An Overview of Migrative Triangular Norms

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Abstract: In this paper we summarize our results obtained recently on continuous triangular norms that are migrative with respect to an arbitrary continuous triangular norm. We start with the original notion of migrativity and completely describe all continuous migrative triangular norms. Since the migrative property excludes both idempotent and nilpotent t-norm classes, the characterization and construction is carried out by solving a functional equation for additive generators of strict t-norms. Then we extend the migrative property by allowing an arbitrary but fixed t-norm in the defining equation instead of the originally used product t-norm. Equivalent forms of this extended migrativity are also provided. Two particular cases when the fixed t-norm is either the minimum or the Łukasiewicz t-norm are studied. In these cases all continuous extended migrative t-norms are characterized and represented. Finally, to reach our main goal, we exploit the ordinal sum structure of continuous t-norms and our former results related to the migrative property. We illustrate the statements by numerical examples and figures.

Key Words: continuous triangular norm, migrative property, functional equations, extended migrative property, minimum t-norm, product t-norm, Łukasiewicz t-norm, ordinal sum.

1 Introduction

Aggregation of information represented by by fuzzy sets (i.e., by membership functions) has been a central matter in intelligent systems where fuzzy rule base and reasoning mechanism are applied since the late 1970s. In most cases, the aggregation operators are defined on a pure axiomatic basis and are interpreted either as logical connectives originated from many-valued logics, with typical examples of triangular norms and conorms [10], or as averaging operators allowing a compensation effect (such as the weighted arithmetic mean, which play a key role in probability and other classical areas).

The research of triangular norms and conorms (t-norms and t-conorms for short) has been expanded since the recognition of their fundamental role in fuzzy set theory. Several important particular subclasses have been identified and fully characterized. Special characteristic properties of such subclasses are typically expressed as functional equations, in addition to the fundamental associativity equation. We refer just to one well-known example, the case of Frank t-norms and t-conorms [9].

Migrativity is a recently introduced and studied property of binary operations defined on the unit interval. The following definition was given in [3].

Definition 1 ([3]). Let α be in ]0, 1[. A binary operation \( T: [0, 1]^2 \rightarrow [0, 1] \) is said to be \( \alpha \)-migrative if we have
\[
T(\alpha x, y) = T(x, \alpha y) \quad \text{for all } x, y \in [0, 1].
\]

Basic results on the migrative property can be found in [5, 7] for t-norms, in [2, 6] for aggregation functions, and in [4, 11] for semicopulas, quasicopulas, and copulas.

The original definition is very much related to the product t-norm. This is extended in [7] by introducing the general notion of the migrative property with respect to any t-norm as follows.

Definition 2 ([7]). Let \( \alpha \) be in \]0, 1[ and \( T_0 \) a fixed triangular norm. A binary operation \( T: [0, 1]^2 \rightarrow [0, 1] \) is said to be \( \alpha \)-migrative with respect to \( T_0 \) (shortly: \( (\alpha, T_0) \)-migrative) if we have
\[
T(T_0(\alpha, x), y) = T(x, T_0(\alpha, y))
\]
for all \( x, y \in [0, 1] \).

Obviously, \( T_0 \) itself is always \( (\alpha, T_0) \)-migrative for any \( \alpha \in [0, 1] \). One obtains Definition 1 when \( T_0(x, y) = T_{\text{p}}(x, y) = xy \) for all \( x, y \in [0, 1] \).

In our papers [5, 7] we studied continuous t-norms that are migrative with respect to one of the
three prototypes \( T_M, T_L \) and \( T_P \). Here \( T_M(x, y) = \min(x, y) \) and \( T_L(x, y) = \max(x + y - 1, 0) \). Any continuous t-norm that is migrative with respect to the Łukasiewicz t-norm \( T_L \) must be nilpotent (i.e., isomorphic to \( T_L \)). Similarly, any continuous t-norm that is migrative with respect to the product t-norm \( T_P \) must be strict (i.e., isomorphic to \( T_P \)). That is, if \( T_0 \) is continuous and Archimedean then any continuous migrative candidate is necessarily Archimedean, from the same subclass as \( T_0 \) belongs to. Simply there is no room for non-Archimedean continuous migrative t-norms in these cases. However, when \( T_0 \) is equal to the minimum \( T_M \), then a continuous \((\alpha, T_M)\)-migrative t-norm is an ordinal sum of two continuous t-norms joining at \( \alpha \) (see the next section and Theorem 6 below for explanation and precise formulation).

In [8] we go beyond the above framework: we study continuous t-norms that are migrative with respect to an arbitrary continuous t-norm \( T_0 \). Obviously, this includes \( T_0 \in \{ T_M, T_L, T_P \} \) as well as any continuous non-Archimedean continuous t-norm (i.e., a continuous ordinal sum) as particular cases. Depending on whether \( \alpha \) is an idempotent element of \( T_0 \) or not, the \((\alpha, T_0)\)-migrative property restricts the ordinal sum structure of \( T \) especially "locally", i.e., at \( \alpha \) or around it. Outside this well-defined neighbourhood of \( \alpha \), the t-norm \( T \) can be arbitrary, under the only condition of keeping it continuous.

The paper is organized as follows. After recalling some preliminaries on ordinal sums and continuous t-norms, in Section 3 we summarize the main results on the original migrative property in Definition 1. In Section 4 we study the extended notion of \((\alpha, T_0)\)-migrativity given in Definition 2. We give full characterization of continuous t-norms that are migrative with respect to \( T_M \) or \( T_L \), the two prototype t-norms beyond the product. In Section 5 we characterize all continuous t-norms that are migrative with respect to an arbitrary continuous t-norm. This requires a careful and constructive merge of our previously summarized results. We describe how to find possible functional forms of these t-norms. We provide also examples, both in functional and in graphical forms. We conclude the paper with some remarks.

## 2 Preliminaries

In this section we briefly recall some facts that are necessary for the understanding of Section 5.

After Alsina et al. [1], we employ the following notations: \( \mathcal{T}_0 \) denotes the set of all continuous t-norms, while \( \mathcal{T}_M \) is the set of all continuous Archimedean t-norms. Recall that a continuous t-norm \( T \) is Archimedean if and only if it satisfies \( T(x, x) < x \) for all \( x \in [0, 1] \). In other words: \( T \in \mathcal{T}_M \) if and only if \( T \in \mathcal{T}_0 \) and \( T \) has only trivial idempotent elements.

Let \( \{T_i\}_{i \in \Gamma} \) be a family of t-norms, and \( \{ [a_i, b_i]\}_{i \in \Gamma} \) be a family of open, non-overlapping, proper subintervals of \([0, 1]\). We introduce the following function \( T : [0, 1]^2 \to [0, 1] \) defined by

\[
T(x, y) = a_i + (b_i - a_i) \cdot T_i \left( \frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i} \right)
\]

if \((x, y) \in [a_i, b_i]^2 \)

otherwise.

Thus defined \( T \) is called the ordinal sum of the summands \((a_i, b_i, T_i)\) \((i \in \Gamma)\). In this case we write

\[
T = \langle \langle a_i, b_i, T_i \rangle \rangle_{i \in \Gamma}.
\]

The cardinality of the index set \( \Gamma \) is either finite or countably infinite. In the extreme case when \( \Gamma = \emptyset \), i.e., when there is no summand, one obtains the minimum \( T_M \). Notice also the particular case of \( \Gamma = \{1\} \) (only one summand) with \( a_1 = 0 \) and \( b_1 = 1 \). Then \( T = \langle (0, 1, T) \rangle \) is a trivial ordinal sum representation of \( T \) itself. Generally speaking, ordinal sum representation of a t-norm is not unique. A t-norm \( T \) is called ordinally irreducible if it has only such a trivial ordinal sum representation.

In the next theorem we collected the most fundamental results on ordinal sums and continuous t-norms that we will exploit in the sequel. The proofs and more details can be found in [1], see Theorem 2.4.2 and Corollary 2.4.5 there.

**Theorem 1** ([1]). Let \( T = \langle \langle a_i, b_i, T_i \rangle \rangle_{i \in \Gamma} \) be an ordinal sum. Then the following statements hold.

(a) \( T \) is a t-norm.

(b) \( T \) is continuous if and only if each \( T_i \) is continuous.

(c) If each \( T_i \) is continuous and Archimedean then \( T(x, x) = x \) holds if and only if \( x \in [0, 1] \setminus \bigcup_{i \in \Gamma} [a_i, b_i] \).

1. Each continuous t-norm \( T \) is isomorphic to an ordinal sum of t-norms, each of which is either \( T_L \) or \( T_P \).

\[\square\]
3 \( \alpha \)-migrative triangular norms

It is obvious from Definition 1 that the product t-norm \( T_P \) is \( \alpha \)-migrative for any \( \alpha \in [0,1] \). In the paper [5] we dealt exclusively with the case when \( T \) was a triangular norm. The main results of [5], where all continuous t-norm solutions of Eq. (1) have been determined, can be summarized in the next theorem. More details and proofs can also be found in [5]. Concerning t-norms, we use the standard notation of [10].

**Theorem 2** ([5]). Let \( \alpha \) be in \( [0,1] \). Suppose \( T \) is a t-norm. Then the following statements hold.

(a) \( T \) is \( \alpha \)-migrative if and only if we have

\[
T(\alpha, y) = \alpha y \quad \text{for all } y \in [0,1].
\]

(b) If \( T \) is continuous and \( \alpha \)-migrative then \( T \) is strict.

(c) Let \( T \) be a strict t-norm with additive generator \( t \). Then \( T \) is \( \alpha \)-migrative if and only if there exists a continuous, strictly decreasing function \( t_0 \) from \( [\alpha, 1] \) to the non-negative reals with \( t_0(\alpha) < +\infty \) and \( t_0(1) = 0 \) such that

\[
t(x) = k \cdot t_0(\alpha) + t_0 \left( \frac{x}{\alpha^k} \right) \quad \text{if } x \in [\alpha^{k+1}, \alpha^k],
\]

where \( k \) is any non-negative integer.

The message of this theorem can be summarized less formally as follows. Since any continuous \( \alpha \)-migrative t-norm \( T \) is necessarily strict, \( T \) has an additive generator function \( t \). Such a \( t \) can be constructed successively by starting from a function \( t_0 \) which is the restriction of any additive generator to the interval \([\alpha,1]\). The migrative property extends \( t_0 \) to the interval \([\alpha^2, \alpha]\) in a unique way, then to the interval \([\alpha^3, \alpha^2]\), then to \([\alpha^4, \alpha^3]\), and so on. We may say that the “last piece” of the graph of \( t \) migrates from the right to the left on the intervals determined by consecutive powers of \( \alpha \).

4 \((\alpha, T_0)\)-migrative triangular norms

In this section we deal with the case when \( T \) in Definition 2 is a t-norm. In the next theorem we give simple but very useful characterizations of \((\alpha, T_0)\)-migrative t-norms.

**Theorem 3.** Let \( \alpha \) be in \([0,1]\) and \( T_0 \) a fixed triangular norm. Then the following statements are equivalent for a t-norm \( T: [0,1]^2 \rightarrow [0,1] \).

(i) \( T \) is \( \alpha \)-migrative with respect to \( T_0 \);

(ii) \( T \) satisfies the following equation for all \( x, y \in [0,1] \):

\[
T_0(T(\alpha, x), y) = T_0(x, T(\alpha, y)).
\]

(iii) \( T \) satisfies the following equation for all \( x \in [0,1] \):

\[
T(\alpha, x) = T_0(\alpha, x).
\]

In the next two subsections we study two particular classes of \((\alpha, T_0)\)-migrative t-norms that correspond to \( T_0 \in \{T_M, T_L\} \). Together with the already revealed case of \( T_0 = T_P \) in Section 3, these results complete the picture when \( T_0 \) is one of the prototype t-norms.

4.1 Continuous t-norms that are \((\alpha, T_M)\)-migrative

Assume that \( T_M = T_M \) in (2). That is, we consider \((\alpha, T_M)\)-migrative continuous triangular norms. In the present case the \((\alpha, T_M)\)-migrative property is read as follows \((x, y \in [0,1])\):

\[
T(\min(\alpha, x), y) = T(x, \min(\alpha, y)),
\]

or equivalently,

\[
T(\alpha, x) = \min(\alpha, x) \quad \text{for all } x \in [0,1].
\]

The description of all \((\alpha, T_M)\)-migrative continuous triangular norms, as solutions to Eq. (7), is given in the following theorem.
Theorem 4. A continuous t-norm \( T \) is \((\alpha, T_M)\)-migrative if and only if there exist two continuous t-norms \( T_1 \) and \( T_2 \) such that \( T \) can be written in the following form:

\[
T(x, y) = \alpha T_1 \left( \frac{x - \alpha}{\alpha}, \frac{y - \alpha}{1 - \alpha} \right)
\]

for \( x, y \in [0, \alpha] \).

\[
T(x, y) = \alpha + (1 - \alpha) T_2 \left( \frac{x - \alpha}{1 - \alpha}, \frac{y - \alpha}{1 - \alpha} \right)
\]

for \( x, y \in [\alpha, 1] \), and

\[
T(x, y) = \min(x, y)
\]

otherwise.

Roughly speaking, this result says that any continuous \((\alpha, T_M)\)-migrative t-norm is an ordinal sum of two arbitrary continuous t-norms, joining at \( \alpha \). In particular, this implies that \( \alpha \) is an idempotent element of \( T \).

4.2 Continuous t-norms that are \((\alpha, T_L)\)-migrative

In this subsection we study \((\alpha, T_0)\)-migrative t-norms when \( T_0 = T_L \). That is, now we have

\[
T_0(x, y) = T_L(x, y) = \max(x + y - 1, 0)
\]

for all \( x, y \in [0, 1] \). The equation that defines \((\alpha, T_L)\)-migrativity is as follows \((x, y \in [0, 1])\):

\[
T(\max(\alpha + x - 1, 0), y) = T(x, \max(\alpha + y - 1, 0)).
\]

We know from Theorem 3 that \( T \) is \((\alpha, T_L)\)-migrative if and only if we have for all \( x \in [0, 1] \) that

\[
T(x, \alpha) = \max(\alpha + x - 1, 0).
\]

The next lemma states that if a continuous t-norm \( T \) is migrative with respect to the \( \check{\text{\L}} \)ukasiewicz t-norm \( T_L \), then \( T \) is necessarily isomorphic to \( T_L \).

Lemma 1. Assume that \( T \) is a continuous t-norm that is \((\alpha, T_L)\)-migrative. Then there exists an automorphism \( \varphi \) of the unit interval such that \( T = T_L^\varphi \). That is, we have for all \( x, y \in [0, 1] \) that

\[
T(x, y) = T_L^\varphi(x, y) = \varphi^{-1}(\max(\varphi(x) + \varphi(y) - 1, 0)).
\]

Taking into account the functional form of \( T \) given in (10) and Theorem 3, we can reformulate \((\alpha, T_L)\)-migrativity of \( T = T_L^\varphi \) in terms of the automorphism \( \varphi \) as follows:

\[
\varphi^{-1}(\max[\varphi(\alpha) + \varphi(x) - 1, 0]) = \max(\alpha + x - 1, 0)
\]

for all \( x \in [0, 1] \).

Lemma 2. Let \( \alpha \in [0, 1] \), \( \varphi \) an automorphism of the unit interval, and \( T = T_L^\varphi \). Then the following statements are equivalent.

(i) \( T = T_L^\varphi \) is \((\alpha, T_L)\)-migrative.

(ii) \( \max[\varphi(\alpha) + \varphi(x) - 1, 0] = \varphi[\max(\alpha + x - 1, 0)] \) for all \( x \in [0, 1] \).

(iii) \( \varphi(\alpha + x - 1) = \varphi(\alpha) + \varphi(x) - 1 \) for all \( x \in [1 - \alpha, 1] \).

Notice that as \( x \) goes from \( 1 - \alpha \) to 1 on the right-hand side of Eq. in (ii), \( x - (1 - \alpha) \) runs from 0 to \( \alpha \) on the left-hand side. Thus, if we know \( \varphi \) on the interval \([1 - \alpha, 1]\) then equation in (ii) defines \( \varphi \) on \([0, \alpha]\) in a unique way. We will exploit this property extensively in the sequel.

For any \( x \in [0, 1] \), any \( k \in \mathbb{N} \), and for any t-norm \( T \), denote \( x_T^{(k)} \) the \( k \)th power of \( x \) with respect to \( T \), defined by \( x_T^{(0)} = 1 \), \( x_T^{(1)} = x \), and \( x_T^{(k)} = T\left(x_T^{(k-1)}, x_T^{(1)}\right) \) for \( k \geq 2 \).

When \( T = T_L \) we use the simplified notation \( x_L^{(k)} \) instead of \( x_L^{(k)} \). Obviously, \( x_L^{(k)} = \max\{kx - (k-1), 0\} \), and thus \( x_L^{(k)} > 0 \) if and only if \( x > \frac{k-1}{k} \).

Let us introduce the number \( n_L(x) \) as follows:

\[
n_L(x) = \max\left\{ k \in \mathbb{N} \mid x \geq \frac{k-1}{k} \right\}, \quad x \in [0, 1].
\]

Notice that \( n_L(x) \) is finite since \( x \in [0, 1] \). Obviously, \( x_L^{(k)} \) is positive for \( k < n_L(x) \). It is also positive for \( k = n_L(x) \), except when \( x = \frac{n_L(x)-1}{n_L(x)} \). Finally, \( x_L^{(k)} = 0 \) for \( k \geq n_L(x)+1 \). Notice also that \( x_L^{(k)} > 0 \) implies

\[
x_L^{(k)} + (1-x) = x_L^{(k-1)}.
\]

We are ready to establish the main result of this subsection: the characterization of continuous t-norms that are \((\alpha, T_L)\)-migrative.
**Theorem 5.** Let $\alpha \in ]0, 1[$ and $n = n_L(\alpha)$. A t-norm $T = T_L^\alpha$ is $(\alpha, S_L)$-migrative if and only if there exist an automorphism $\psi_0$ of the unit interval and a real number $\gamma$ such that \[
\frac{n-1}{n} \leq \gamma \leq \frac{n+1}{n+1}.
\]

$$\psi_0 \left( \frac{1 - \alpha - \alpha L(n)}{1 - \alpha} \right) = \frac{1 - \gamma - \gamma L(n)}{1 - \gamma}$$  \hspace{1cm} (12)

holds, and $\varphi$ can be written in the following form for $k \leq n$:

$$\varphi(x) = \gamma_L^{(k)} + (1 - \gamma)\psi_0 \left( \frac{x - \alpha L(n)}{1 - \alpha} \right)$$  \hspace{1cm} (13)

if $x \in \left[ \alpha L^{(k)}, \alpha L^{(k-1)} \right]$, and

$$\varphi(x) = \gamma_L^{(n)} + \gamma - 1 + (1 - \gamma)\psi_0 \left( \frac{x - \alpha L(n) + \alpha - 1}{1 - \alpha} \right)$$  \hspace{1cm} (14)

if $x \in \left[ 0, \alpha L^{(n)} \right]$.

Notice that, although (11) is true, $\gamma_L^{(n)} + \gamma - 1$ cannot be written as $\gamma_L^{(n+1)}$ since $\gamma_L^{(n)} + \gamma - 1 < 0$ while $\gamma_L^{(n+1)} = 0$. This is why we have to handle this case separately in (14).

In the following example we illustrate the above construction numerically.

**Example 1.** We would like to construct an automorphism $\varphi$ such that $T = T_L^\alpha$ is $(\alpha, S_L)$-migrative with $\alpha = 7/10$. Then we have $\alpha_L^{(2)} = 2\alpha - 1 = 2/5$, $\alpha_L^{(3)} = 3\alpha - 2 = 1/10$, and $\alpha_L^{(4)} = \max(4\alpha - 3, 0) = \max(-2/10, 0) = 0$. Thus, $n = n_L(7/10) = 3$. We want to find an appropriate automorphism $\psi_0$ and $\gamma$ such that (12) is satisfied. We have

$$\frac{1 - \alpha - \alpha L^{(3)}}{1 - \alpha} = \frac{3 - 4\alpha}{1 - \alpha} = \frac{2/10}{3/10} = \frac{2}{3}.$$

Let us choose the value of $\gamma$ between 2/3 and 3/4, say $\gamma = 17/25$. Then $\frac{1 - \gamma - (3\gamma - 2)}{1 - \gamma} = \frac{7}{8}$, so we need a $\psi_0$ that satisfies

$$\psi_0 \left( \frac{2}{3} \right) = \frac{7}{8}.$$

We consider the $\psi_0$ that is linear between points $(0, 0)$ and $(2/3, 7/8)$, as well as between $(2/3, 7/8)$ and $(1, 1)$. Elementary calculations show that

$$\psi_0(x) = \begin{cases} 
\frac{21}{16}x & \text{if } 0 \leq x \leq 2/3, \\
\frac{3}{8}x + \frac{5}{8} & \text{if } 2/3 \leq x \leq 1.
\end{cases}$$

Finally, define an automorphism $\varphi = \bigvee_{i=1}^4 \varphi_i$. According to Theorem 5, the corresponding t-norm $T_L^\alpha$ is $(\alpha, S_L)$-migrative.

In accordance with Example 1, we illustrate the proof and the construction graphically in Figure 2. On the right side one can see the automorphism $\psi_0$, where the bold point represents Eq. (12). On the left side we show the structure of the automorphism $\varphi$. In the upper-right rectangle we start with the appropriately transformed $\psi_0$. Then we proceed according to (13). Notice how we can guarantee $\varphi(0) = 0$ by (12).

### 5 $\alpha$-migrative triangular norms with respect to continuous ordinal sums

In this section we go beyond the above framework: we study continuous t-norms that are migrative with...
respect to an arbitrary continuous t-norm \( T_0 \). Obviously, this includes \( T_0 \in \{ T_M, T_L, T_P \} \) as well as any continuous non-Archimedean t-norm as particular cases.

5.1 Characterization of continuous migrative t-norms with respect to continuous ordinal sums

Now we start our study on continuous t-norms \( T \) that are \( \alpha \)-migrative with respect to a fixed continuous t-norm \( T_0 \). According to Theorem 1, such t-norms can be written as ordinal sums

\[
T = \langle (a_i, b_i, T_i) \rangle_{i \in \Gamma}, \\
T_0 = \langle (a_{0j}, b_{0j}, T_{0j}) \rangle_{j \in \Gamma_0},
\]

where \( T_i, T_{0j} \in T_0 \) for all \( i \in \Gamma \) and \( j \in \Gamma_0 \).

Then, obviously, for any \( \alpha \in [0, 1] \) there are two exhaustive and mutually exclusive cases: first, when \( \alpha \) is an idempotent element of \( T_0 \); second, when there exists a \( k \in \Gamma \) such that \( \alpha \in [a_{0k}, b_{0k}] \).

The following theorem characterizes the first case in terms of migrativity with respect to the minimum t-norm.

**Theorem 6.** Suppose \( T_0 \) is a continuous t-norm and \( \alpha \in [0, 1] \) is an idempotent element of \( T_0 \). Then the following statements are equivalent for a continuous t-norm \( T \):

(i) \( T \) is \((\alpha, T_0)\)-migrative;

(ii) \( T \) is \((\alpha, T_M)\)-migrative;

(iii) there exist continuous t-norms \( T_1 \) and \( T_2 \) such that \( T = \langle (0, \alpha, T_1), (\alpha, 1, T_2) \rangle \).

\[ \square \]

Therefore, whether the fixed t-norm \( T_0 \) is the minimum or any other continuous t-norm with \( \alpha \) as an idempotent element, the set of continuous \((\alpha, T_0)\)-migrative t-norms and the set of continuous \((\alpha, T_M)\)-migrative t-norms are the same.

Turning to the second case, the following theorem characterizes continuous \((\alpha, T_0)\)-migrative t-norms when \( \alpha \) is not an idempotent element of \( T_0 \).

**Theorem 7.** Suppose \( T_0 = \langle (a_{0j}, b_{0j}, T_{0j}) \rangle_{j \in \Gamma_0} \) is a continuous t-norm and \( \alpha \in [a_{0k}, b_{0k}] \) for some \( k \in \Gamma_0 \). Then the following statements are equivalent for a continuous t-norm \( T \):

(i) \( T \) is \((\alpha, T_0)\)-migrative;

(ii) There exist t-norms \( T_1, T_3 \in T_{C_0} \) and \( T_2 \in T_{Ar} \) such that

\[
(a) \ T = \langle (0, a_{0k}, T_1), (a_{0k}, b_{0k}, T_2), (b_{0k}, 1, T_3) \rangle,
\]

and

\[
(b) \ T_2 \text{ is } \left( \frac{\alpha - a_{0k}}{b_{0k} - a_{0k}} \right) - \text{migrative with respect to } T_{0k}.
\]

\[ \square \]

According to this theorem, the \((\alpha, T_0)\)-migrative property is restrictive on a continuous \( T \) mainly in a neighbourhood of \( \alpha \). This is just \( \alpha \) itself if \( \alpha \) is an idempotent element of \( T_0 \), and it is the square \([a_{0k}, b_{0k}]^2\) otherwise. Outside this neighbourhood \( T \) can arbitrarily be defined so that it is in harmony with the ordinal sum representation of continuous t-norms.

5.2 Possible forms of \( T_2 \)

After clarifying the structure of continuous migrative t-norms with respect to continuous ordinal sums, we concentrate on the summand in the middle of \( T = \langle (0, a_{0k}, T_1), (a_{0k}, b_{0k}, T_2), (b_{0k}, 1, T_3) \rangle \). More exactly, on the continuous Archimedean triangular norm \( T_2 \).

Let us start with \( T_{0k} \). According to Theorem 1 (d), there exists an automorphism \( \psi \) of the unit interval such that

\[
T_{0k}(x, y) = \psi^{-1}(T_L(\psi(x), \psi(y))), \quad (15)
\]

or

\[
T_{0k}(x, y) = \psi^{-1}(T_P(\psi(x), \psi(y))) \quad (16)
\]

holds for all \( x, y \in [0, 1] \).

We introduce \( \beta := \left( \frac{\alpha - a_{0k}}{b_{0k} - a_{0k}} \right) \) just for simplifying the notation in what follows.

5.2.1 When \( T_{0k} \) is nilpotent

Consider the case when \( T_{0k} \) is nilpotent; i.e., when it has the form of (15). Then, by Theorem 3, \( T_2 \) is \((\beta, T_{0k})\)-migrative if and only if for all \( x \in [0, 1] \) we have

\[
T_2(\beta, x) = \psi^{-1}(\max(\psi(\beta) + \psi(x) - 1, 0)) \quad (17)
\]

**Lemma 3.** Assume that \( T_2 \) is \((\beta, T_{0k})\)-migrative. Then there exists an automorphism \( \xi \) of the unit interval such that

\[
T_2(x, y) = \xi^{-1}(T_L(\xi(x), \xi(y))) \quad \text{holds for all } x, y \in [0, 1].
\]

\[ \square \]
Then Eq. (17) can be reformulated as follows:
\[ \xi^{-1} \left( \max(\xi(x) - 1, 0) \right) = \psi^{-1} \left( \max(\psi(\beta) + x - 1, 0) \right) \]
for all \( x \in [0, 1] \). Let us introduce \( y := \psi(x) \) and \( \lambda := \psi(\beta) \), \( \varphi := \xi \circ \psi^{-1} \). Then, obviously, \( \varphi \) is an automorphism of the unit interval, and this last equation can be written as follows: for all \( y \in [0, 1] \) we have
\[ \max(\varphi(\lambda) + \varphi(y) - 1, 0) = \varphi(\max(\lambda + y - 1, 0)). \tag{18} \]

According to [7, Lemma 6], this is equivalent to the \( \lambda \)-migrativity of \( T^\varphi_L \). Therefore, we can apply Theorem 8 from [7] to give the functional form of the automorphism \( \varphi \). Then the unknown \( \xi \) can be expressed as follows:
\[ \xi(x) = \varphi(\psi(x)), \quad x \in [0, 1], \]
which finally gives the functional form of \( T_2 \) when \( T_{ok} \) is nilpotent.

5.2.2 When \( T_{ok} \) is strict

We deal with the other possible case when \( T_{ok} \) is strict; i.e., when it has the form of (16). Then, by Theorem 3, \( T_2 \) is \( (\beta, T_{ok}) \)-migrative if and only if
\[ T_2(\beta, x) = \psi^{-1}(\psi(\beta) \cdot x) \quad \text{for all } x \in [0, 1]. \tag{19} \]

Lemma 4. Assume that \( T_2 \) is \( (\beta, T_{ok}) \)-migrative. Then there exists an automorphism \( \xi \) of the unit interval such that
\[ T_2(x, y) = \xi^{-1}(T_P(\xi(x), \xi(y))) = \xi^{-1}(\xi(x) \cdot \xi(y)) \]
holds for all \( x, y \in [0, 1] \).

Then Eq. (19) can be reformulated as follows:
\[ \xi^{-1}(\xi(x)) = \psi^{-1}(\psi(\beta) \cdot x) \]
for all \( x \in [0, 1] \). Let us introduce \( y := \psi(x) \) and \( \lambda := \psi(\beta) \), and \( \varphi := \xi \circ \psi^{-1} \). Then, obviously, \( \varphi \) is an automorphism of the unit interval, and this last equation can be written as follows:
\[ \varphi(\lambda) \cdot \varphi(y) = \varphi(\lambda \cdot y), \quad y \in [0, 1]. \tag{20} \]

In our paper [5] we completely solved the original migrativity problem for continuous t-norms. The characterization and construction was given in terms of additive generators of strict t-norms. Knowing however that if \( \xi : [0, 1] \rightarrow [0, \infty] \) is an additive generator of a strict t-norm \( T \) then \( \varphi(x) := \exp(-t(x)) \) is an automorphism such that \( T(x, y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y)) \), one can reformulate the original results of [5, Theorem 3] in terms of automorphisms, similarly to Eq. (28) in [7]. Therefore, one can determine the functional form of the automorphism \( \varphi \). Then the unknown \( \xi \) can be expressed as follows:
\[ \xi(x) = \varphi(\psi(x)), \quad x \in [0, 1], \]
which finally gives the functional form of \( T_2 \) when \( T_{ok} \) is strict.

5.3 Examples

We would like to illustrate the results by some examples. Let us start from the following t-norm \( T_0 \) that we use in all the examples.
\[ T_0 = (\langle 1/6, 1/3, T_L \rangle, \langle 1/3, 2/3, T_P \rangle, \langle 5/6, 1, T_P \rangle). \tag{21} \]
For more formal identifiability with notations we used above, we write
\[ a_{01} = 1/6, b_{01} = 1/3, a_{02} = 1/3, b_{02} = 2/3, \]
\[ a_{03} = 5/6, b_{03} = 1, \]
\[ T_{01} = T_L, T_{02} = T_P, T_{03} = T_P, \quad \Gamma_0 = \{1, 2, 3\}. \]

In the forthcoming three examples we construct \( (\alpha, T_0) \)-migrative t-norms for the three possible cases above.

Example 2. Let \( \alpha = 2/3 \) and consider \( T_0 \) in (21). Then \( T_0(\alpha, \alpha) = \alpha \), so any \( (2/3, T_0) \)-migrative t-norm \( T \) is written as
\[ T = \langle 0, 2/3, T_1 \rangle, \langle 2/3, 1, T_2 \rangle, \]
by Theorem 6. The summand t-norms \( T_1 \) and \( T_2 \) can be any continuous t-norms. For instance, \( T_1 = T_2 = T_0 \) is as good as \( T_1 = T_P \) and \( T_2 = T_L \), or as \( T_1 = T_2 = T_M \).

Example 3. Let \( \alpha = 5/12 \) and consider \( T_0 \) in (21). Then \( \alpha \in |a_{02}, b_{02}| \), so \( k = 2 \) and \( \beta = (\alpha - a_{02}) / (b_{02} - a_{02}) = 1/4 \) in Theorem 7. By the same theorem we know that any \( (5/12, T_0) \)-migrative t-norm \( T \) is written as
\[ T = \langle 0, 1/3, T_1 \rangle, \langle 1/3, 2/3, T_2 \rangle, \langle 2/3, 1, T_3 \rangle, \]
with continuous \( T_1 \) and \( T_3 \), and with continuous Archimedean \( T_2 \), which is \( (1/4, T_P) \)-migrative. Then \( T_2(x, y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y)) \) with an appropriate automorphism \( \varphi \). This automorphism can be given as follows (see Eq. (28) in [7]):
\[ \varphi(x) = \psi_0 \left( \frac{x}{(1/4)^k} \right) \cdot (\varphi(1/4))^{k} \]
if \( x \in (1/4)^{k+1}, (1/4)^k \), where \( k = 0, 1, 2, \ldots \) and \( \psi_0 \) is any automorphism of the unit interval.
Example 4. Let $\alpha = 1/4$ and consider $T_0$ in (21). Then $\alpha \in [a_{01}, b_{01}]$, so $k = 1$ and $\beta = \left( \frac{\alpha - a_{01}}{a_{01} - b_{01}} \right) = 1/2$ in Theorem 7. This theorem states that any $(1/4, T_0)$-migrative t-norm $T$ is written as
\[ T = (\langle 0, 1/6, T_1 \rangle, \langle 1/6, 1/3, T_2 \rangle, \langle 1/3, 1, T_3 \rangle), \]
with continuous $T_1$ and $T_3$, and with continuous Archimedean $T_2$, which is $(1/2, T_L)$-migrative. Then $T_2(x, y) = \varphi^{-1}(\max(\varphi(x) + \varphi(y) - 1, 0))$ with an appropriate automorphism $\varphi$. We construct such a $\varphi$ now, by applying results in [5, Theorem 3] as follows.

Consider any automorphism $\psi_0$ of the unit interval and define $\varphi$ as follows:
\[ \varphi(x) = \begin{cases} 
\frac{x}{2} + \frac{1}{2} \cdot \psi_0(2x - 1) & \text{if } x \in [1/2, 1], \\
\frac{1}{2} \cdot \psi_0(2x) & \text{if } x \in [0, 1/2]. 
\end{cases} \]

Then $T_2 = T_{2,L}^\varphi$ is $(1/2, T_L)$-migrative, so the ordinal sum construction with three summands yields a continuous t-norm $T$ which is $(1/4, T_0)$-migrative.

6 Concluding remarks

In this paper we summarized our recent results on migrative triangular norms. First we considered the original notion (Definition 1) and completely characterized continuous t-norms that are migrative. It turned out that only strict t-norms can own this property. Next we studied the extended notion (Definition 2) in general, and for the prototype t-norms $T_M$ and $T_L$ in particular. We could characterize continuous migrative t-norms with respect to the minimum and the Łukasiewicz t-norm. Finally, based mainly on these results, we could characterize continuous $\alpha$-migrative t-norms with respect to an arbitrary continuous t-norm (i.e., a continuous ordinal sum).

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References: