdot)\).

Let \(y^1_\lambda(\tau, t) = \frac{\partial x^1_\lambda(\tau, t)}{\partial t^\alpha}|_{\epsilon = 0}\) be the infinitesimal deformation of the \((m + 1)\)-sheet \(x_\epsilon(\tau, t)\) induced by the previous control variation (from now on, \(\alpha, \beta, ..., \lambda, \mu, ..., \mu = 1, ..., m\)). In this case, Lemma 1.2 and Lemma 1.3 become

**Lemma 4.1** Let \(\varphi : [[0], [s]] \times (-\delta, \delta)^m \rightarrow R, \varphi = \varphi(\tau, t, \epsilon), \) be a differentiable parametrized function. Then

\[
\frac{\partial}{\partial \epsilon} \int_{[[s - \epsilon], [s]]} \varphi(\tau, t, \epsilon)d\tau dt|_{\epsilon = 0} = \int_{[s]} \varphi(\tau, t, 0)n_\lambda(\tau, t)d\tau d\sigma,
\]

\[
\frac{\partial}{\partial \epsilon} \int_{[[s - \epsilon], [s]]} \varphi(\tau, t, \epsilon)d\tau dt|_{\epsilon = 0} = \int_{(s)} \varphi(\tau_0, t, 0)n_\lambda(\tau_0, t)d\sigma,
\]

where \(n = (n_\lambda)\) denotes the covector corresponding to the unit normal vector field of the hypersurface \([s]\) and \(d\sigma\) is the volume element of \(s\).

**Lemma 4.2** The infinitesimal deformation \(y^1_\lambda\) induced by the needle-shaped control variation satisfies the following relations:

\[
y^1_\lambda(\tau, t) = 0, \quad i(\tau, t) \in [[0], [s]],
\]

\[
\int_{[s]} y^1_\lambda n_\mu d\sigma = \int_{[s]} X^i_\mu(\tau, t, x^*, u) - X^i_\mu(\tau, t, x^*, u^*)|n_\lambda d\tau d\sigma,
\]

\[
\frac{\partial y^1_\lambda}{\partial t^\alpha} = \frac{\partial X^1_\alpha}{\partial x^i}(\tau, t, x^*, u^*)y^1_\lambda, \quad \text{on } ([s], [t_0]),
\]

\(\forall \lambda = 1, ..., m, \forall \alpha = 0, ..., m, \forall i = 1, ..., n\).

We introduce a mixed control Hamiltonian \((H, H_0)\), defined by an interior Hamiltonian, respectively a terminal Hamiltonian:

\[
H(\tau, t, x, p_0, p, u) = p_0 X(\tau, t, x, u)
\]

\[
+ p_0^0 X^1_\alpha(\tau, t, x, u) \text{ on } [[0), [t_0]];
\]

\[
H_0(t, x, q, u) = q(t, x, u) + q^1_\lambda X^1_\lambda(t, x, u) \text{ on } [[0], (t_0)].
\]
Theorem 4.3 Suppose $u^*(\cdot)$ is an optimal control for (MixPDE), (MixJ) and that $x^*(\cdot)$ is the corresponding optimal $(m + 1)$-sheet. Then there exist the dual functions $p_0^\alpha, p_1^\alpha : [0, \tau_0] \rightarrow R$ such that

\[
(\text{MixPDE}) \quad \frac{\partial x^i}{\partial \tau}(\tau, t) = \frac{\partial H}{\partial p_i^\alpha}(\tau, t, x^*, p_0^\alpha, p^*, u^*),
\]

\[
(\text{MixADJ}) \quad \frac{\partial p_i^\alpha}{\partial \tau}(\tau, t) = -\frac{\partial H}{\partial x^i}(\tau, t, x^*, p_0^\alpha, p^*, u^*)
\]

and

\[
(\text{MixM}) \quad \frac{\partial H}{\partial u^\alpha}(\tau, t, x^*, p^*, u^*) = 0.
\]

Finally, the boundary conditions

\[
(\tau_0) \quad p_i^\alpha(\tau_0, t) = \frac{\partial H_0}{\partial x^i}(\tau_0, t, x^*(\tau_0, t), q^*(t), u^*(\tau_0, t))
\]

\[+ \frac{\partial q_i^\lambda}{\partial \lambda}(t), \forall t \in \Omega_{\tau_0},
\]

are satisfied, where the boundary dual functions $q_i^\lambda : \Omega_{\tau_0} \rightarrow R$ are solutions for

\[
(\text{MixM0}) \quad \frac{\partial H_0}{\partial u^\alpha}(t, x^*(\tau_0, t), q^*(t), u^*(\tau_0, t)) = 0.
\]

Proof. The mixed control Hamiltonian $(H, H_0)$, $H = p_0 X + p_1 X^\alpha, H_0 = g + q_1 X^\lambda$ satisfies

\[
(1) \quad \frac{\partial (p_0^\alpha x^i)}{\partial \tau} = H(\tau, t, x^*, p_0^\alpha, p^*, u^*)
\]  

\[-p_0 X(\tau, t, x^*, u^*) + \frac{\partial p_0^\alpha}{\partial x^i} x^i, \text{ on } [0, t_0];
\]

\[
(1') \quad \frac{\partial (q_1^\lambda x^i)}{\partial \lambda} = H_0(t, x^*, q, u^*) - g(t, x^*, u^*)
\]  

\[+ \frac{\partial q_i^\lambda}{\partial \lambda} x^i, \text{ on } [0, t_0];
\]

\[
(2) \quad \frac{\partial (p_1^\alpha x^i)}{\partial \tau} = H(\tau, t, x^*, p_0^\alpha, p^*, u^*)
\]  

\[+ \frac{\partial p_0^\alpha}{\partial x^i} x^i, \text{ on } [s - \epsilon, s];
\]

\[
(2') \quad \frac{\partial (q_1^\lambda x^i)}{\partial \lambda} = H_0(t, x^*, q, u^*) - g(t, x^*, u^*)
\]  

\[+ \frac{\partial q_i^\lambda}{\partial \lambda} x^i, \text{ on } [s, t_0].
\]

Therefore, by taking the difference $(2) - (1)$ on $[s - \epsilon, [s]]$ and integrating afterwards, we obtain

\[
\int_{[s]} [(x^i - x^*)] p_0^\alpha n_\lambda d\tau d\sigma
\]

\[= \int_{[s - \epsilon, [s]]} \{ H(\tau, t, x^*, p_0^\alpha, p, u^*) - H(\tau, t, x^*, p_0^\alpha, p, u^*)
\]

\[-p_0 X(\tau, t, x^*, u^*) - X(\tau, t, x^*, u^*) \} n_\mu d\tau d\sigma.
\]

Computing the derivative with respect to $\epsilon_\mu$ (see Lemma 4.1), we obtain

\[
(4) \quad \int_{[s]} y_\mu^i p_0^\alpha n_\lambda d\tau d\sigma = \int_{[s]} \{ H(\tau, t, x^*, p_0^\alpha, p, u^*)
\]

\[-H(\tau, t, x^*, p_0^\alpha, p, u^*)
\]

\[-p_0 X(\tau, t, x^*, u^*) - X(\tau, t, x^*, u^*) \} n_\mu d\tau d\sigma.
\]

Similar arguments concerning $(H_0)$ give

\[
(5) \quad \int_{(s)} y_\mu^i q_1^\lambda n_\lambda d\sigma = \int_{(s)} \{ H_0(t, x^*, q, u^*)
\]

\[-H_0(t, x^*, q, u^*) - g(t, x^*, u^*) \} n_\mu d\sigma.
\]

We chose the costate vector $p^* \in \mathbb{R}^n$ as solution for the adjoint PDE (MixADJ) with boundary conditions $(\tau_0)$. On the multi-interval $[s, [t_0]]$, we have

\[
\frac{\partial (p_0^\alpha y_\mu^i)}{\partial \tau} = -p_0 \frac{\partial X}{\partial x^i}(\tau, t, x^*, u^*) y_\mu^i \text{ on } [s, [t_0]].
\]

Then,

\[
\int_{[s], [t_0]} p_0 \frac{\partial X}{\partial x^i}(\tau, t, x^*, u^*) y_\mu^i d\tau dt
\]

\[= - \int_{[s], [t_0]} \frac{\partial (p_0^\alpha y_\mu^i)}{\partial t} d\tau dt.
\]
= -\int_{[s],(t_0)} y_{\mu}'(\tau_0, t)p^1_{\mu} (\tau_0, t)dt + \int_{[s]} y_{\mu}' p^1_{\mu} n_{\lambda} d\tau d\sigma

and, using relation (4), we obtain

(6) \quad 0 = \int_{[s]} [H(\tau, t, x^*, p_0, p, u) - H(\tau, t, x^*, p_0, p, u^*)]\{n_{\mu}d\tau d\sigma

- \int_{[s]} p_0 [X(\tau, t, x^*, u) - X(\tau, t, x^*, u^*)]\{n_{\mu}d\tau d\sigma

- \int_{[s]} p_0 \frac{\partial X}{\partial x^3}(\tau, t, x^*, u^*)y_{\mu}' d\tau dt

- \int_{[s],(t_0)} \frac{\partial H_0}{\partial x^3}(t, x^*(\tau_0, t), q^*(t), u^*(\tau_0, t))

+ \frac{\partial q_{\lambda}}{\partial t^\lambda}(t) y_{\mu}'(\tau_0, t)dt

We repeat the arguments from the last part for \(H_0\). The result is

\frac{\partial (q^*_{\lambda} y_{\mu}^i)}{\partial t^\lambda} = \left[ \frac{\partial H_0}{\partial x^3}(t, x^*, q^*, u^*)

- \frac{\partial q}{\partial x^3}(t, x^*, u^*) \frac{\partial q^*_{\lambda}}{\partial t^\lambda}(t) \right] y_{\mu}'

and it follows

\frac{\partial (q^*_{\lambda} y_{\mu}^i)}{\partial t^\lambda} = - \frac{\partial q}{\partial x^3}(t, x^*, u^*) y_{\mu}' on ((s), (t_0)].

Then,

\int_{[s],(t_0)} \frac{\partial q}{\partial x^3}(t, x^*, u^*) y_{\mu}' dt

= - \int_{[s],(t_0)} \frac{\partial (q^*_{\lambda} y_{\mu}^i)}{\partial t^\lambda} dt

= \int_{(s)} y_{\mu}' p^1_{\mu} n_{\lambda} d\tau d\sigma,

consequently

(7) \quad 0 = - \int_{[s],(t_0)} \frac{\partial q}{\partial x^3}(t, x^*, u^*) y_{\mu}' dt

+ \int_{(s)} \{H_0(t, x^*, q, u) - H_0(t, x^*, q, u^*) - g(t, x^*, u) + g(t, x^*, u^*)\} n_{\mu} d\tau d\sigma

Since \(u^*\) is an optimal control, it follows that \(\epsilon = 0\) is a maximum point for the function

\epsilon \rightarrow \int_{[[0],[\sigma-\epsilon]]} X(\tau, t, x^*, u^*)d\tau dt

+ \int_{[[\sigma-\epsilon],[\sigma]]} X(\tau, t, x, u)d\tau dt

+ \int_{[[\sigma],[\tau_0]]} g(t, x, u)d\tau dt + \int_{[[s-\epsilon),(s)]]} g(t, x, u^*)dt

and

(8)

- \int_{[s]} X(\tau, t, x^*, u^*)n_{\mu}d\tau d\sigma + \int_{[s]} X(\tau, t, x^*, u)\{n_{\mu}d\tau d\sigma

+ \int_{[s]} \frac{\partial X}{\partial x^3}(t, x^*, u^*)y_{\mu}' d\tau dt - \int_{(s)} g(t, x^*, u^*)n_{\mu} d\sigma

+ \int_{(s)} g(t, x^*, u) n_{\mu} d\sigma + \int_{[(s),(t_0)]} \frac{\partial q}{\partial x^3}(t, x^*, u^*) y_{\mu}' dt \leq 0.

Combining (6) and (7) and taking \(p_0 = 1\), we conclude that

0 \geq \int_{[s]} [H(\tau, t, x^*, p_0^*, p^*, u)

- H(\tau, t, x^*, p_0^*, p^*, u^*)]d\tau d\sigma

+ \int_{[s]} [H_0(t, x^*, q, u) - H_0(t, x^*, q^*, u^*)]d\sigma.

We have obtained the maximum principle inequality in functional integral form. By writing the corresponding mixed Euler-Lagrange PDEs (see Section 2), we obtain

\frac{\partial H}{\partial u}(\tau, t, x^*(\tau, t), p_0^*(\tau, t), p^*(\tau, t), u^*(\tau, t)) = 0,

\forall (\tau, t) \in [s], \forall s \in \Omega_{0t};

\frac{\partial H_0}{\partial u}(\tau_0, t, x^*(\tau_0, t), q^*(t), u^*(\tau_0, t)) = 0,

\forall (\tau_0, t) \in (s), \forall s \in \Omega_{0t}.

\Box

Remark 4.4 When proving the multitime maximum principle corresponding to an optimal control problem, we need to consider a suitable multi-time order and proper multi-time intervals on which to define the needle-shaped variations. Nevertheless, we can eliminate this difficulty by taking interior controls and by substituting the needle-shaped variations with control variations of type \(u_\epsilon(t) = u^*(t) + \epsilon \nu(t)\). Then, a simplified multitime maximum principle can be proved, using techniques specific to calculus of variations (see [14], [15], [17], [26], [27]).
5 Example

Let us consider a smooth piece-wise curve $\gamma = \gamma(t) : [0, 1] \rightarrow R^3$. We take the set of all the geodesic deformations $x : [0, 1] \times [0, 1] \rightarrow R^3$ of $\gamma$, fixing the endpoints and we try to find a deformation as above such that $t \rightarrow x(1, t)$ has minimal energy. Therefore, we should consider the following optimal control problem:

$$\min \int_0^1 \delta_{ij} u^j(t) u^j(t) dt;$$

subject to

$$J(x(0, t) = \gamma(t); x(\tau, 0) = \gamma(0); x(\tau, 1) = \gamma(1),$$

subject to

$$\left(\begin{array}{c}
\frac{\partial x^i}{\partial \tau} (\tau, t) = u^i_0(t); \\
\frac{\partial x^i}{\partial t} (\tau, t) = u^i(\tau, t).
\end{array}\right)$$

Using the corresponding mixed Hamiltonian

$$(H, H_0) = (p^0_i u^i_0 + p^i u^i, \delta_{ij} u^j u^j + q_i u^i)$$

and applying the mixed multitime maximum principle, we obtain

$$(M) \quad 0 = \frac{\partial H}{\partial u^0_i} \Rightarrow p^0_i = 0; \quad 0 = \frac{\partial H}{\partial u^i} \Rightarrow p_i = 0;$$

$$0 = \frac{\partial H_0}{\partial u^i} \Rightarrow q_i = -2\delta_{ij} u^j.$$

On the other hand, the terminal conditions are

$$0 = p^0_i(1, t) = \frac{\partial H_0}{\partial x^i} + \frac{dq_i}{dt},$$

therefore $\frac{\partial u^i}{\partial \tau}(1, t) = 0$ and $x(1, t) = (1 - t)\gamma(0) + t \gamma(1)$. Also, by using the first set of equation from (PDE), together with the initial condition, we obtain

$$x(\tau, t) = \tau u_0(t) + \gamma(t).$$

Therefore, $u_0(t) = (1 - t)\gamma(0) + t \gamma(1) - \gamma(t)$ and the optimal geodesic deformation keeping the endpoints fixed is

$$x(\tau, t) = (1 - \tau)\gamma(t) + \tau[(1 - t)\gamma(0) + t \gamma(1)].$$

References:


