Linear instability of curved shallow mixing layers

I. EGLITE
Department of Engineering Mathematics
Riga Technical University
1 Meza street, Riga
LATVIA
irina.eglite@gmail.com
A. A. KOLYSHKIN
Department of Engineering Mathematics
Riga Technical University
1 Meza street, Riga
LATVIA
akoliskins@rbs.lv

Abstract: - Linear stability of slightly curved shallow mixing layers is analyzed in the present paper. Two base flow profiles are used in the analysis. One of the profiles corresponds to stably curved mixing layer while the second profile represents unstably curved mixing layer. Linear stability problem is solved numerically by means of a collocation method based on Chebyshev polynomials. Results of numerical computations show that the curvature has a stabilizing effect on stably curved mixing layer. The critical values of the bed-friction number increase as the curvature increases for the case of unstably curved mixing layer.

Key-Words: - shallow mixing layer, curvature, linear stability

1 Introduction
Curved shallow mixing layers are widespread in nature and engineering. Such flows occur in compound and composite channels and at river junctions. From a practical point of view it is important to understand mass and momentum exchange between fast and slow fluid streams in a shallow mixing layer. This is a complicated problem which can be solved by numerical integration of nonlinear shallow water equations. Experimental investigations also can shed some light on the development of mixing layer. Shallow mixing layers are studied experimentally in [1]-[3]. Experimental results in [1]-[3] showed that bottom friction reduces the growth of the mixing layer width.

Methods of linear stability theory are often used in order to predict the onset of instability in a mixing layer. Linear stability analysis of shallow mixing layers is performed in several papers [4]-[8]. It is shown in [4]-[8] that shear instability in shallow mixing layers is governed by the bed-friction number defined as the ratio of the bed friction force to shear force at the shear layer.

Another type of instability that can occur in mixing layers is the centrifugal instability [9]-[10]. Linear stability of slightly curved mixing layers in deep water is analyzed in [9]. It is shown in [9] that in a stably curved mixing layer the curvature reduces the growth rate of the shear mode. On the other hand, shear modes become more unstable in an unstably curved mixing layer.

In the present paper the stability of a slightly curved shallow mixing layer is analyzed. It is assumed that the radius of curvature is much larger than the thickness of a curved mixing layer. Rigid-lid assumption is used in the analysis. It is shown in [6] that the rigid-lid assumption works well (in terms of the stability characteristics of the flow) for small Froude numbers. Small Froude number assumption is also adopted in the present study. Linear stability problem is solved numerically by means of a collocation method based on Chebyshev polynomials. Results of numerical computations are presented.

2 Mathematical formulation of the problem
Shallow water equations under the rigid-lid assumption in the presence of a small curvature have the form
\begin{align}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} - \frac{c_f}{2h} u \sqrt{u^2 + v^2}, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} - \frac{c_f}{2h} v \sqrt{u^2 + v^2},
\end{align}

where \( u \) and \( v \) are the depth-averaged velocity components in the \( x \) and \( y \) directions, \( x \) and \( y \) are the directions along the streamline and perpendicular to the streamline, respectively, \( p \) is the pressure, \( h \) is water depth, \( c_f \) is the friction coefficient and \( \frac{1}{R} = \frac{\delta}{R} \ll 1 \) is a small parameter.

Here \( R \) is the radius of curvature of the centerline of the curved mixing layer and \( \delta \) is the thickness of the mixing layer.

Introducing the stream function by the relations

\begin{align}
u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x},
\end{align}

and eliminating the pressure we rewrite (1)-(3) in the form

\begin{align}
(\Delta \psi)_x + \psi_y (\Delta \psi)_x - \psi_x (\Delta \psi)_y + \frac{2}{R} \psi_x \psi_{xy} + \frac{c_f}{2h} \Delta \psi \sqrt{\psi_x^2 + \psi_y^2} + \frac{c_f}{2h} \psi \sqrt{\psi_x^2 + \psi_y^2} \psi_y^2 \psi_{yy} + 2 \psi_x \psi_y \psi_{xy} + 2 \psi_x^2 \psi_{xx} = 0.
\end{align}

The parallel flow assumption is used in the present study. As pointed out in [9] this approximation is the leading-order solution in a multiple-scale expansion which takes into account slow flow divergence.

Following [9] we consider two base flow profiles of the form

\begin{align}
U(y) &= \frac{1}{2} (1 + \tanh y), \\
\text{and} \\
U(y) &= \frac{1}{2} (1 - \tanh y).
\end{align}

Velocity profile (6) corresponds to stably curved mixing layer (in this case the high-speed stream is on the outside of the low-speed stream). Profile (7) represents the opposite situation (the high-speed stream is on the inside of the low-speed stream). It is shown in [11] that experimentally observed base flow velocity profile has similar shape to that of the plane mixing layer. As a result base flow profiles (6) and (7) are adopted in the present study.

Consider a perturbed flow of the form

\begin{align}
u = U + u', \quad v = v', \quad p = P + p',
\end{align}

where the quantities with primes represent small perturbations. Substituting (8) into (4)-(5), dropping the primes and linearizing the resulting equation in the neighborhood of the base flow we obtain the following equation

\begin{align}
\psi_{xx} + \psi_{yy} + \psi_{0y} (\psi_{xx} + \psi_{yy}) - \psi_{0yy} \psi_x + \frac{c_f}{2h} (\psi_{0y} \psi_{xx} + 2 \psi_{0y} \psi_y + 2 \psi_{0y} \psi_{yy}) + \frac{2}{R} \psi_{0y} \psi_{xy} = 0,
\end{align}

where \( \psi_{0y} = U \).

Following the method of normal modes we assume a perturbation of the form

\begin{align}
\psi(x, y, t) = \phi(y) \exp[i(k(x - ct))],
\end{align}

where \( \phi(y) \) is the amplitude of the normal perturbation, \( k \) is the wave number and \( c \) is the phase speed of the perturbation. Substituting (10) into (9) we obtain

\begin{align}
\phi''[ik(U - c) + SU] + \phi'' \left( \frac{2}{R} ik U + SU_y \right) + \phi(ik^3 c - ik^3 U - ik U_y - k^2 SU / 2) = 0,
\end{align}

where \( S = \frac{c_f \delta}{h} \) is the bed-friction number and \( \delta \) is the width of the mixing layer.

The boundary conditions are

\begin{align}
\phi(\pm \infty) = 0.
\end{align}

The eigenvalues, \( c = c_r + ic_i \), determine the linear stability of base flow (6) or (7). The base flow is said to be linearly stable if all \( c_i < 0 \), and unstable, if at least one \( c_i > 0 \).
3 Numerical method

Collocation method based on Chebyshev polynomials is used to solve eigenvalue problem (11), (12). The transformation

$$r = \frac{2}{\pi} \arctan y$$

(13)

is applied to map the interval \((-\infty, +\infty)\) into the interval \((-1, 1)\). The solution to (11), (12) in terms of the new variable \(r\) is sought in the form

$$\varphi(r) = \sum_{m=0}^{N} a_m (1-r^2) T_m(r),$$

(14)

where \(T_m(r) = \cos m \arccos r\) are the Chebyshev polynomials of the first kind of order \(m\) and \(a_m\) are unknown coefficients. The factor \(1-r^2\) is added in (14) in order to satisfy zero boundary conditions at \(r = \pm 1\). The collocation points are

$$r_j = \cos \frac{\pi kj}{N}, \quad j = 0, 1, ..., N.$$  \hspace{2cm} (15)

Substituting (13) and (14) into (11) and discretizing the resulting equation at collocation points (15) we obtain the generalized eigenvalue problem of the form

$$(A - cB) a = 0,$$  \hspace{2cm} (16)

where \(A\) and \(B\) are complex-valued matrices and \(a = (a_0, a_1, ..., a_N)^T\). Problem (16) is solved by means of IMSL routine DVGLCG.

4 Numerical results

The results of numerical computations for the case of stably curved shallow mixing layer (base flow velocity profile (6)) are shown in Fig. 1. Three marginal stability curves are shown in Fig. 1 for the three values of the parameter \(1/R\), namely, \(1/R = 0, 0.025\) and 0.05, respectively (from top to bottom). The region of instability is below the curves.

As can be seen from Fig. 1 curvature has a stabilizing influence on stably curved shallow mixing layer: the critical values of the bed-friction number \(S\) decrease as the parameter \(1/R\) increases.

Marginal stability curves for unstably curved shallow mixing layer (base flow profile (7)) are shown in Fig. 2.

The results shown in Fig. 2 indicate that the increase of the parameter \(1/R\) has a destabilizing influence on unstably curved base flow profile (7): the critical values of the bed-friction number increase for larger \(1/R\).

5 Conclusion

Linear stability of slightly curved shallow mixing layers is investigated in the present paper. Stability analysis is performed for two base flow profiles one of which is stably curved while the other is unstably curved. Linear stability problem is solved numerically by means of a collocation method.
Results of numerical computations show that the curvature stabilizes the flow in the case of stably curved mixing layer while for unstably curved mixing layer the curvature has a destabilizing effect on the flow.

6 Acknowledgement

The work has been supported by the European Social Fund within the project “Support for the implementation of doctoral studies at Riga Technical University”.

References: