On Possibilistic Correlation Coefficient and Ratio for Fuzzy Numbers

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Abstract: In this paper we show some properties of possibilistic correlation coefficient and correlation ratio for marginal possibility distributions. We will show some examples for their computation from some joint possibility distributions.

Key–Words: Fuzzy number, Possibility distribution, Correlation coefficient, Correlation ratio

1 Introduction

In probability theory the expected value of functions of random variables plays a fundamental role in defining the basic characteristic measures of probability distributions. For example, the variance, covariance and correlation of random variables can be computed as the expected value of their appropriately chosen real-valued functions. In possibility theory we can use the principle of expected value of functions on fuzzy sets to define variance, covariance and correlation of possibility distributions. Marginal possibility distributions are determined from the joint one by the principle of ‘falling integrals’ and marginal possibility distributions are determined from the joint possibility distribution by the principle of ‘falling shadows’. Probability distributions can be interpreted as carriers of incomplete information [12], and possibility distributions can be interpreted as carriers of imprecise information. In 1987 Dubois and Prade [3] defined an interval-valued expectation of fuzzy numbers, viewing them as consonant random sets. They also showed that this expectation remains additive in the sense of addition of fuzzy numbers. In possibility theory we can use the principle of average value of appropriately chosen real-valued functions to define mean value, variance, covariance and correlation of possibility distributions. Namely, we can equip each level set of a possibility distribution (represented by a fuzzy number) with a uniform probability distribution, then apply their standard probabilistic calculation, and then define measures on possibility distributions by integrating these weighted probabilistic notions over the set of all membership grades [1, 4]. These weights (or importances) can be given by weighting functions.

Definition 1 A function \( g : [0, 1] \to \mathbb{R} \) is said to be a weighting function if \( g \) is non-negative, monotone increasing and satisfies the following normalization condition \( \int_0^1 g(\gamma) \, d\gamma = 1 \).

Different weighting functions can give different (case-dependent) importances to level-sets of possibility distributions. We should note here that the choice of uniform probability distribution on the level sets of possibility distributions is not without reason. We suppose that each point of a given level set is equally possible and then we apply Laplace’s principle of Insufficient Reason: if elementary events are equally possible, they should be equally probable (for more details and generalization of principle of Insufficient Reason see [2], page 59). In this paper we will discuss the concepts of possibilistic correlation coefficient and ratio for marginal possibility distributions of joint possibility distributions.

Definition 2 A fuzzy number \( A \) is a fuzzy set \( \mathbb{R} \) with a normal, fuzzy convex and continuous membership function of bounded support.

The family of fuzzy numbers is denoted by \( \mathcal{F} \). Fuzzy numbers can be considered as possibility distributions. A fuzzy set \( C \) in \( \mathbb{R}^2 \) is said to be a joint possibility distribution of fuzzy numbers \( A, B \in \mathcal{F} \), if it satisfies the relationships

\[
\max\{x \mid C(x, y)\} = B(y),
\]

and

\[
\max\{y \mid C(x, y)\} = A(x),
\]

for all \( x, y \in \mathbb{R} \). Furthermore, \( A \) and \( B \) are called the marginal possibility distributions of \( C \). A \( \gamma \)-level
set (or $\gamma$-cut) of a fuzzy number $A$ is a non-fuzzy set denoted by $[A]^\gamma$ and defined by

$$[A]^\gamma = \{ t \in X \mid A(t) \geq \gamma \},$$

if $\gamma > 0$ and cl(supp$A$) if $\gamma = 0$, where cl(supp$A$) denotes the closure of the support of $A$. Let $A \in F$ be fuzzy number with a $\gamma$-level set denoted by $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$ and let $U_\gamma$ denote a uniform probability distribution on $[A]^\gamma$, $\gamma \in [0, 1]$.

2 A Correlation Coefficient for Marginal Possibility Distributions

In 2004 Fullér and Majlender [4] introduced the notion of covariance between marginal distributions of a joint possibility distribution $C$ as the expected value of their interactivity function on $C$. That is, the $g$-weighted measure of interactivity between $A \in F$ and $B \in F$ (with respect to their joint distribution $C$) is defined by their measure of possibilistic covariance [4], as

$$\text{Cov}_g(A, B) = \int_0^1 \text{cov}(X_\gamma, Y_\gamma) g(\gamma) \, d\gamma,$$

where $X_\gamma$ and $Y_\gamma$ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0, 1]$, and $\text{cov}(X_\gamma, Y_\gamma)$ denotes their possibilistic covariance.

It is easy to see that the possibilistic covariance is an absolute measure in the sense that it can take any value from the real line. To have a relative measure of interactivity between marginal distributions Fullér, Mezei and Várlaki introduced the principle of correlation (normalized covariance) in 2011 (see [9]) that improves the earlier definition introduced by Carlsson, Fullér and Majlender in 2005 (see [1]).

**Definition 3** The $g$-weighted possibilistic correlation coefficient of $A, B \in F$ (with respect to their joint distribution $C$) is defined by

$$\rho_g(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) g(\gamma) \, d\gamma,$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)} \sqrt{\text{var}(Y_\gamma)}}$$

and, where $X_\gamma$ and $Y_\gamma$ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0, 1]$, and $\text{cov}(X_\gamma, Y_\gamma)$ denotes their possibilistic covariance.

In other words, the $g$-weighted possibilistic correlation coefficient is nothing else, but the $g$-weighted average of the possibilistic correlation coefficients $\rho(X_\gamma, Y_\gamma)$ for all $\gamma \in [0, 1]$.

In 2011 Harmati [10] proved that the correlation coefficient depends only on the joint possibility distribution, but not directly on its marginal distributions. In other words, exact knowledge of the marginal possibility distributions does not give any restrictions for the possibilistic correlation coefficient and also for the $g$-weighted possibilistic correlation coefficient.

If $A$ and $B$ are non-interactive fuzzy numbers then their joint possibility distribution is defined by $C = A \times B$. Since all $[C]^\gamma$ are rectangular and the probability distribution on $[C]^\gamma$ is defined to be uniform we get $\text{cov}(X_\gamma, Y_\gamma) = 0$, for all $\gamma \in [0, 1]$. So $\text{Cov}_g(A, B) = 0$ and $\rho_g(A, B) = 0$ for any weighting function $g$. That is, non-interactivity entails zero correlation.

Fuzzy numbers $A$ and $B$ are said to be in perfect correlation, if there exist $q, r \in \mathbb{R}$, $q \neq 0$ such that their joint possibility distribution is defined by [1]

$$C(x_1, x_2) = A(x_1) \cdot \chi_{\{qx_1+r=x_2\}}(x_1, x_2),$$

where $\chi_{\{qx_1+r=x_2\}}$, stands for the characteristic function of the line

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid qx_1 + r = x_2\}.$$

If $A$ and $B$ have a perfect positive (negative) correlation then from $\rho(X_\gamma, Y_\gamma) = 1$ ($\rho(X_\gamma, Y_\gamma) = -1$) (see [1] for details), for all $\gamma \in [0, 1]$, we get $\rho_g(A, B) = 1$ ($\rho_g(A, B) = -1$) for any weighting function $g$.

We emphasize here that zero correlation does not always imply non-interactivity. Let $A, B \in F$ be fuzzy numbers, let $C$ be their joint possibility distribution, and let $\gamma \in [0, 1]$. Suppose that $[C]^\gamma$ is symmetrical, i.e. there exists $a \in \mathbb{R}$ such that

$$C(x, y) = C(2a - x, y),$$

for all $x, y \in [C]^\gamma$ (the line defined by $\{(a, t) \mid t \in \mathbb{R}\}$ is the axis of symmetry of $[C]^\gamma$). In this case $\text{cov}(X_\gamma, Y_\gamma) = 0$ and $\rho_g(A, B) = 0$ for any weighting function $g$, (see [5]).

We should note here that there exist several other ways to define correlation coefficient for fuzzy numbers, e.g. Liu and Kao [15] used fuzzy measures to define a fuzzy correlation coefficient of fuzzy numbers and they formulated a pair of nonlinear programs to find the $\alpha$-cut of this fuzzy correlation coefficient, then, in a special case, Hong [11] showed an exact calculation formula for this fuzzy correlation coefficient.
3 A Correlation Ratio for Marginal Possibility Distributions

In statistics, the correlation ratio is a measure of the relationship between the statistical dispersion within individual categories and the dispersion across the whole population or sample. The correlation ratio was originally introduced by Karl Pearson [18] as part of analysis of variance and it was extended to random variables by Andrei Nikolaevich Kolmogorov [13] as,

$$\eta^2(X|Y) = \frac{D^2[E(X|Y)]}{D^2(X)},$$

where \( X \) and \( Y \) are random variables. If \( X \) and \( Y \) have a joint probability density function, denoted by \( f(x, y) \), then we can compute \( \eta^2(X|Y) \) using the following formulas

$$E(X|Y) = \int_{-\infty}^{\infty} xf(x|y) dx$$

and

$$D^2[E(X|Y)] = E(E(X|y) - E(X))^2,$$

and where,

$$f(x|y) = \frac{f(x,y)}{f(y)}.$$ It measures a functional dependence between random variables \( X \) and \( Y \). It takes on values between 0 (no functional dependence) and 1 (purely deterministic dependence). In probability theory the joint distribution function is always connected with its marginals by a copula (Sklar theorem [19]), but it is not always true for joint possibility distributions [6]. In 2010 Fullér, Mezei and Várlikó introduced the definition of possibilistic correlation coefficient for marginal possibility distributions (see [6]).

**Definition 4** Let us denote \( A \) and \( B \) the marginal possibility distributions of a given joint possibility distribution \( C \). Then the \( g \)-weighted possibilistic correlation ratio of marginal possibility distribution \( A \) with respect to marginal possibility distribution \( B \) is defined by

$$\eta^2_g(A|B) = \int_0^1 \eta^2(X_\gamma|Y_\gamma) g(\gamma) d\gamma$$

where \( X_\gamma \) and \( Y_\gamma \) are random variables whose joint distribution is uniform on \([C]_\gamma\) for all \( \gamma \in [0,1] \), and \( \eta^2(X_\gamma|Y_\gamma) \) denotes their probabilistic correlation ratio.

So the \( g \)-weighted possibilistic correlation ratio of the fuzzy number \( A \) on \( B \) is nothing else, but the \( g \)-weighted average of the probabilistic correlation ratios \( \eta^2(X_\gamma|Y_\gamma) \) for all \( \gamma \in [0,1] \).

4 Examples

In this Section we will compute the correlation coefficient and the correlation ratio for \( A \) and \( B \) when their joint possibility distribution, \( C \), is defined by \( C(x, y) = T(A(x), B(y)) \), where \( T \) is a t-norm [20]. We will consider the Mamdani t-norm [17], Łukasiewicz t-norm [16] and Larsen t-norm [14]. For simplicity we will consider fuzzy numbers of same shape.

\[
A(x) = \begin{cases} 
  x & \text{if } 0 \leq x \leq 1 \\
  0 & \text{otherwise}
\end{cases},
\]

\[
B(y) = \begin{cases} 
  y & \text{if } 0 \leq y \leq 1 \\
  0 & \text{otherwise}
\end{cases}.
\]

Recall that \( X \) and \( Y \) are random variables then their correlation coefficient is computed from,

$$\rho(X,Y) = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - E^2(X)\sqrt{E(Y^2) - E^2(Y)}}}$$

and their correlation ratio is defined by,

$$\eta^2(X,Y) = \frac{E(E^2(X|Y)) - E^2(X)}{E^2(X) - E^2(X)}.$$

4.1 Mamdani t-norm

In this case the joint possibility distribution \( C(x, y) \) is defined by the min operator,

\[ C(x, y) = \min\{A(x), B(y)\}. \]

Then a \( \gamma \)-level set of \( C \) is,

\[ [C]_{\gamma} = \{(x, y) \in \mathbb{R}^2 | \gamma \leq x, y \leq 1\}. \]

The joint density function of a uniform distribution on \([C]_{\gamma}\) is

\[
f(x,y) = \begin{cases} 
  \frac{1}{T_\gamma} & \text{if } (x,y) \in [C]_\gamma \\
  0 & \text{otherwise}
\end{cases},
\]

where

\[ T_\gamma = (1-\gamma)^2 \]

denotes the area of the \( \gamma \)-level set. The marginal density functions are obtained from \( f(x,y) \) by

\[
f_1(x) = \int f(x,y) dy
\]

that is,

\[
f_1(x) = \begin{cases} 
  \frac{1}{T_\gamma} (1-\gamma) & \text{if } \gamma \leq x \leq 1 \\
  0 & \text{otherwise}
\end{cases}.
\]
The expected values are,

\[ E(X_\gamma) = E(Y_\gamma) = \int_{\gamma}^{1} x f_1(x) \, dx = \frac{1 + \gamma}{2}. \]

and,

\[ E(X^2_\gamma) = E(Y^2_\gamma) = \int_{\gamma}^{1} x^2 f_1(x) \, dx = \frac{1 + \gamma + \gamma^2}{3}. \]

The expected value of the product is

\[ E(X_\gamma Y_\gamma) = \int_{\gamma}^{1} \int_{\gamma}^{1} x y f(x, y) \, dy \, dx = \frac{(1 + \gamma)^2}{4}. \]

The conditional expected value is

\[ E(X|Y) = \int_{\gamma}^{1} x f(x|y) \, dx = \int_{\gamma}^{1} x \frac{f(x, y)}{f_2(y)} \, dx = \int_{\gamma}^{1} x \frac{1}{1 - \gamma} \, dx = \frac{1 + \gamma}{2}. \]

and,

\[ E(E^2(X|Y)) = \int_{\gamma}^{1} E^2(X|Y) f_2(y) \, dy = \int_{\gamma}^{1} \left( \frac{1 + \gamma}{2} \right)^2 \frac{1}{T_\gamma} (1 - \gamma) \, dy = \frac{(1 + \gamma)^2}{4}. \]

From these results it follows the correlation ratio is also equal to zero. It is not surprising, since in this case \( A \) and \( B \) are non-interactive fuzzy numbers.

### 4.2 Łukasiewicz t-norm

In this example we define the joint possibility distribution by the Łukasiewicz \( t \)-norm, that is,

\[ C(x, y) = \max\{A(x) + B(y) - 1, 0\}. \]

Then a \( \gamma \)-level set of \( C \) is

\[ [C]^\gamma = \{(x, y) \in \mathbb{R}^2 | \gamma \leq x, y \leq 1, x + y \geq 1 + \gamma\}. \]

The expected values are,

\[ E(X_\gamma) = \int_{\gamma}^{1} \frac{1}{T_\gamma} (x - \gamma) \, dx = \int_{\gamma}^{1} \frac{1}{T_\gamma} (\gamma + 1 - \gamma) \, dx = \frac{2 + \gamma}{3}. \]

The joint density function of a uniform distribution on \([C]^\gamma\) is

\[ f(x, y) = \begin{cases} \frac{1}{T_\gamma}, & \text{if } (x, y) \in [C]^\gamma, \\ 0, & \text{otherwise}, \end{cases} \]

where

\[ T_\gamma = \frac{(1 - \gamma)^2}{2} \]

denotes the area of the \( \gamma \)-level set. The marginal density function

\[ f_1(x) = \begin{cases} \frac{1}{T_\gamma} (x - \gamma), & \text{if } \gamma \leq x \leq 1, \\ 0, & \text{otherwise}. \end{cases} \]

The expected value of the product is,

\[ E(X_\gamma Y_\gamma) = \int_{\gamma}^{1} \int_{\gamma}^{1} x y f(x, y) \, dy \, dx = \frac{(\gamma + 5)(\gamma + 1)}{12}. \]

From these we can compute the correlation coefficient:

\[ \rho(X_\gamma, Y_\gamma) = -\frac{1}{2}. \]

The conditional density function is:

\[ f(x|y) = \frac{f(x, y)}{f_2(y)} = \begin{cases} \frac{1}{y - \gamma}, & \text{if } \gamma \leq y \leq 1, \\ 0, & \text{otherwise}. \end{cases} \]

The conditional expected value:

\[ E(X|Y) = \int_{1+\gamma-y}^{1} x \frac{1}{y - \gamma} \, dx = \frac{2 + \gamma - y}{2}. \]

The conditional expected value is

\[ E(E^2(X|Y)) = \int_{\gamma}^{1} \left( \frac{2 + \gamma - y}{2} \right)^2 \frac{1}{T_\gamma} (y - \gamma) \, dy = \frac{3\gamma^2 + 10\gamma + 11}{24}. \]
With this result we can compute the correlation ratio and we get
\[ \eta^2(X_\gamma, Y_\gamma) = \frac{1}{4} \Rightarrow \eta(X_\gamma, Y_\gamma) = \frac{1}{2} \]
In this example \( \eta^2 = \rho^2 \), because of the linear relationship between \( A \) and \( B \).

### 4.3 Larsen t-norm

In this case we define the joint possibility distribution \( C \) by the product \( t \)-norm, \( C(x, y) = A(x)B(y) \). Then a \( \gamma \)-level set of \( C \) is,
\[ [C]_\gamma = \{(x, y) \in \mathbb{R}^2| 0 \leq x, y \leq 1, \ xy \geq \gamma \}. \]
The joint density function of a uniform distribution on \( [C]_\gamma \) is
\[ f(x, y) = \begin{cases} \frac{1}{T_\gamma} & \text{if } (x, y) \in [C]_\gamma, \\ 0 & \text{otherwise}, \end{cases} \]
where \( T_\gamma = 1 - \gamma + \gamma \ln \gamma \) denotes the area of the \( \gamma \)-level set. The marginal density function
\[ f_1(x) = \begin{cases} \frac{1}{T_\gamma} \frac{x - \gamma}{x} & \text{if } \gamma \leq x \leq 1, \\ 0 & \text{otherwise}. \end{cases} \]
The expected values are,
\[ E(X_\gamma) = E(Y_\gamma) = \int^{1}_\gamma x \frac{1}{T_\gamma} \frac{x - \gamma}{x} dx = \frac{1}{2} \frac{(1 - \gamma)^2}{T_\gamma}. \]
\[ E(X^2_\gamma) = E(Y^2_\gamma) = \int^{1}_\gamma x^2 \frac{1}{T_\gamma} \frac{x - \gamma}{x} dx = \frac{1}{6} \frac{1}{T_\gamma} (1 - \gamma)^2 (\gamma + 2). \]
The expected value of the product is,
\[ E(X_\gamma Y_\gamma) = \int^{1}_\gamma \int^{1}_{\gamma/x} \frac{1}{T_\gamma} dy dx = \frac{1}{4} \frac{1}{T_\gamma} (1 - \gamma^2 + 2\gamma^2 \ln \gamma). \]
The conditional expected value is,
\[ E(X|Y) = \frac{1}{2} \frac{y + \gamma}{y}. \]

\[ E(E^2(X|Y)) = \frac{1}{8} \frac{1}{T_\gamma} (\gamma^3 + 2\gamma^2 - 5\gamma + 2 - 2\gamma \ln \gamma). \]

From these expression we could determine the correlation coefficient and the correlation ratio for any \( \gamma \). The final formulas are very long and difficult, so they are omitted here.

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