Existence of fuzzy equilibria for fuzzy abstract economies with Q'–majorized correspondences

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Abstract: In this paper, by using an existence theorem of maximal elements for Q’–majorized correspondences and Kim and Lee’s existence theorems of best proximity pairs (2006), we prove the existence of fuzzy equilibrium pairs for fuzzy abstract economies with Q’–majorized preference correspondences.

Key–Words: Q’–majorized correspondences, upper semicontinuous correspondences, free fuzzy abstract economy, fuzzy abstract economy, fuzzy equilibrium pair. ■

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1 Introduction

The classical model of abstract economy ([4]) was generalized in [2] by Kim and Lee, who defined the free abstract economy and proved existence theorems of best proximity pairs and equilibrium pairs.

L. Zadeh initiated the theory of fuzzy sets [6] as a framework for phenomena which can not be characterized precisely. In [1] the authors introduced the concept of a fuzzy game and proved the existence of equilibrium for 1-person fuzzy game.

In this paper, by using Kim and Lee’s existence theorems of best proximity pairs [2] and an existence theorem of maximal elements for Q’–majorized correspondences, we prove the existence of fuzzy equilibrium pairs for abstract fuzzy economies.

The paper is organized in the following way: Section 2 contains preliminaries and notation. The fuzzy equilibrium pair theorems are stated in Section 3 and the existence of equilibrium for free fuzzy abstract economies is studied in section 4.

2 Preliminaries and notation

Definition 1. Let X, Y be topological spaces and T : X → 2^Y be a correspondence

1. T is said to be upper semicontinuous if for each x ∈ X and each open set V in Y with T(x) ⊂ V, there exists an open neighborhood U of x in X such that T(x) ⊂ V for each y ∈ U.

2. T is said to be lower semicontinuous (shortly l.s.c) if for each x ∈ X and each open set V in Y with T(x) ∩ V ≠ ∅, there exists an open neighborhood U of x in X such that T(y) ∩ V ≠ ∅ for each y ∈ U.

3. The correspondence T is defined by T(x) = {y ∈ Y : (x, y) ∈ cl_{X×Y}Gr(T)} (the set cl_{X×Y}Gr(T) is called the adherence of the graph of T). It is easy to see that clT(x) ⊂ T(x) for each x ∈ X.

Definition 2. Let X, Y be topological spaces and T : X → 2^Y be a correspondence. An element x ∈ X is named maximal element for T if T(x) = ∅.

For each i ∈ I, let X_i be a nonempty subset of a topological space E_i and T_i : X := ∏_{i ∈ I} X_i → 2^{Y_i} a correspondence. Then a point x ∈ X is called a maximal element for the family of correspondences \{T_i\}_{i ∈ I} if T_i(x) = ∅ for all i ∈ I.

Now we prove the following fixed point theorem.

Theorem 1. Let Γ = (X_i, A_i, P_i, B_i)_{i ∈ I} be an abstract economy, where I is a (possibly uncountable) set of agents such that for each i ∈ I :

(1) X_i is a non-empty convex set in a Hausdorff locally convex space E_i, X := ∏_{i ∈ I} X_i is paracompact and D_i is a non-empty, convex, compact subset of X_i;

(2) B_i is lower semicontinuous with non-empty convex values and for each x ∈ X, A_i(x) ≠ ∅, A_i(x) ⊂ B_i(x) and clB_i(x) ∩ D_i ≠ ∅;
(3) the correspondence $A_i \cap P_i : X \rightarrow 2^{D_i}$ is lower semi-continuous;

(4) for each $x \in X$, $x_i \notin (\text{co}A_i \cap \text{co}P_i)(x)$.

Then there exists an equilibrium point $x^* \in D$ for $\Gamma$, i.e., for each $i \in I$, $x_i^* \in B_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

Proof. For each $i \in I$ let $W_i := \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$. By condition (3) we have that $W_i$ is open in $X$.

For each $i \in I$ and $x \in X$, let

$$F_i(x) = \begin{cases} \text{co}(A_i \cap P_i)(x), & \text{if } x \in W_i, \\ B_i(x), & \text{if } x \in X \setminus W_i; \end{cases}$$

For each closed set $V$ in $D_i$, the set

$$\{x \in X : F_i(x) \subset V\} = \{x \in W_i : \text{co}(A_i \cap P_i)(x) \subset V\} \cup \{x \in X \setminus W_i : B_i(x) \subset V\}.$$

By condition (3), the set

$$\{x \in X : \text{co}(A_i \cap P_i)(x) \subset V\}$$

is closed in $X$. Since $W_i$ is open in $X$, $X \setminus W_i$ is closed in $X$ and hence the set

$$\{x \in X : B_i(x) \subset V\}$$

is closed in $X$ because $B_i : X \rightarrow 2^{D_i}$ is lower semi-continuous. Therefore, the set $\{x \in X : F_i(x) \subset V\}$ is closed in $X$. Therefore, $F_i : X \rightarrow 2^{D_i}$ is lower semi-continuous. $F_i$ also has non-empty and convex values.

Define $F : X \rightarrow 2^D$ by $F(x) = \prod_{i \in I} F_i(x)$.

The correspondence $F$ is l.s.c with non-empty closed convex values and there exists a set $D$ such that $G(x) \cap D \neq \emptyset$ for each $x \in X$.

By Corollary 3.2 in [3], it follows that exists $x^* \in D$ such that $x^* \in F(x^*)$, i.e., for each $i \in I$, $x_i^* \in \mathcal{F}_i(x^*)$.

If there exists some $i_0 \in I_0$ such that $x_i^* \in W_i$, $x_i^* \in \mathcal{F}_i(x^*)$, by the definition of $F_i$, then $x_i^* \in (\text{co}A_i \cap \text{co}P_i)(x^*)$, which contradicts assumption 5.

Therefore, $x_i^* \notin W_i$ for all $i \in I_0$, i.e. $A_i(x^*) \cap P_i(x^*) = \emptyset$ for all $i \in I_0$. By the definition of $F_i$, we must have that $x_i^* \in B_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$ for all $i \in I_0$.

Theorem 2 is an existence theorem for maximal elements that is a consequence of Theorem 1.

**Theorem 2.** Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game where $I$ is an index set such that for each $i \in I$, the following conditions hold:

1) $X_i$ is a nonempty convex compact subset of a Hausdorff locally convex topological vector space $E$ and $X := \prod_{i \in I} X_i$;
2) $P_i : X \rightarrow 2^{X_i}$ is lower semi-continuous;
3) for each $x \in X$, $x_i \notin \text{co}P_i(x)$

Then there exists a point $\pi \in X$ such that $P_i(\pi) = \emptyset$ for all $i \in I$, i.e. $\pi$ is a maximal element of $\Gamma$.

**Definition 3.** [3] Let $X$ be a topological space and $Y$ be a non-empty subset of a vector space $E$, $\theta : X \rightarrow E$ be a mapping and $T : X \rightarrow 2^Y$ be a correspondence.

(1) $T$ is said to be of class $Q_0$ (or $Q'$) if

(a) for each $x \in X$, $\theta(x) \notin T(x)$ and
(b) $T$ is lower semicontinuous with open and convex values in $Y$;

(2) A correspondence $T_x : X \rightarrow 2^Y$ is said to be a $Q_0$-majorant of $T$ at $x$ if there exists an open neighborhood $N(x)$ of $x$ such that

(a) For each $z \in N(x)$, $T(z) \subset T_x(z)$ and
(b) $T_x$ is l.s.c. with open and convex values;

(3) $T$ is said to be $Q_0$-majorized if for each $x \in X$ with $T(x) \neq \emptyset$ there exists a $Q_0$-majorant $T_x$ of $T$ at $x$.

**Notation.** Let $X$ and $Y$ be any two subsets of a normed space $E$ with a norm $\| \cdot \|$, and the metric $d(x, y)$ is induced by the norm. We use the following notation:

$$\text{Prox}(X, Y) := \{(x, y) \in X \times Y : d(x, y) = d(X, Y) = \inf\{d(x, y) : x \in X, y \in Y\}\};$$

$X_0 := \{x \in X : d(x, y) = d(X, Y) \text{ for some } y \in Y\};$

$Y_0 := \{y \in Y : d(x, y) = d(X, Y) \text{ for some } x \in X\}$.

If $X$ and $Y$ are non-empty compact and convex subsets of a normed linear space, then it is easy to see that $X_0$ and $Y_0$ are both non-empty compact and convex.

Let $I$ be a finite (or an infinite) index set. For each $i \in I$, let $X_i$ and $Y_i$ be a nonempty subsets of a normed space $E$ with a norm $\| \cdot \|$, and the metric $d(x, y)$ is induced by the norm. Then, we can use the following notation: for each $i \in I$,

$$X_0^i := \{x \in X_i : \text{for each } i \in I, \exists y_i \in Y_i \text{ such that } d(x, y_i) = d(X, Y_i) = \inf\{d(x, y) : x \in X_i, y \in Y_i\}\};$$

$$Y_0^i := \{y \in Y_i : \text{there exists } x \in X_i \text{ such that } d(x, y) = d(X, Y_i)\}.$$

When $|I| = 1$, it is easy to see that $X_0 = X_0^0$ and $Y_0 = Y_0^0$.

**Notation.** Let $E$ and $F$ be two Hausdorff topological vector spaces and $X \subset E$, $Y \subset F$ be two nonempty convex subsets. We denote by $\mathcal{F}(Y)$ the collection of fuzzy sets on $Y$. A mapping from $X$ into $\mathcal{F}(Y)$ is called a fuzzy mapping. If $F : X \rightarrow \mathcal{F}(Y)$ is a fuzzy mapping, then for each $x \in X$, $F(x)$ (denoted by $F_x$ in this sequel) is a fuzzy set in $\mathcal{F}(Y)$ and $F_x(y)$ is the degree of membership of point $y$ in $F_x$. 

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A fuzzy mapping $F : X \rightarrow \mathcal{F}(Y)$ is called convex, if for each $x \in X$, the fuzzy set $F_x$ on $Y$ is a fuzzy convex set, i.e., for any $y_1, y_2 \in Y$, $t \in [0, 1]$, $F_x((1-t)y_1 + ty_2) \geq \min\{F_x(y_1), F_x(y_2)\}$.

In the sequel, we denote by

$$(A) q = \{y \in Y : A(y) \geq q\}, q \in [0, 1]$$

the $q$-cut set of $A \in \mathcal{F}(Y)$.

### 3 Existence of fuzzy equilibrium pairs for fuzzy abstract economies

In this section we describe the fuzzy equilibrium pair for a fuzzy abstract economy. We prove the existence of fuzzy equilibrium of fuzzy abstract economies in several cases.

Let $I$ be a nonempty set (the set of agents). For each $i \in I$, let $X_i$ be a nonempty topological vector space representing the set of actions and define $X := \prod_{i \in I} X_i$; let $A_i, B_i : X \rightarrow \mathcal{F}(X_i)$ be the constraint fuzzy correspondences and $P_i : X \rightarrow \mathcal{F}(X_i)$ the preference fuzzy correspondence, $a_i, b_i : X \rightarrow (0, 1]$ fuzzy constraint functions and $p_i : X \rightarrow (0, 1]$ fuzzy preference function.

**Definition 4.** An abstract fuzzy economy is defined as an ordered family $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i \in I}$.

If $A_i, B_i, P_i : X \rightarrow 2^{X_i}$ are classical correspondences, then the previous definition can be reduced to the standard definition of abstract economy due to Yuan [5].

We introduce the following concept of fuzzy equilibrium.

**Definition 5.** A fuzzy equilibrium pair for $\Gamma$ is defined as a pair of points $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in (B_{\overline{x}_i})_{b_i}(\overline{y}_i)$ and $(A_{\overline{x}_i})_{a_i}(\overline{x}_i) \cap (P_{\overline{x}_i})_{p_i}(\overline{y}_i) = \emptyset$, where $(A_{\overline{x}_i})_{a_i}(\overline{x}_i) = \{z \in \overline{y}_i : A_i(z) \geq a_i(\overline{x}_i)\}$, $(B_{\overline{x}_i})_{b_i}(\overline{y}_i) = \{z \in \overline{y}_i : B_i(z) \geq b_i(\overline{x}_i)\}$, $(P_{\overline{x}_i})_{p_i}(\overline{y}_i) = \{z \in \overline{y}_i : P_i(z) \geq p_i(\overline{y}_i)\}$.

We state some new fuzzy equilibrium existence theorems for fuzzy abstract economies.

**Theorem 3.** Let $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i \in I}$ be a fuzzy abstract economy such that for each $i \in I$:

1. $X_i$ is a nonempty compact convex subset of a locally convex Hausdorff topological vector space $E$;
2. $P_i$ is such that $x \rightarrow (P_{\overline{x}_i})_{p_i}(x) : X \rightarrow 2^{X_i}$ is $Q'_{\overline{x}_i}$-majorized on $X$ and has nonempty convex values;
3. $A_i, B_i$ are such that $x \rightarrow (B_{\overline{x}_i})_{b_i}(x) : X \rightarrow 2^{X_i}$ is lower semicontinuous such that each $(B_{\overline{x}_i})_{b_i}(x)$ is a closed convex subset of $X_i$, $(A_{\overline{x}_i})_{a_i}(x)$ is nonempty convex and $(A_{\overline{x}_i})_{a_i}(x) \subset (B_{\overline{x}_i})_{b_i}(x)$ for each $x \in X$.

Then there exists a fuzzy equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in (B_{\overline{x}_i})_{b_i}(\overline{y}_i)$ and $(A_{\overline{x}_i})_{a_i}(\overline{x}_i) \cap (P_{\overline{x}_i})_{p_i}(\overline{y}_i) = \emptyset$.

**Proof.** For each $i \in I$, $x \rightarrow (B_{\overline{x}_i})_{b_i}(x)$ is lower semicontinuous and it has non-empty, convex and closed values. By Theorem 3.2 in [3], there exists $\overline{x}_i \in (B_{\overline{x}_i})_{b_i}(\overline{y}_i)$ for each $i \in I$. It remains to show that there exists a point $\overline{x}_i \in X$ such that $\overline{x}_i \in (B_{\overline{x}_i})_{b_i}(\overline{y}_i)$ and $(A_{\overline{x}_i})_{a_i}(\overline{x}_i) \cap (P_{\overline{x}_i})_{p_i}(\overline{y}_i) = \emptyset$ for each $i \in I$.

Since $X$ is paracompact and $x \rightarrow (P_{\overline{x}_i})_{p_i}(x)$ is $Q'_{\overline{x}_i}$-majorized, by Lemma 4.1 in [3], there exists a correspondence $\varphi_i : X \rightarrow 2^{X_i}$ of class $Q'_{\overline{x}_i}$ such that $(P_{\overline{x}_i})_{p_i}(x) \subset \varphi_i(x)$ for each $x \in X$. Then $\varphi_i$ is lower semicontinuous with nonempty open convex values and $x_i \notin \varphi_i(x)$ for each $x \in X$.

Define $T_i : X \rightarrow 2^{X_i}$,

$$T_i(y) = \left\{ (A_{\overline{x}_i})_{a_i}(\overline{x}_i) \cap \varphi_i(y), \text{if } y_i \in cl(P_{\overline{x}_i})_{p_i}(\overline{y}_i); \right.\left. \varphi_i(y), \text{if } y_i \notin cl(P_{\overline{x}_i})_{p_i}(\overline{y}_i). \right\}$$

By Theorem 1.1 ([5], pag. 23), it follows that $T_i$ is lower semicontinuous on $X$. Then $clT_i$ is lower semicontinuous, it has convex values and $x_i \notin T_i(x)$.

By Theorem 2, there exists $\overline{y}_i \in X$ such that $clT_i(\overline{y}_i) = \emptyset$ for each $i \in I$.

For each $y \in X$, $\varphi_i(y)$ is a non-empty subset of $X_i$. We have $\overline{y}_i \in cl(P_{\overline{x}_i})_{p_i}(\overline{y}_i)$ and $cl((A_{\overline{x}_i})_{a_i}(\overline{x}_i) \cap \varphi_i(\overline{y}_i)) = \emptyset$. It follows that $(A_{\overline{x}_i})_{a_i}(\overline{x}_i) \cap \varphi_i(\overline{y}_i) = \emptyset$.

Since $(P_{\overline{x}_i})_{p_i}(\overline{y}_i) \subset \varphi_i(\overline{y}_i)$ we have that $(A_{\overline{x}_i})_{a_i}(\overline{x}_i) \cap (P_{\overline{x}_i})_{p_i}(\overline{y}_i) = \emptyset$. Hence, $\overline{x}_i \in (B_{\overline{x}_i})_{b_i}(\overline{y}_i)$, $\overline{y}_i \in (P_{\overline{x}_i})_{p_i}(\overline{y}_i)$ and $(A_{\overline{x}_i})_{a_i}(\overline{x}_i) \cap (P_{\overline{x}_i})_{p_i}(\overline{y}_i) = \emptyset$ for each $i \in I$; and then $(\overline{x}, \overline{y})$ is an equilibrium pair for $\Gamma$. □

**Corollary 1.** Let $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i \in I}$ be a fuzzy abstract economy such that for each $i \in I$:

1. $X_i$ be a nonempty compact convex subset of a locally convex Hausdorff topological vector space $E$;
2. $P_i$ is such that $x \rightarrow (P_{\overline{x}_i})_{p_i}(x) : X \rightarrow 2^{X_i}$ is lower semicontinuous on $X$, has nonempty open convex values and $x_i \notin (P_{\overline{x}_i})_{p_i}(\overline{y}_i)$ for each $x \in X$;
3. $A_i, B_i$ are such that $x \rightarrow (B_{\overline{x}_i})_{b_i}(x) : X \rightarrow 2^{X_i}$ is lower semicontinuous such that each $(B_{\overline{x}_i})_{b_i}(x)$ is a closed convex subset of $X_i$, $(A_{\overline{x}_i})_{a_i}(x)$ is nonempty convex and $(A_{\overline{x}_i})_{a_i}(x) \subset (B_{\overline{x}_i})_{b_i}(x)$ for each $x \in X$;

Then there exists a fuzzy equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in (B_{\overline{x}_i})_{b_i}(\overline{y}_i)$ and $(A_{\overline{x}_i})_{a_i}(\overline{x}_i) \cap (P_{\overline{x}_i})_{p_i}(\overline{y}_i) = \emptyset$.

**Theorem 4.** Let $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i \in I}$ be a fuzzy abstract economy such that for each $i \in I$:
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(1) $X_1$ be a non-empty compact convex subset of a locally convex Hausdorff topological vector space $E$;

(2) $P_i$ is such that $x \rightarrow (P_{ix})_{p_i(x)} : X \rightarrow 2^{X_i}$ is $Q'_i$-majorized on $X$ and has nonempty values;

(3) $A_i, B_i$ are such that $x \rightarrow (B_{ix})_{b_i(x)} : X \rightarrow 2^{X_i}$ is upper semicontinuous such that each $(B_{ix})_{b_i(x)}$ is a closed convex subset of $X_1$, $(A_i)_{a_i(\bar{\tau})}$ is nonempty convex and $(A_i)_{b_i(x)} \subset (B_{ix})_{b_i(x)}$ for each $x \in X$;

Then there exists a fuzzy equilibrium pair $(\bar{\tau}, \bar{\eta}) \in X \times X$ such that for each $i \in I$, $\bar{\tau}_i \in (\overline{B_{ix}})_{b_i(x)}$ and $(A_i)_{a_i(\bar{\tau})} \cap (P_{ix})_{p_i(\bar{\tau})} = \emptyset$.

4 Existence of fuzzy equilibrium pairs for free fuzzy abstract economies

Let $I$ be a nonempty set (the set of agents). For each $i \in I$, let $X_i$ be a non-empty set of manufacturing commodities, and $Y_i$ be a non-empty set of selling commodities. Define $X := \Pi_{i \in I} X_i$, let $A_i, B_i : X \rightarrow \mathcal{F}(Y_i)$ be the fuzzy constraint correspondences, $P_i : Y := \Pi_{i \in I} Y_i \rightarrow \mathcal{F}(Y_i)$ the fuzzy preference correspondence, $a_i, b_i : X \rightarrow (0, 1]$ fuzzy constraint functions and $p_i : Y \rightarrow (0, 1]$ fuzzy preference function. We consider that $X_i$ and $Y_i$ are non-empty subsets of a normed linear space $E$.

Definition 6. A free fuzzy abstract economy is defined as an ordered family $\Gamma = (X_i, Y_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i \in I}$.

Definition 7. A fuzzy equilibrium pair for $\Gamma$ is defined as a pair of points $(\bar{\tau}, \bar{\eta}) \in X \times X$ such that for each $i \in I$, $\bar{\tau}_i \in (B_{ix})_{b_i(x)}$ with $d(\bar{\tau}_i, \bar{\eta}_i) = d(X_i, Y_i)$ and $(A_i)_{a_i(\bar{\tau})} \cap (P_{ix})_{p_i(\bar{\tau})} = \emptyset$, where $(A_i)_{a_i(\bar{\tau})} = \{z \in Y_i : A_i(z) \geq a_i(\bar{\tau})\}$, $(B_{ix})_{b_i(x)} = \{z \in Y_i : B_i(z) \geq b_i(x)\}$ and $(P_{ix})_{p_i(\bar{\tau})} = \{z \in Y_i : P_i(z) \geq p_i(\bar{\tau})\}$.

If $A_i, B_i, P_i : X \rightarrow 2^{X_i}$ are classical correspondences and $A_i = B_i$, then we get the definition of free abstract economy and equilibrium pair defined by W.K. Kim and K. H. Lee in [2].

Whenever $X_i = X$ for each $i \in I$, for the simplicity, we may assume $A_i : X \rightarrow \mathcal{F}(Y_i)$ instead of $A_i : \Pi_{i \in I} X_i \rightarrow \mathcal{F}(Y_i)$ for the free abstract fuzzy economy $\Gamma = (X, Y_i, A_i, P_i, a_i, b_i, p_i)_{i \in I}$ and equilibrium pair. In particular, when $I = \{1, 2... n\}$, we may call $\Gamma$ a free n-person fuzzy game.

The economic interpretation of an equilibrium pair for $\Gamma$ is based on the requirement that for each $i \in I$, minimize the travelling cost $d(x_i, y_i)$, and also, maximize the preference $P_i$ on the constraint set $A_i$. Therefore, it is contemplated to find a pair of points $(\bar{\tau}, \bar{\eta}) \in X \times Y$ such that for each $i \in I$, $\bar{\tau}_i \in (B_{ix})_{b_i(x)}$ and $(A_i)_{a_i(\bar{\tau})} \cap (P_{ix})_{p_i(\bar{\tau})} = \emptyset$.

Corollary 2. Let $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i \in I}$ be a fuzzy abstract economy such that for each $i \in I$:

(1) $X_i$ be non-empty compact convex subsets of a locally convex Hausdorff topological vector space $E$;

(2) $P_i$ is such that $x \rightarrow (P_{ix})_{p_i(x)} : X \rightarrow 2^{X_i}$ is lower semicontinuous on $X$, has nonempty closed convex values and $x_i \notin (P_{ix})_{p_i(x)}$ for each $x \in X$;

(3) $A_i, B_i$ are such that $x \rightarrow (B_{ix})_{b_i(x)} : X \rightarrow 2^{X_i}$ is upper semicontinuous such that each $(B_{ix})_{b_i(x)}$ is a closed convex subset of $X_i$, $(A_i)_{a_i(\bar{\tau})}$ is nonempty convex and $(A_i)_{b_i(x)} \subset (B_{ix})_{b_i(x)}$ for each $x \in X$;

Then there exists a fuzzy equilibrium pair $(\bar{\tau}, \bar{\eta}) \in X \times X$ such that for each $i \in I$, $\bar{\tau}_i \in (B_{ix})_{b_i(x)}$ and $(A_i)_{a_i(\bar{\tau})} \cap (P_{ix})_{p_i(\bar{\tau})} = \emptyset$.

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and $\| \mathbf{x}_i - \mathbf{y}_i \| = d(X_i, Y_i)$, where $d(X_i, Y_i) = \inf \{ \| \mathbf{x}_i - \mathbf{y}_i \| : x_i \in X_i, y_i \in Y_i \}.$

When in addition $X_i = Y_i$ and $A_i, B_i, P_i : X \to 2^Y_i$ are classical correspondences for each $i \in I$, then the previous definitions can be reduced to the standard definitions of equilibrium theory in economics due to Yannelis and Prabhakar [4].

To prove our equilibrium theorems we need the following results.

**Definition 8** [2] Let $X$ and $Y$ be two non-empty subsets of a normed linear space $E$, and let $T : X \to 2^Y$ be a correspondence. Then the pair $(\mathbf{x}, T(\mathbf{x}))$ is called the best proximity pair [2] for $T$ if $d(\mathbf{x}, T(\mathbf{x})) = d(\mathbf{x}, \mathbf{y}) = d(X, Y)$ for some $\mathbf{y} \in T(\mathbf{x})$. Then the best proximity pair theorem seeks an appropriate solution which is optimal. In fact, the best proximity pair theorem analyzes the conditions which the problem of minimizing the real-valued function $x \to d(x, T(x))$ has a solution.


**Theorem 5.** For each $i \in I = \{1, \ldots, n\}$, let $X$ and $Y$ be non-empty compact and convex subsets of a normed linear space $E$, and let $T_i : X \to 2^{Y_i}$ be an upper semicontinuous correspondence in $X^0$ such that $T_i(x)$ is nonempty closed and convex subset of $Y_i$ for each $x \in X$. Assume that $T_i(x) \cap Y_i^0 \neq \emptyset$ for each $x \in X^0$.

Then there exists a system of best proximity pairs $(\mathbf{x}, T(\mathbf{x})) \subseteq X \times Y$, i.e., for each $i \in I$, $d(\mathbf{x}, T(\mathbf{x})) = d(X, Y)$.

**Definition 9** [2]. The set $A_i = \{y \in Y : y \in A(x)\}$ and $d(x, y) = d(X, Y)$ is named the best proximity set of the correspondence $A : X \to 2^Y$ at $x$.

In general, $A_i$ might be an empty set. If $(x, A_i(x))$ is a proximity pair for $A$ and $A(x)$ is compact, then $A_i$ must be non-empty.

Theorem 6 is an existence theorem of pair equilibrium for a free n-person fuzzy game with upper semi-continuous constraint correspondences and $Q'_b$-majorized preference correspondences.

**Theorem 6.** Let $\Gamma = (X, Y, A_i, B_i, P_i, a_i, b_i, p_i)_{i \in I}$ be a free n-person fuzzy game such that for each $i \in I = \{1, 2, \ldots, n\}$:

1. $X$ and $Y_i$ are non-empty compact and convex subsets of normed linear space $E$.
2. $A_i, B_i : X \to \mathcal{F}(Y_i)$ are such that $x \to (B_i(x))_b(x) : X \to 2^{Y_i}$ is upper semicontinuous in $X^0$, $(B_i(x))_b(x)$ is a closed convex subset of $Y_i$, $(B_i(x))_b(x) \cap Y_i^0 \neq \emptyset$ for each $x \in X^0$, $(A_i(x))_a(x)$ is nonempty, convex and $(A_i(x))_a(x) \subset (B_i(x))_b(x)$ for each $x \in X$.
3. $P_i : Y := \prod_{i \in I} Y_i \to \mathcal{F}(Y_i)$ is such that $y \to (P_i(y))_p(y) : Y \to 2^{Y_i}$ is $Q'_b$-majorized; $(P_i(y))_p(y)$ is nonempty for each $y \in Y$.

Then there exists a fuzzy equilibrium pair of points $(\mathbf{x}, \mathbf{y}) \in X \times Y$ such that $(P_i(y))_p(y) = \{y_i \mid y_i \in Y_i\}$.
points \((\underline{v}, \underline{w}) \in X \times Y\) such that for each \(i \in I\),
\[(\underline{v})_i \in (B_i(\underline{v}))_i \text{ with } d(\underline{v}, \underline{w}) = d(X_i, Y_i)\]
and \((A_i(\underline{v}))_i \cap (P_i(\underline{w}))_i = \emptyset\).

5 Conclusion

We developed the equilibrium theory in the framework of fuzzyness, introducing a new concept of equilibrium. We worked with two models of fuzzy abstract economy, both of them having \(Q'\)-majorized correspondences.

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